# 446.328 Mechanical System Analysis
# 기계시스템해석

## - lecture 9, 10, 11-

Dongjun Lee (이동준)

Department of Mechanical & Aerospace Engineering
Seoul National University

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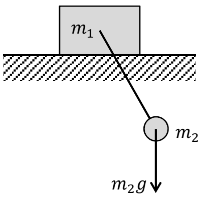
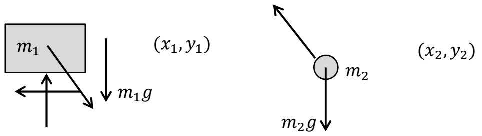
## Lagrange Equation and Generalized Coordinates



what would you do if you use Newton method?



* vector-based; many coordinates & forces; error-prone…

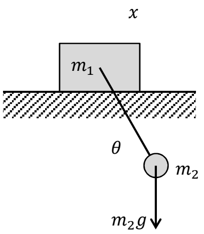
* $x_1, x_2, y_1, y_2$ are not independent: under 2-constraints

Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

- $q = (q_1, q_2, .., q_n)$: generalized coordinates
  - * minimum, yet, enough # of parameters to completely describe system
  - * degree-of-freedom = # of generalized coordinates ($= n$)
- $L(q, \dot{q}) = KE - PE = T - V$ is called Lagrangian

# Lagrange Equation Example



$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

generalized coordinates $q_1 = x; q_2 = \theta$

Lagrangian $L(q,\dot{q}) = T(q,\dot{q}) - V(q)$

kinetic energy

potential energy: depends only on *q* (e.g., gravity, spring energy)

kinematic relations $x = x_1, x_2 = x + l sin\theta, y_2 = l(1 - cos\theta)$

kinetic energy

$$T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2(\dot{x}_2^2 + \dot{y}_2^2)$$

$$= \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2[\dot{x}^2 + 2\dot{x}\dot{\theta}lcos\theta + l^2\dot{\theta}^2]$$

potential energy

$$V = m_2 g y_2 = m_2 g l(1 - cos\theta)$$

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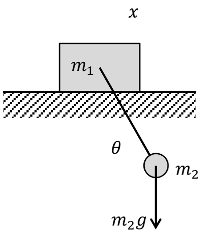
# Lagrange Equation Example

$$[m_1 + m_2]\ddot{x} + m_2 l\cos\theta\cdot\ddot{\theta} - m_2 l\sin\theta\,\dot{\theta}^2 = 0$$

$$m_2 l\cos\theta\,\ddot{x} + m_2 l^2\ddot{\theta} - m_2 l\sin\theta\cdot\cancel{\dot{x}\dot{\theta}} + m_2\cancel{\dot{x}l}\dot{\theta}\sin\theta + m_2 g l\sin\theta = 0$$

# Lagrange Equation Example



$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0}$$

Lagrangian $L(q,\dot{q}) = T(q,\dot{q}) - V(q)$

$$T = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2[\dot{x}^2 + 2\dot{x}\dot{\theta}lcos\theta + l^2\dot{\theta}^2]$$

$$V = m_2 g y_2 = m_2 g l(1 - cos\theta)$$

equation of motion:

$$\boxed{M(q)\ddot{q} + C(q,\dot{q})\dot{q} + \frac{\partial V(q)}{\partial q} = 0}$$

*correctness checks: 1) matrix $\dot{M}(q) - 2C(q,\dot{q})$ should be skew-symmetric;

2) $M(q)$ should be symmetric all real positive eigenvalues

3) $\frac{\partial V}{\partial q} = [\frac{\partial V}{\partial q_1}, \frac{\partial V}{\partial q_2}, \ldots\ldots, \frac{\partial V}{\partial q_n}]$ ?

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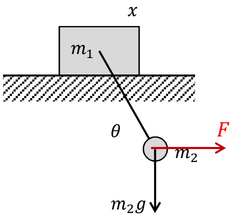
# Forced Lagrange Equation & Generalized Force



$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = \tau_i}$$

generalized force ($\neq F$)

* virtual displacements of $\delta x_1, \delta y_1, \delta x_2, \delta y_2$ should satisfy constraints (e.g., $\delta y_1 = 0, \ldots$)

* yet, $\delta q_1, \delta q_2$ are unconstrained

virtual work $\delta W = F\delta x_2$

$$= F(\delta x + l\,cos\theta \cdot \delta\theta) = F\delta x + Flcos\theta\delta\theta = \tau_1\delta x + \tau_2\delta\theta$$

