446.328 Mechanical System Analysis
기계시스템해석

- lecture 19, 20, 21-

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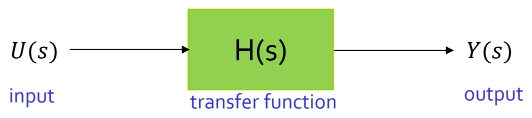
Frequency Response

transfer function (input-to-output)



- often, we want to know (or design) system's response when the input signal contains certain frequency components, e.g.,

$$u(t) = A_1 \sin(w_1 t + \phi_1) + A_2 \sin(w_2 t + \phi_2) + \cdots$$

↖ excitation mode

- e.g., microphone/headphone, engine mount, earthquake-proof structure,...)

⇒ frequency response of $H(s)$

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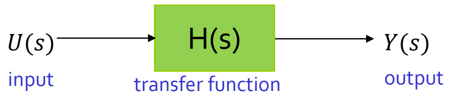
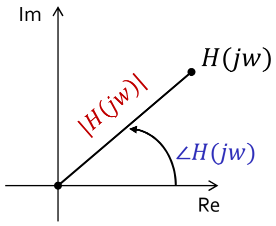
Frequency Response of TF

transfer function (input-to-output)



$u(t) = A \sin wt$ → for stable $H(s)$, in steady-state → $y(t) = A|H(jw)| \sin(wt + \angle H(jw))$

gain (→ $|H(jw)|$), same frequency as input (→ w), phase (→ $\angle H(jw)$)



ex) $\ddot{x} + 3\dot{x} + 2x = u$

$u(t) = A \sin t \;\Rightarrow\; y(t) \to \frac{A}{\sqrt{10}} \sin(t - \tan^{-1} 3)$

$u(t) = A \sin \sqrt{2}t \;\Rightarrow\; y(t) \to \frac{\sqrt{2}A}{6} \sin(t - \frac{\pi}{2})$

ex) $\dot{x} + 2x = u$

$u(t) = \sum_{k=1}^{n} A_k \sin(w_k t + \phi_k)$

$\Rightarrow y(t) \to \sum_{k=1}^{n} A_k |H(jw_k)| \sin(w_k t + \phi_k + \angle H(jw_k))$

* if $H(s)$ is stable and in steady-state

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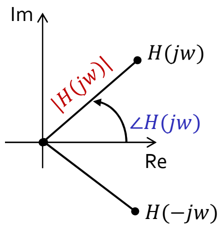
Derivation of Frequency Response

$$Y(s) = H(s)U(s) = H(s)\frac{Aw}{s^2 + w^2} = \frac{N(s)}{D(s)} \frac{Aw}{s^2 + w^2}$$

$$= \frac{c_1}{s + jw} + \frac{c_2}{s - jw} + \frac{\alpha(s + a) + \beta}{(s + a)^2 + b^2} + \frac{\gamma_1}{s + p} + \frac{\gamma_2}{(s + p)^2} + \cdots + H(s)[initial]$$

if $H(s)$ is stable (i.e., all poles are in LHP), in steady-state,

$$Y(s) = \frac{c_1}{s + jw} + \frac{c_2}{s - jw} \qquad c_2 = H(jw)\frac{A}{2j} \qquad c_1 = H(-jw)\frac{A}{-2j}$$



$$H(jw) = Me^{j\phi} \;\rightarrow\; H(-jw) = \overline{H(jw)} = Me^{-j\phi}$$

$$y(t) \to A|H(jw)| \sin(wt + \phi)$$

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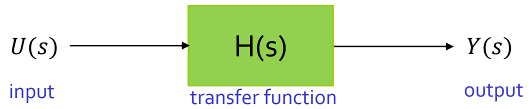
Bode Plot

transfer function (input-to-output)



- steady-state frequency response when $H(s)$ is stable
- gain plot: $\log_{10} w$ vs $20\log_{10}|H(jw)|$ [dB] (decibel)
- phase plot: $\log_{10} w$ vs $\angle H(jw)$
- decade: 10 times of w (e.g., from 5 [rad/s] to 50 [rad/s])
- decibel: $20\log_{10}|H(jw)|$ $\rightarrow$ allows us to "add" Bode plots

