Home Problem Set #2

1. State the two Schrödinger equations for electrons in a periodic potential field (Kronig-Penny model). Use for their solutions, instead of the Bloch function, the trial solution

$$\Psi(x) = Ae^{ikx}$$

Discuss the result. (*Hint*: For free electrons $V_o = 0$)

S.E. I (no potential) : $\frac{d^2\Psi}{dx^2} + \frac{2mE}{\hbar^2}\Psi = 0$

S.E. II (v=v_0) : $\frac{d^2\Psi}{dx^2} + \frac{2m(E-v_0)}{\hbar^2}\Psi = 0$

The trial solution $\Psi(x) = Ae^{ikx}$ 대입. ($\frac{d\Psi}{dx} = A_i e^{ikx}, \frac{d^2\Psi}{dx^2} = -Ak^2e^{ikx}$)

I : $-Ak^2e^{ikx} + \frac{2mE}{\hbar^2}Ae^{ikx} = 0$ $\quad\quad \therefore k^2 = \frac{2mE}{\hbar^2}$

II: $-Ak^2e^{ikx} + \frac{2m(E-v_0)}{\hbar^2}Ae^{ikx} = 0$ $\quad\quad \therefore k^2 = \frac{2m(E-v_0)}{\hbar^2}$ $\quad\quad v_0 = 0$

Bloch function $\Psi(x) = u(x)\cdot e^{ikx}$ 에서 u(x)는 lattice의 peridicity를 알려주는 주기함수이다. u(x)가 constant A일 경우 periodic potential이 없는 ($v_0 = 0$) 상황이 되어 free electron과 같은 경우가 된다.

2. The differential equation for an undamped vibration is $a\frac{d^2u}{dx^2} + bu = 0$ (1)

whose solution is $u = Ae^{ikx} + Be^{-ikx}$ (2)

Where $k = \sqrt{b/a}$

Prove that (2) is indeed a solution of (1)

$u = Ae^{ikx} + Be^{-ikx}$ (2) $\quad\quad a\frac{d^2u}{dx^2} + bu = 0$ (1) $\quad\quad k = \sqrt{b/a}$

(2)식을 (1)에 대입

$\frac{du}{dx} = A_i ke^{ikx} - B_i ke^{-ikx}$, $\quad\quad \frac{d^2u}{dx^2} = -A_i k^2e^{ikx} - B_i k^2e^{-ikx}$

$\therefore a\cdot\left(-A_i k^2e^{ikx} - B_i k^2e^{-ikx}\right) + b\left(Ae^{ikx} + Be^{-ikx}\right)$

$$= \left(-aA\cdot\frac{b}{a} + bA\right)e^{ikx} + \left(-aB\cdot\frac{b}{a} + bB\right)e^{-ikx} = 0$$

3. The total electron energy of a hydrogen atom is given by

$$E = \frac{me^4}{2(4\pi\varepsilon_0\hbar)^2}\frac{1}{n^2} = -13.6\cdot\frac{1}{n^2}(eV)$$

where $n = 1, 2, 3, \ldots.$

(a) Derive the above relation in a semiclassical way by assuming that the centripetal force of an electron, mv^2/r, is counterbalanced by the Coulombic attraction force, $-e^2/4\pi\varepsilon_0 r^2$, between the nucleus and the orbiting electron. Use Bohr's postulate which states that the angular momentum $L = mvr$ (v = linear electron velocity and r = radius of the orbiting electron) is a multiple integer of Planck's constant (i.e., $n\hbar$). (*Hint*: The kinetic energy of the electron is $E = mv^2/2$)

$F = \frac{mv^2}{r} = \frac{e^2}{4\pi\varepsilon_0 r^2}$ (방향 고려 부호 생략), $v^2 = \frac{e^2}{4\pi\varepsilon_0 mr}$

Bohr's postulate $L = mvr = n\hbar,\quad r = n\hbar/mv$ $v^2 = \frac{e^2 mv}{4\pi\varepsilon_0 mn\hbar}$

$$E = KE + PE\ \ (PE = -\int F(r)dr = -e^2/\,4\pi\varepsilon_0 r)$$

$$= \frac{1}{2}mv^2 - \frac{e^2}{4\pi\varepsilon_0 r} = \frac{1}{2}mv^2 - mv^2 = -\frac{1}{2}mv^2 = -\frac{m}{2}\cdot\frac{e^4}{(4\pi\varepsilon_0 r)^2}\cdot\frac{1}{n^2}$$

$$= \frac{-9.11\times10^{-31}\times(1.6\times10^{-19})^4}{2\times(4\pi\times8.85\times10^{-12}\times6.626\times10^{-34}/2\pi)^2}\cdot\frac{1}{n^2} = -2.17\times10^{-18}\cdot\frac{1}{n^2}\,(\mathrm{J})$$

$$= -13.6\cdot\frac{1}{n^2}\,(\mathrm{eV})$$

(b) Derive the above relation by solving the Schrodinger equation for the electron wave of the hydrogen atom.