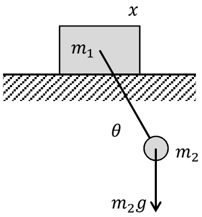
from kinematic relation $x = x_1, x_2 = x + lsin\theta, y_2 = l(1 - cos\theta)$

equation of motion

$$\boxed{M(q)\ddot{q} + C(q,\dot{q})\dot{q} + \frac{\partial V(q)}{\partial q} = \tau}$$

* generalized force $\tau_i$ doesn't include: potential force; zero-work force
→ no need to compute no-slip friction, constraint force, internal force, ….!

## Equilibrium and Linearization



$$q = (x, \theta)$$

$$\boxed{M(q)\ddot{q} + C(q,\dot{q})\dot{q} + \frac{\partial V(q)}{\partial q} = 0}$$

equilibrium: if you start there, you will stay there

$$(\dot{q}, q) = (0, q_e) \text{ s.t. } \frac{\partial V}{\partial q} = 0 \text{ at } q = q_e$$

two isolated equilibria:

$\dot{q} = 0,\ q_e = (x', 0)$: downward position;

$\dot{q} = 0,\ q_e = (x', \pi)$ : upward position

$x'$ can be arbitrary: symmetry w.r.t. $x$

- linearization about $(\dot{q}, q) = (0, (x', 0))$

$$\frac{m_1 m_2 l^2}{m_1 + m_2}\ddot{\theta}_d + m_2 g l \theta_d = 0 \quad \rightarrow \text{marginally stable}$$

- linearization about $(\dot{q}, q) = (0, (x', \pi))$

$$\frac{m_1 m_2 l^2}{m_1 + m_2}\ddot{\theta}_d - m_2 g l \theta_d = 0 \quad \rightarrow \text{unstable (saddle)}$$

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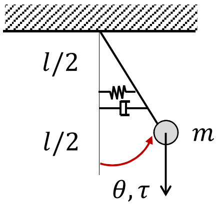
## Example: Pendulum with Spring/Damper



damper/spring acting
only horizontally

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = \tau_i}$$

$$T = \frac{1}{2}m\left(l\dot{\theta}\right)^2 \qquad V = mgl(1 - cos\theta) + \frac{1}{2}k\left(\frac{l}{2}sin\theta\right)^2$$

$$q = \theta$$

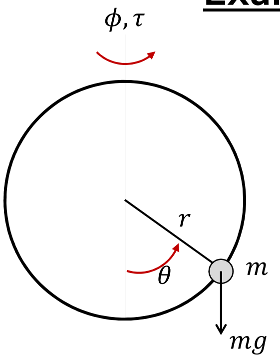
$$x = \frac{l}{2}sin\theta$$

$$\delta W = \tau\delta\theta - c\dot{x}\delta x = \left[\tau - \frac{cl^2}{4}\cos^2\theta\,\dot{\theta}\right]\delta\theta$$

$$ml^2\ddot{\theta} + \frac{cl^2}{4}\cos^2\theta\,\dot{\theta} + \frac{kl^2}{4}\text{sin}\theta\text{cos}\theta + mgl\sin\theta = \tau$$

$$\boxed{ml^2\ddot{\theta} + \frac{cl^2}{4}\dot{\theta} + \frac{kl^2}{4}\theta + mgl\theta = \tau}$$

# Example: Bead on a Hoop



$$q_1 = \theta, q_2 = \phi$$

$$T = \frac{1}{2}m\left(r\dot{\theta}\right)^2 + \frac{1}{2}m\left(r\sin\theta\dot{\phi}\right)^2$$

$$V = mgr(1 - \cos\theta)$$

$$\delta W = \tau\delta\phi = 0\delta q_1 + \tau_2\delta q_2$$

$$\boxed{\begin{gathered} mr^2\ddot{\theta} - mr^2 s\theta c\theta\dot{\phi}^2 + mgrs\theta = 0 \\ mr^2 s^2\theta\ddot{\phi} + 2mr^2 s\theta c\theta\dot{\phi}\dot{\theta} = \tau \end{gathered}}$$

$$\boxed{M(q)\ddot{q} + C(q,\dot{q})\dot{q} + \frac{\partial V(q)}{\partial q} = 0}$$