0 [dB] $\rightarrow$ $|H(jw)| = 1$ $\rightarrow$ no amplification/attenuation

10 [dB] $\rightarrow$ $|H(jw)| = 10^{0.5} \approx 3$ $\rightarrow$ 3 times amplification

-10 [dB] $\rightarrow$ $|H(jw)| = 10^{-0.5} \approx 0.3$ $\rightarrow$ attenuation

-3 [dB] $\rightarrow$ $|H(jw)| = 10^{-3/20} \approx 1/\sqrt{2}$

- if $u(t) = A\cos 0$, $y(t) \rightarrow A|H(j0)|\cos(\angle H(j0)) = AH(0)$, when $H(0) > 0$

$\nwarrow$ dc-gain

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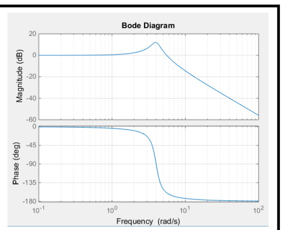
Bode Plot Example



$\ddot{x} + \dot{x} + 16x = 16u(t)$, $\quad u(t) = \sin wt$

1. $w = 10rad/s \Rightarrow \mathrm{M} \approx -14.5dB\ (\approx 0.2),\ \phi \approx -173^o$

2. $w = 1rad/s \Rightarrow M \approx 0dB\ (\approx 1),\ \phi \approx 0^o$

3. $w = 4rad/s \Rightarrow M \approx 12dB\ (\approx 4),\ \phi \approx -90^o$

4. $w = 0rad/s \Rightarrow M = 0dB\ (= 1),\ \phi = 0^o$

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Simple Bode Plot Examples

1. $H(s) = K \rightarrow H(jw) = K \rightarrow |H(jw)| = K, \angle H(jw) = 0$

2. $H(s) = \frac{1}{s} \rightarrow H(jw) = \frac{1}{wj} \rightarrow |H(jw)| = \frac{1}{w}$

$\rightarrow 20\log_{10}|H(jw)| = -20\log_{10} w$

-20dB/decade roll-off: low-freq. attenuation

$\angle H(jw) = -\frac{\pi}{2}$

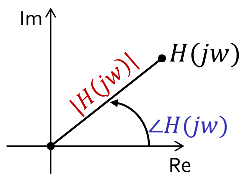
3. $H(s) = s \rightarrow H(jw) = wj \rightarrow |H(jw)| = w$

$\rightarrow 20\log_{10}|H(jw)| = 20\log_{10} w$

+20dB/decade: noisy high-freq. amplification

$\angle H(jw) = \frac{\pi}{2}$

phase-lead? acausal system



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Bode Plot of 1st Order System

$$H(s) = \frac{p}{s+p} \rightarrow H(jw) = \frac{p}{p+jw} = \frac{p}{p^2+w^2}(p-jw)$$

$$|H(jw)| = \frac{p}{\sqrt{w^2+p^2}}, \ \angle H(jw) = -\tan^{-1}\left(\frac{w}{p}\right)$$

Bode Plot

- gain: $|H(jw)| = 20\log_{10}\frac{p}{\sqrt{w^2+p^2}} = 20\log_{10} p - 10\log_{10}(p^2+w^2) \leq 0$ [dB]
- phase: $-\frac{\pi}{2} \leq -\tan^{-1}\left(\frac{w}{p}\right) \leq 0$ [rad]
- asymptotes
  - (i) $w \ll p$: gain ≈ 0 [dB], phase ≈ 0
  - (ii) $w \gg p$: gain $\approx 20\log_{10} p - 20\log_{10} w$, phase $\approx -90^o$
  - (iii) $w = p$: gain $\approx -20\log_{10}\sqrt{2} \approx -3$ [dB], phase $= -45^o$

roll-off: -20dB/decade

bandwidth $w \leq p$
$w_c = p$: cut-off frequency

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Bode Plot of 2nd Order System w/ $\zeta < 1$

$$H(s) = \frac{w_n^2}{s^2+2\zeta w_n s+w_n^2} \rightarrow H(jw) = \frac{w_n^2}{-w^2+2\zeta w_n wj+w_n^2} = w_n^2 \frac{(w_n^2-w^2)-2\,\zeta\, w\_n\, wj}{\left(w_n^2-w^2\right)^2+4\zeta^2 w_n^2 w^2}$$

$$|H(jw)| = \frac{w_n^2}{\sqrt{\left(w_n^2-w^2\right)^2+4\zeta^2 w^2 w_n^2}} \qquad -\pi \le \angle H(jw) = -\tan^{-1}\frac{2\zeta w_n w}{w_n^2-w^2} \le 0$$