(Hint, refer to Concepts of Modern Physics, Ed. Arthur Beiser or other books)

(hydrogen atom.pdf 파일 첨부)

4. Consider an electron trapped in a one-dimensional box with infinitely high potential. The box extends from $x = 0$ to $x = a$. The electron has the mass m and the total energy E.

(a) Find the energy eigenvalue of the electron by solving Schrödinger equation,

$$\frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2}E\psi = 0$$

$\frac{d^2\Psi}{dx^2} + \frac{2mE}{\hbar^2}E\Psi = 0 \quad k = \frac{\sqrt{2mE}}{\hbar},\ \ \frac{d^2\Psi}{dx^2} = -k^2\Psi$

$\Psi = A\sin kx + B\cos kx$ ($Ae^{ikx} + Be^{-ikx}$와 같은 형태)

x=0 에서 $\Psi = 0$, $\therefore \Psi(0) = B = 0$

x=a 에서 $\Psi = 0$, $\therefore \Psi(a) = A\sin ka = 0$, A≠0이어야 물리적 의미를 가지므로

ka=nπ (n=1, 2, 3…)

$$\therefore k = \frac{n\pi}{a},\quad E_n = \frac{\hbar^2k^2}{2m} = \frac{n^2\pi^2\hbar^2}{2ma^2}\quad (n = 1, 2, 3 \ldots)$$

(b) If the lowest energy possible for the electron is 10.00 eV, what are the two higher energies the particle can have?

The lowest energy: 10.00eV (n=1)

The two higher energies: $E_n \propto n^2$ 이므로 $E_2 = 40.00\text{eV}, E_3 = 90.00\text{eV}$

(c) Find the eigenfunction ψ_n corresponding to E_n.

(c) $\psi_n = A\sin kx = A\sin\frac{n\pi x}{a}$, normalization : $\int_{-\infty}^{\infty}|\psi_n|^2\,dx = 1$

$\therefore \int_0^a A^2\sin^2\frac{n\pi x}{a}\,dx = A^2\int_0^a\left(\frac{1}{2} - \frac{1}{2}\cos\frac{2n\pi x}{a}\right)dx$

$= A^2\left|\frac{1}{2}x - \frac{1}{2}\cdot\frac{a}{2n\pi}\cancel{\sin\frac{2n\pi x}{a}}\right|_0^a = A^2\cdot\frac{a}{2} = 1.\quad \therefore A = \sqrt{\frac{2}{a}}$

$\therefore \psi_n = \sqrt{\frac{2}{a}}\sin\frac{n\pi x}{a}$

(d) Find the probability density that the electron can be found between x = 0 and a/3 for the second excited sate (n =3).

(d) · the probability density $(0 < x \le \frac{a}{3}) \Rightarrow \int_0^{\frac{a}{3}}|\psi_3|^2\,dx\ (n=3)$

$\therefore \int_0^{\frac{a}{3}}\frac{2}{a}\cdot\sin^2\frac{3\pi x}{a}\,dx = \frac{2}{a}\cdot\int_0^{\frac{a}{3}}\left(\frac{1}{2} - \frac{1}{2}\cos\frac{6\pi x}{a}\right)dx$

$= \frac{2}{a}\cdot\left|\frac{1}{2}x - \cancel{\frac{a}{12\pi}\sin\frac{6\pi x}{a}}\right|_0^{\frac{a}{3}} = \frac{\cancel{2}}{\cancel{a}}\cdot\frac{1}{\cancel{2}}\cdot\frac{\cancel{a}}{3} = \frac{1}{3}$

5. An electron of mass m and kinetic energy $\boldsymbol{E > 0}$ approaches a potential drop V_0

(a) Write down the equations of two regions (x≥0, x≤0) from time-independent Schrödinger equation.

(a) $x \ge 0$ $\qquad$ $x \le 0$

$\frac{d^2\psi}{dx^2} + \frac{2m(E+V_0)}{\hbar^2}\psi = 0$ $\qquad$ $\frac{d^2\psi}{dx^2} + \frac{2mE}{\hbar^2}\psi = 0$

(b) Assume there is no incoming wave to the right, so you can get three amplitude constants. Let A : incident amplitude, B : reflected amplitude, C : transmitted amplitude.