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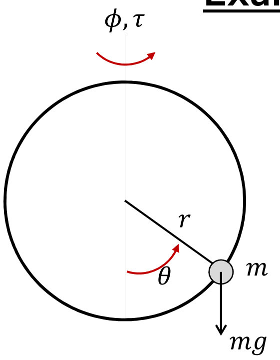
# Example: Bead on a Hoop



$$q_1 = \theta, q_2 = \phi$$

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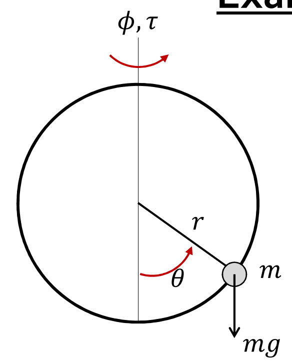
# Example: Bead on a Hoop



$$\boxed{\begin{gathered} mr^2\ddot{\theta} - mr^2 s\theta c\theta\dot{\phi}^2 + mgrs\theta = 0 \\ mr^2 s^2\theta\ddot{\phi} + 2mr^2 s\theta c\theta\dot{\phi}\dot{\theta} = \tau \end{gathered}}$$

- choose the control

$\tau = 2mr^2 s\theta c\theta\dot{\phi}\dot{\theta} + mr^2\, s^2\theta[\dot{\Omega} - b(\dot{\phi} - \Omega)]$

$\Rightarrow \dot{\phi} \to \Omega$ as long as $\sin\theta \neq 0$

- with $\dot{\phi} = \Omega$

$$\boxed{\ddot{\theta} + \sin\theta\left(\frac{g}{r} - \Omega^2\cos\theta\right) = 0}$$

- three equilibria (with $\dot{\theta} = 0$)

$$\boxed{\theta = 0,\quad \theta = \pi,\quad \theta = \theta_e \quad \text{w/} \quad \cos\theta_e = \frac{g}{r\Omega^2} \quad \text{when} \quad r\Omega^2 \geq g}$$

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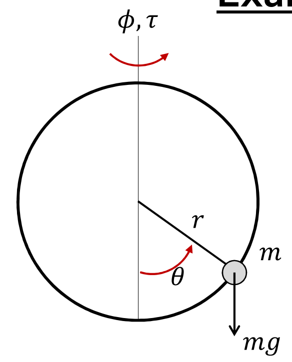
# Example: Bead on a Hoop



- with $\dot{\phi} = \Omega$ $\quad \boxed{\ddot{\theta} + \sin\theta\left(\frac{g}{r} - \Omega^2\cos\theta\right) = 0}$

1. $\theta = 0, \quad \theta \approx \epsilon$

$$\ddot{\epsilon} + \epsilon\left(\frac{g}{r} - \Omega^2\right) = 0$$

marginally stable if $r\Omega^2 < g$; unstable if $r\Omega^2 \geq g$

2. $\theta = \pi, \quad \theta \approx \pi + \epsilon$

$$\ddot{\epsilon} - \epsilon\left(\frac{g}{r} + \Omega^2\right) = 0 \quad \rightarrow \quad \text{unstable}$$

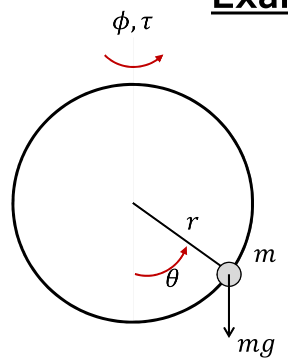
3. $\theta = \theta_e, \quad \theta \approx \theta_e + \epsilon, \quad \theta_e = \frac{g}{r\Omega^2}$ when $r\Omega^2 \geq g$

$$\ddot{\epsilon} + \Omega^2\sin^2\theta_e\epsilon = 0 \quad \rightarrow \quad \text{maginally stable}$$

$sin(\theta_e + \epsilon) \approx sin\theta_e + \epsilon cos\theta_e$

$\cos(\theta_e + \epsilon) \approx cos\theta_e - \epsilon sin\theta_e$

# Example: Bead on a Hoop



1. $\theta = 0$

   marginally stable if $r\Omega^2 < g$; unstable if $r\Omega^2 \geq g$

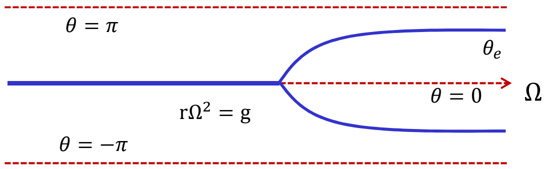
2. $\theta = \pi \rightarrow$ unstable

3. $\theta = \theta_e = \frac{g}{r\Omega^2}$ w/ $r\Omega^2 \geq g$ → marginally stable