Bode Plot

- asymptotes

(i) $w \ll w_n$: gain $= 0$ [dB], phase $= 0$ roll-off: -40dB/decade

(ii) $w \gg w_n$: gain $= 40\log_{10} w_n - 40\log_{10} w$, phase $= -180^o$

(iii) $w = w_n$: gain $\approx -20\log_{10}\left(\frac{1}{2\zeta}\right) \ge 0$ [dB] if $\zeta \le 0.5$, phase $= -90^o$

- resonance (i.e., peak of $|H(jw)|$): $w\left(w^2 - w_n^2(1-2\zeta^2)\right) = 0$ no resonance if $\zeta > \frac{1}{\sqrt{2}}$

$w_r = w_n\sqrt{1-2\zeta^2}$, $M_{pw} = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$ if $\zeta \le \frac{1}{\sqrt{2}} \approx 0.707$

recall $w_d = w_n\sqrt{1-\zeta^2}$ and $M_p = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}}$

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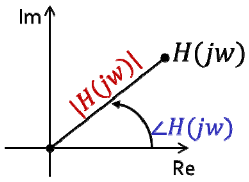
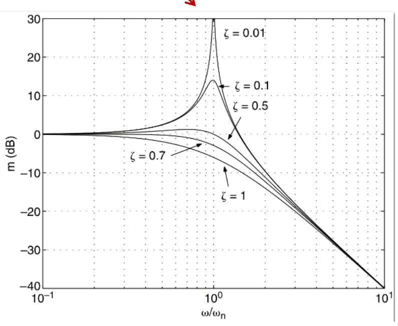
Bode Plot of 2nd Order System w/ $\zeta < 1$

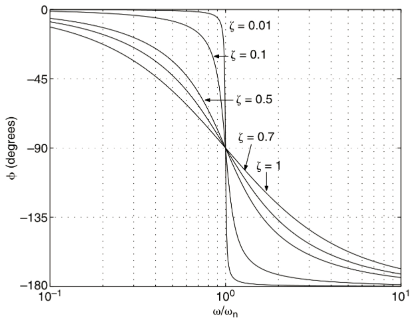
$$H(s) = \frac{w_n^2}{s^2+2\zeta w_n s+w_n^2}$$

polar plot



$M_{pw} \rightarrow \infty, if\ \zeta = 0$





$w = w_n$: gain $\approx -20\log_{10}\left(\frac{1}{2\zeta}\right) \ge 0$ [dB] if $\zeta \le 0.5$, phase $= -90^o$

$w_r = w_n\sqrt{1-2\zeta^2}$, $M_{pw} = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$ if $\zeta \le \frac{1}{\sqrt{2}} \approx 0.707$

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Additive Property of Bode Plot

$$H(s) = \frac{k(s+z)}{s(s+p)(s^2+\alpha s+\beta)}$$

$$bode[H(s)] = bode[k] - bode\left[\frac{1}{s+z}\right] + bode\left[\frac{1}{s}\right] + bode\left[\frac{1}{s+p}\right] + bode\left[\frac{1}{s^2+\alpha s+\beta}\right]$$

$$H(s) = H_1(s)H_2(s)\ldots H_n(s)$$

$$\Rightarrow 20\log_{10}|H(jw)|$$

$$= 20\log_{10}|H_1(jw)| + 20\log_{10}|H_1(jw)| \ldots + 20\log_{10}|H_n(jw)|$$

$$\Rightarrow \angle H(jw) = \angle H_1(jw) + \angle H_2(j2) + \cdots \angle H_n(jw)$$

$$\therefore\ bode[H] = bode[H_1] + bode[H_2] + \cdots bode[H_n]$$

$$H(s) = 1/H'(s)$$

$$\Rightarrow 20\log_{10}|H(jw)| = -20\log_{10}|H'(jw)|, \quad \angle H(jw) = -\angle H'(jw)$$

