SEOUL NATIONAL UNIVERSITY
SCHOOL OF MECHANICAL AND AEROSPACE ENGINEERING

SYSTEM CONTROL **Fall 2014**

HW #6 **Assigned: November 11 (Tu)**
Due: November 20 (Th)

1. Consider the following system:

$$\dddot{y} + 6\ddot{y} + 11\dot{y} + 6y = 6u$$

Obtain a state-space representation of this system in a diagonal canonical form.

Sol) The transfer function representation of this system is

$$\frac{Y(s)}{U(s)} = \frac{6}{s^3 + 6s^2 + 11s + 6} = \frac{6}{(s+1)(s+2)(s+3)}$$

The partial-fraction expansion of Y(s)/U(s) is

$$\frac{Y(s)}{U(s)} = \frac{3}{s+1} + \frac{-6}{s+2} + \frac{3}{s+3}$$

Then, a diagonal canonical form of the system is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 3 & -6 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

2. Given the system equation

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Find the solution in terms of the initial conditions $x_1(0)$, $x_2(0)$, and $x_3(0)$.

Sol) The given state matrix is in the Jordan canonical form. The eigenvalues are

$$\lambda_1 = 2,\ \lambda_2 = 2,\ \lambda_3 = 2$$

Since

$$e^{At} = \begin{bmatrix} e^{2t} & te^{2t} & \frac{1}{2}t^2e^{2t} \\ 0 & e^{2t} & te^{2t} \\ 0 & 0 & e^{2t} \end{bmatrix}$$

We have

$$x(t) = e^{At}x(0)$$

Or

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} e^{2t} & te^{2t} & \frac{1}{2}t^2e^{2t} \\ 0 & e^{2t} & te^{2t} \\ 0 & 0 & e^{2t} \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix}$$

※ General A

$AP_i = \lambda_i P_i$, where λ_i is eigenvalue and P_i is corresponding eigenvector.

$$A\begin{bmatrix} P_1 & P_2 & P_3 & \dots & P_n \end{bmatrix} = \begin{bmatrix} P_1 & P_2 & P_3 & \dots & P_n \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 & \ddots \\ 0 & \lambda_2 & \ddots & 0 \\ 0 & \ddots & \ddots & 0 \\ \ddots & 0 & 0 & \lambda_n \end{bmatrix}$$

$AP = P\Lambda$

$A = P\Lambda P^{-1} \ \dots \ A^n = P\Lambda^n P^{-1}$

Hence, $e^{At} = Pe^{\Lambda t}P^{-1}$ → more useful

3. Consider the system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

The output is given by

$$y = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

(a) Show that the system is not completely observable.

Sol) The rank of

$$\begin{bmatrix} C^T & (CA)^T & (CA^2)^T \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 \\ 1 & 5 & 16 \\ 1 & 2 & 4 \end{bmatrix}$$

is two, because

$$\begin{vmatrix} 1 & 2 & 4 \\ 1 & 5 & 16 \\ 1 & 2 & 4 \end{vmatrix} = 0, \quad \begin{vmatrix} 1 & 2 \\ 1 & 5 \end{vmatrix} = 3$$

Hence, the system is not completely observable.

(b) Show that the system is completely observable if the output is given by

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Sol) The rank of

$$\begin{bmatrix} C^T & (CA)^T & (CA^2)^T \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 & 2 & 4 & 4 \\ 1 & 2 & 5 & 13 & 16 & 44 \\ 1 & 3 & 2 & 6 & 4 & 12 \end{bmatrix}$$

is three, because the determinant of a 3x3 matrix consisting of the first, fourth, and sixth column is

$$\begin{vmatrix} 1 & 2 & 4 \\ 1 & 13 & 44 \\ 1 & 6 & 12 \end{vmatrix} = -72$$

Hence, the system is completely observable.

4. Consider the system defined by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -1 & -2 & -2 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} u$$

Is the system completely state controllable?

Sol) The rank of

$$\begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 2 & -4 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & -5 \end{bmatrix}$$

is three, because

$$\begin{vmatrix} 2 & -4 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & -5 \end{vmatrix} = -10$$

Hence, the system is completely state controllable.

5. Consider the system defined by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} u$$

By using the state-feedback control $u = -Kx$, it is desired to have the closed-loop poles at $s = -2 \pm j4,\ s = -10$. Determine the state-feedback gain matrix K.

Sol) Define a gain matrix K as

$$K = [k_1 \quad k_2 \quad k_3]$$

The closed-loop system with a state-feedback control defined by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -k_1 & -k_2 & 1-k_3 \\ -1-k_1 & -5-k_2 & -6-k_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = A' \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

The characteristic equation of closed loop system can be obtained as below.

$$|sI - A'| = \begin{vmatrix} s & -1 & 0 \\ k_1 & s+k_2 & -1+k_3 \\ 1+k_1 & 5+k_2 & s+6+k_3 \end{vmatrix}$$

$$= s^3 + (k_2 + k_3 + 6)s^2 + (k_1 + 7k_2 - 5k_3 + 5) + 7k_1 - k_3 + 1$$

Desired closed-loop poles are $s = -2 \pm j4,\ s = -10$. Therefore, characteristic equation should be

$$s^3 + 14s^2 + 60s + 200$$

Hence, a gain matrix K is

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 7 & -5 \\ 7 & 0 & -1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 8 \\ 55 \\ 199 \end{bmatrix} \Rightarrow \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 7 & -5 \\ 7 & 0 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 8 \\ 55 \\ 199 \end{bmatrix} = \begin{bmatrix} 28.7831 \\ 5.5181 \\ 2.4819 \end{bmatrix}$$