- if $r\Omega^2 < g$: two equilibria & oscillate about $\theta = 0$
- if $r\Omega^2 \geq g$: three equilibria & oscillate about $\theta = \theta_e$

* bifurcation: parameter change induces sudden change in the system's qualitative behaviors

$$sin(\theta_e + \epsilon) \approx sin\theta_e + \epsilon cos\theta_e$$
$$\cos(\theta_e + \epsilon) \approx cos\theta_e - \epsilon sin\theta_e$$



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# Potential Energy and Conservative Force

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = \tau_i}$$

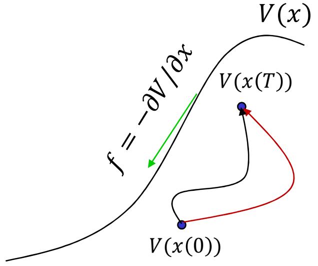
- conservative force & potential function

$$f(x) = -\frac{\partial V(x)}{\partial x} \quad V(x)\text{: potential function}$$

$$\int_0^T f\,\dot{x}\,dt = -V\big(x(T)\big) + V\big(x(0)\big) \leq V(x(0))$$

work done by potential:
path-independent!

conceptually



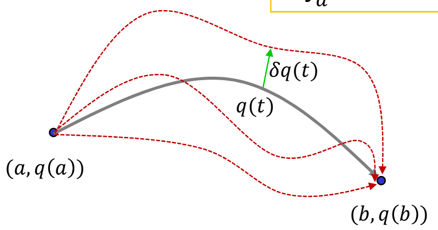
examples:

gravity $V(y) = mg(y - y_o)$, $f = -mg$

spring force $V(x) = \frac{1}{2}k(x - x_o)^2$, $f = -k(x - x_o)$

# Hamilton's Least Action Principle

Hamilton's principle $$\boxed{\delta\int_a^b L(q,\dot{q},t)dt = 0}$$ → calculus of variations



separate from $t$

$$q(\alpha,t) = q(t) + \alpha\eta(t)$$
$$= q(t) + \delta q(t)$$

variation

$$\eta(a) = \eta(b) = 0$$

using $$\delta L = \sum\left[\left(\frac{\partial L}{\partial q_i}\right)\delta q_i + \left(\frac{\partial L}{\partial \dot{q}_i}\right)\delta\dot{q}_i + \left(\frac{\partial L}{\partial t}\right)\delta t\right]$$

$\delta\dot{q}_i = \alpha\dot{\eta} = \frac{d}{dt}\delta q_i$ and integration-by-part

$$\boxed{\delta\int_a^b Ldt = \sum\int_a^b\left[\left(\frac{\partial L}{\partial q_i}\right) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\right]\delta q_i dt = 0, \qquad \forall\delta q_i}$$

Lagrange equation w/o external force

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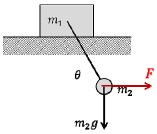
# Lagrange-D'Alembert Equation

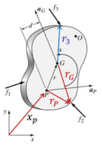
D'Alembert principle

$\boxed{\sum\vec{M}_A = \sum(\vec{M}_A)_{eff}}$

$$\boxed{\sum_{j=1}^{r}\left(f_j - m_j\frac{d^2x_j}{dt^2}\right)\delta x_j = 0}$$

force





$$f_j = f_{j,constraint} + f_{j,potential} + f_{j,external}$$

$x_j = x_j(q_1, q_2, \dots, q_n)$ : particle position in 3D

$$\delta W = \sum_{j=1}^{r}[f_{j,ext} + f_{j,pot}]\delta x_j = \sum_{i=1}^{n}\tau_{i,ext}\delta q_i - \delta V = \sum_{j=1}^{r}\frac{d}{dt}(m_j\dot{x}_j\delta x_j) - \delta T$$

$$\delta\int_a^b Ldt + \int_a^b\sum_{i=1}^{n}\tau_{i,ext}\delta q_i dt = \sum_{j=1}^{r}m_j\dot{x}_j\delta x_j\Big|_a^b = 0$$

generalized force

Lagrange-D'Alembert equation $$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = \tau_{i,ext}}$$

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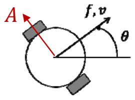
# Constraints and Lagrange Multiplier



can't define
2 generalized
coordinates
...