$$\therefore\ bode[1/H] = -bode[H]$$

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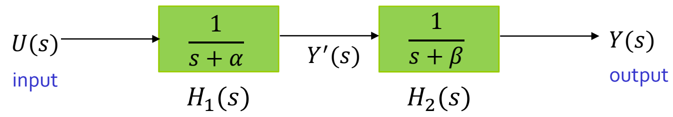
Bode Plot of 2nd Order System w/ $\zeta \geq 1$

$$H(s) = \frac{1}{s^2+as+b} = \frac{1}{\alpha\beta}\frac{\alpha}{s+\alpha}\frac{\beta}{s+\beta}$$

$$bode[H(s)] = bode\left[\frac{\alpha}{s+\alpha}\right] + bode\left[\frac{\beta}{s+\beta}\right] - bode\left[\frac{1}{\alpha\beta}\right]$$



$$u(t) = A\sin wt \ \Rightarrow\ y'(t) \to A|H_1(jw)|\sin\big(wt + \angle H_1(jw)\big)$$

$$\Rightarrow y(t) \to A|H_1(jw)||H_2(jw)|\sin(wt + \angle H_1(jw) + \angle H_2(jw))$$

$$|H(jw)| = |H_1(jw)||H_2(jw)| \ \Rightarrow\ 20\log_{10}|H| = 20\log_{10}|H_1| + 20\log_{10}|H_2|$$

$$\angle H(jw) = \angle H_1(jw) + \angle H_2(jw)$$

$$\Rightarrow bode[H] = bode[H_1] + bode[H_2]$$

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Example 7.3

$$H(s) = \frac{s+1}{s(s^2+s+16)}$$

$$bode[H(s)] = -bode\left[\frac{1}{s+1}\right] + bode\left[\frac{1}{s}\right] + bode\left[\frac{16}{s^2+s+16}\right] - bode[16]$$

- low frequency: $H(s)$ behaves like an integrator
- high frequency: $H(s)$ behaves like a standard 2nd order system

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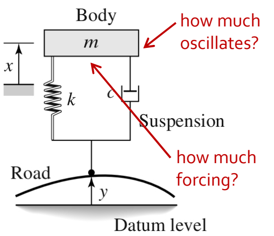
Base Motion & Transimissibility



system dynamics

$$m\ddot{x} = f_t = -c(\dot{x}-\dot{y}) - k(x-y)$$

displacement transmissibility

$$T_D(s) = \frac{X(s)}{Y(s)} = \frac{cs+k}{ms^2+cs+k} \leftarrow \text{body motion}$$

force transmissibility

$$T_F(s) = \frac{F_t(s)}{kY(s)} = \frac{ms^2(cs+k)}{k(ms^2+cs+k)} \leftarrow \text{chassis force}$$

ex) $m = 1kg, c = \frac{1Ns}{m}, k = \frac{4N}{m}$

$$T_D(s) = \frac{cs+k}{k}\frac{k}{ms^2+cs+k} \qquad T_F(s) = \frac{m}{k}s^2 \cdot T_D(s)$$

$T_D(s)$ peak 7.16dB ≈ 2.28 @$w = 1.87$rad/s $\Rightarrow$ approx: $|T_D(jw_n)| = \sqrt{5} \approx 2.24$

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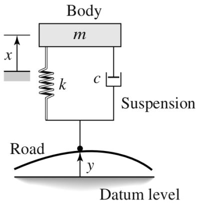
Base Motion & Transimissibility

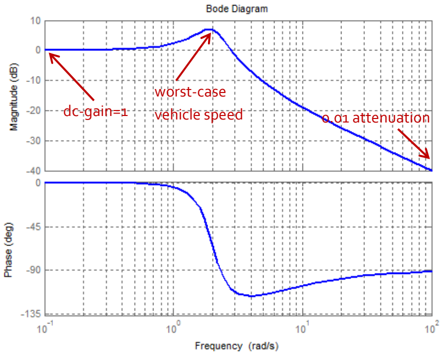


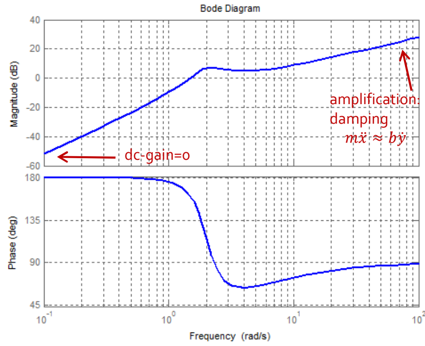
$$m\ddot{x} = -b(\dot{x} - \dot{y}) - k(x - y) = f_t$$

