* Verify that if $V(x)$ is Rayleigh distributed, then $\langle V(x)^2 \rangle = \frac{4}{\pi}(\langle V(x) \rangle)^2$

sol) If a random variable $X \sim \chi_2^2$ (chi-square with 2 degree of freedom) then $\sqrt{X}$ is Rayleigh distributed. Also, if a r.v $R \sim \text{Rayleigh}(1)$ then R^2 has a chi-square distribution with two degrees of freedom : $R^2 \sim \chi_2^2$ $(\sigma = 1)$

- r.v. X : Rayleigh distribution

$$f_X(x) = \frac{x}{\sigma^2} \exp\left(-\frac{x^2}{2\sigma^2}\right), \quad \underline{\underline{x \ge 0}}$$

Let $Y = X^2$, $f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right|$, $(dy = 2x\,dx)$

$$\Rightarrow f_Y(y) = \left.\frac{f_X(x)}{2x} - \frac{f(-x)}{2x}\right|_{x=\pm\sqrt{y}} = \left.\frac{f_X(x)}{2x}\right|_{x=\sqrt{y}} \quad (\because x \ge 0)$$

$$\therefore \langle V(x)^2 \rangle = E[X^2] = E[Y]$$

$$= \int_{-\infty}^{\infty} y \cdot f_Y(y)dy = \int_0^{\infty} y \cdot \frac{f_X(\sqrt{y})}{2\sqrt{y}} dy$$

$$= \int_0^{\infty} \frac{y}{2\cancel{\sqrt{y}}} \cdot \frac{\cancel{\sqrt{y}}}{\sigma^2} \exp\left(-\frac{y}{2\sigma^2}\right) dy = \int_0^{\infty} \frac{y}{2\sigma^2} \exp\left(-\frac{y}{2\sigma^2}\right) dy$$

(Let, $\frac{y}{2\sigma^2} = t \Rightarrow dy = 2\sigma^2 dt$)

$$= \int_0^{\infty} t \cdot \exp(-t) \cdot 2\sigma^2 dt = 2\sigma^2 \underbrace{\int_0^{\infty} t \cdot \exp(-t)dt}_{\text{Gamma function form}}.$$

(Gamma function $\Gamma(z) = \int_0^{\infty} x^{z-1} e^{-x} dx$ $(z > 0)$)

$$= 2\sigma^2 \underbrace{\Gamma(2)}_{1!} = 2\sigma^2$$

- $\langle V(x) \rangle = E[X] = \int_0^{\infty} x \cdot f_X(x)dx = \int_0^{\infty} x \cdot \frac{x}{\sigma^2} \exp\left(-\frac{x^2}{2\sigma^2}\right) dx$

$$= \frac{\sqrt{2\pi}}{\sigma} \cdot \frac{1}{2} \underbrace{\int_{-\infty}^{\infty} x^2 \cdot \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{x^2}{2\sigma^2}\right) dx}$$

↳ $N(0, \sigma^2)$의 second moment,

$$= \frac{\sqrt{2\pi}}{\sigma} \cdot \frac{1}{2} \cdot \sigma^2$$

Normal distribution, $E[Z^2] = Var[Z] = \sigma^2$.

$$= \sigma\sqrt{\frac{\pi}{2}}$$