Pfaffian constraints $\boxed{A(q)\dot{q} = 0 \quad \text{i.e.,} \quad \sum_{i=1}^{n} a_i(q)\,\delta q_i = 0}$

**examples:**

pendulum $A(q) = [x \;\; y]$ from $x^2 + y^2 = r^2$ with $q = (x, y)$

wheeled robot $A(q) = [\sin\theta \;\; -\cos\theta \;\; 0]$ from no-slip condition

with $= (x, y, \theta)$

* these $q$ is not free anymore: needs to satisfy constraints $A(q)\delta q = 0$

$$\delta \int_a^b L\,dt + \int_a^b \sum_{i=1}^{n} \tau_{i,ext}\delta q_i dt - \int_a^b \lambda \sum_{i=1}^{n} a_i(q)\delta q_i dt = 0$$

Lagrange multiplier

$\delta q_i$ now constrained

similar to generalized force!

$$\boxed{M(q)\ddot{q} + C(q,\dot{q})\dot{q} + \frac{\partial V}{\partial q} + A^T(q)\lambda = \tau_{ext}}$$

unconstrained generalized coordinates

constraint force

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# Constraints and Lagrange Multiplier

Pfaffian constraints $\boxed{A(q)\dot{q} = 0 \quad \text{i.e.,} \quad \sum_{i=1}^{n} a_i(q)\,\delta q_i = 0}$

Lagrange multiplier

$$\boxed{M(q)\ddot{q} + C(q,\dot{q})\dot{q} + \frac{\partial V}{\partial q} + A^T(q)\lambda = \tau_{ext}}$$

unconstrained generalized coordinates

constraint force

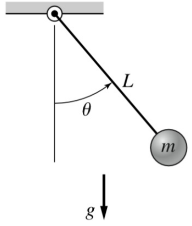
* Lagrange multiplier $\lambda$ should possess the value, that enforces constraints

1. differentiate $A(q)\dot{q} = 0$ to $A\ddot{q} + \dot{A}\dot{q} = 0$
2. plug in the dynamics via $\ddot{q}$ s.t.

$$AM^{-1}\left[\tau - C\dot{q} - \frac{\partial V}{\partial q} - A^T\lambda\right] + \dot{A}\dot{q} = 0$$

3. compute $\lambda = (AM^{-1}A^T)^{-1}[\dot{A}\dot{q} + AM^{-1}\left(\tau - C\dot{q} - \frac{\partial V}{\partial q}\right)]$

## Example: Pendulum



$A(q) = [x \;\; y]$ from $x^2 + y^2 = r^2$ with $q = (x, y)$

* can't see the constraint force if choose $q = \theta$

equation of motion

$$m\ddot{x} + \lambda x = 0 \qquad m\ddot{y} + mg + \lambda y = 0$$
$$\lambda = \left(mgy + m(\dot{x}^2 + \dot{y}^2)\right)/\mathrm{r}^2$$

holonomic constraints:

$$A(q)\dot{q} = 0 \;\rightarrow\; h(q) = x^2 + y^2 = r^2$$

DOF reduced by 1-DOF with $q_1 = \theta$

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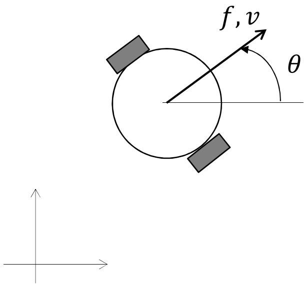
## Example: Wheeled Mobile Robot



$A(q) = [\sin\theta \;\; -\cos\theta \;\;\; 0]$ from no-slip condition

$A(q)\dot{q} = 0$ cannot be integrated to $h(q)$

equation of motion

$$m\ddot{x} + \lambda\sin\theta = f\cos\theta$$
$$m\ddot{y} - \lambda\cos\theta = f\sin\theta$$
$$I\ddot{\theta} = \tau$$
$$\lambda = m(\dot{x}cos\theta + \dot{y}sin\theta)\dot{\theta}$$

nonholonomic constraints:

- $A(q)\dot{q} = 0$ cannot be integrated into $h(q) = 0$
- configuration DOF not reduced w/ $q = (x, y, \theta)$

## Next Lecture

- state-space formulation