$$T_D(s) = \frac{X(s)}{Y(s)} = \frac{bs + k}{ms^2 + bs + k}$$

$$T_F(s) = \frac{F_t(s)}{kY(s)} = \frac{ms^2(bs + k)}{k(ms^2 + bs + k)}$$

$$w = \frac{2\pi v}{L}$$





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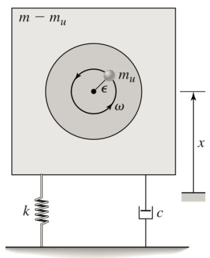
Rotating Unbalance



ϵ: eccentricity
m_u: unbalance mass
e.g.: lathe, rotor, etc...

system dynamics

$$m\ddot{x} + c\dot{x} + kx = m_u \epsilon w^2 \sin wt$$

vibration displacement

$$X(s) = \frac{1}{ms^2+cs+k} L[m_u \epsilon w^2 \sin wt]$$

$$x(t) \rightarrow m_u \epsilon w^2 |H_D(jw)| \sin(wt + \angle H_D(jw))$$

$$|H_D(jw)| = \left|\frac{1}{ms^2+cs+k}\right|_{s=jw}$$

transmitted force

$$F_t(s) = \frac{cs+k}{ms^2+cs+k} L[m_u \epsilon w^2 \sin wt]$$

$$f_t(t) \rightarrow m_u \epsilon w^2 |H_F(jw)| \sin(\mathrm{wt} + \angle H_F(jw))$$

$$|H_F(jw)| = \left|\frac{cs+k}{ms^2+cs+k}\right|_{s=jw}$$

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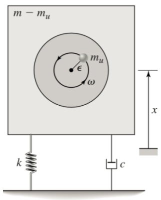
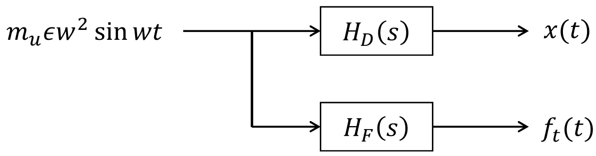
Rotating Unbalance





what is $|x(t)|_{max}$ and $|f_t(t)|_{max}$ given rotation speed w?

vibration displacement

$$x(t) \to m_u \epsilon |w^2 H_D(jw)| \sin(wt + \angle H_D(jw)) \qquad |H_D(jw)| = \left|\frac{1}{ms^2+cs+k}\right|_{s=jw}$$

transmitted force

$$f_t(t) \to m_u \epsilon |w^2 H_F(jw)| \sin(wt + \angle H_F(jw)) \qquad |H_F(jw)| = \left|\frac{cs+k}{ms^2+cs+k}\right|_{s=jw}$$

- bode plot of $s^2 H_D(s)$ and $s^2 H_F(s)$ to take into account w^2
- effect of damping c?
- problematic if c is small, yet, $w \approx w_n$

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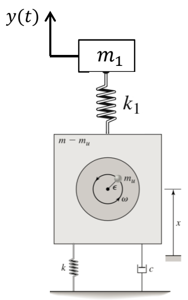
Dynamic Vibration Absorber

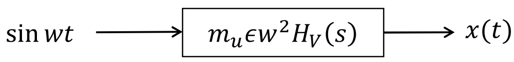
absorber.m
rotate_m_u_absorber.m



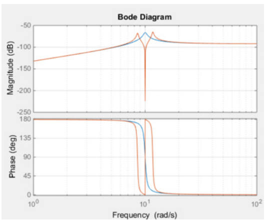
$$m\ddot{x} + b\dot{x} + kx = -k_1(x-y) + u, \qquad u = m_u \epsilon w^2 \sin wt$$
$$m_1 \ddot{y} = -k_1(y-x)$$

$H_v(s)$ with zero at $\pm j\sqrt{k_1/m_1}$

$$X(s) = \frac{m_1 s^2 + k_1}{(ms^2+bs+k+k_1)(m_1 s^2 + k_1) - k_1^2} U(s)$$



- if we set m_1, k_1 s.t. $\sqrt{k_1/m_1} = w_n$, $x(t) \to 0$ even if $w = w_n$



$$Y(s) = \frac{k_1}{(ms^2+bs+k+k_1)(m_1 s^2 + k_1) - k_1^2} U(s)$$

- zero vibration at $w_n = \sqrt{k_1/m_1}$; two extra peaks nearby though
- at $w = w_n$, $y(t) \to -\frac{1}{k_1} u(t) \Rightarrow$ force cancelation!

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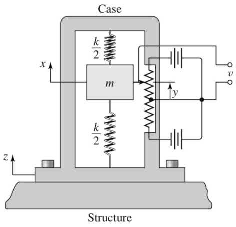
Seismograph



$y = x - z$: direct measurement
$b/2$: damping top & bottom
want to measure earthquake $z(t)$ from $y(t)$

system dynamics:

$$m\ddot{x} = -b\dot{y} - ky = m(\ddot{y} + \ddot{z})$$

$$Y(s) = -\frac{s^2}{s^2 + 2\zeta w_n s + w_n^2} Z(s)$$

- if $w \gg w_n \Rightarrow y(t) \approx -z(t)$
- want to have a small $w_n \Rightarrow$ large m & soft k (i.e., $x(t) \approx 0$)

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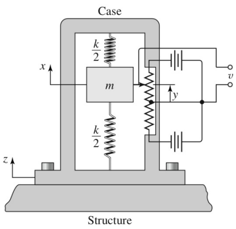
Accelerometer



$y = x - z$: direct measurement
$b/2$: damping top & bottom
want to measure $\ddot{z}(t)$ from $y(t)$

system dynamics:

$$m\ddot{y} + b\dot{y} + ky = -m\ddot{z} = -ma$$

$$Y(s) = -\frac{1}{s^2 + 2\zeta w_n s + w_n^2} A(s)$$

- if $w \ll w_n \Rightarrow y(t) \approx -\frac{1}{w_n^2} a(t)$
- want to have a high $w_n \Rightarrow$ small m & stiff k

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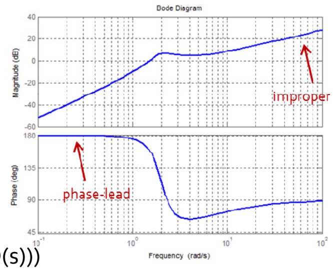
More Comments on Bode Plots



- phase-lead isn't forecasting of a future:
  merely a phase-shift in the sinusoid output
  in steady-state

$$y(t) = A|H(jw)|\sin(wt + \angle H(jw))$$

- $H(s)$ is improper if $\lim_{w\to\infty}|H(jw)| \to \infty$ (i.e., o(N(s))>o(D(s)))
  $\Rightarrow$ not realizable since it requires differentiation

- we may define Bode plot purely mathematically by $(20\log_{10}|H(jw)|, \angle H(jw))$:
  this then represents steady-state gain/phase if $H(s)$ is stable

- Bode gain-phase relation: $|G(jw)|$ uniqely determines "minimum" $\angle G(jw)$

$$\angle G(jw_o) = \frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{d\ln|G(jw)|}{d\ln w}\ln\left|\frac{w+w_o}{w-w_o}\right|\frac{dw}{w}$$

slop   sampling around w_o

* can't design $|G(jw)|$ and $\angle G(jw)$ independent
* non-minimum phase: $|G(jw)|$, but non-minimum $\angle G(jw)$

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Example: Non-Minimum Phase System

non_minimum.m

$$H(s) = \frac{s-10}{s^2+6s+10} \quad \Rightarrow \quad bode[H(s)] = bode\left[\frac{s-10}{10}\right] + bode\left[\frac{10}{s^2+6s+10}\right]$$

* same gain with $H(s) = \frac{s+10}{s^2+6s+10}$, but non-minimum phase shift
* $H(s)$ with RHP zeros $\Rightarrow$ *non-minimum phase*
  phase-shift is not minimum among the systems with the same gain plot
* $H(s)$ without RHP zeros (& delay) $\Rightarrow$ *minimum-phase* system

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Next Lecture

- control

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