Determine the reflection coefficient R. $\left(R = \dfrac{|B|^2}{|A|^2}\right)$

(b) $x \le 0$

$k_1 = \dfrac{\sqrt{2mE}}{\hbar}$, $\psi_1(x) = A e^{ik_1 x} + B e^{-ik_1 x}$

B → reflected amplitude

A → incident amplitude (정확한 의미의 amplitude ≒ $|A|^2$ 입니다.)

$x \ge 0$

$k_2 = \dfrac{\sqrt{2m(E+V_0)}}{\hbar}$, $\psi_2(x) = C e^{ik_2 x} + \cancel{D} e^{-ik_2 x}$ (assuming there is no incoming wave to the right)

Boundary condition, $x=0$ 에서 $\begin{cases} \psi_1(0) = \psi_2(0) & \text{— (1)} \\ \left.\dfrac{d\psi_1}{dx}\right|_{x=0} = \left.\dfrac{d\psi_2}{dx}\right|_{x=0} & \text{— (2)} \end{cases}$

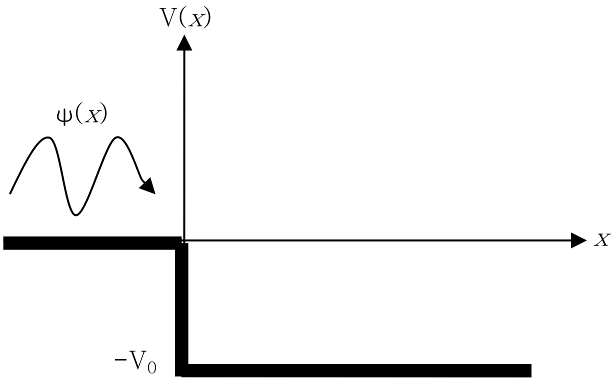
(1) : $A + B = C$

(2) : $Aik_1 - Bik_1 = Cik_2 \rightarrow \dfrac{(A-B)k_1}{k_2} = C$ → (1)에 대입.

$A + B = \dfrac{(A-B)k_1}{k_2}$ $\quad k_2(A+B) = k_1(A-B)$

$(k_1 - k_2)A = (k_1 + k_2)B$ $\quad \therefore \dfrac{B}{A} = \dfrac{k_1 - k_2}{k_1 + k_2}$

$\therefore R = \dfrac{|B|^2}{|A|^2} = \left(\dfrac{k_1 - k_2}{k_1 + k_2}\right)^2 = \left(\dfrac{\sqrt{E} - \sqrt{E+V_0}}{\sqrt{E} + \sqrt{E+V_0}}\right)^2$



(c) What is R if $E = V_0 / 3$?

Calculate the transmission coefficient T in this case from $R + T = 1$.

(c) $E = V_0/3 \rightarrow V_0 = 3E$

$\therefore R = \left(\dfrac{\sqrt{E} - 2\sqrt{E}}{\sqrt{E} + 2\sqrt{E}}\right)^2 = \left(\dfrac{-1}{3}\right)^2 = \dfrac{1}{9}$ $\quad \therefore T = 1 - R = \dfrac{1}{9}$

6. For an electron in a one dimensional crystal, assuming it experiences one dimensional periodic potential (Kronig-Penny model) shown below, ($E < V_0$)

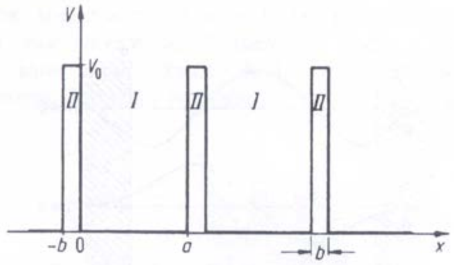


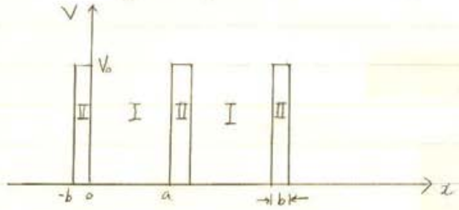
Figure 4.9. One-dimensional periodic potential distribution (simplified) (**Kronig-Penney model**).

(a) Using the Bloch function $\psi(x) = u(x) \cdot e^{ikx}$ as the wave function of the electron, show that solution of the Schrodinger equations in regions I and II leads to

$$\frac{\gamma^2 - \alpha^2}{2\alpha\gamma}\sinh(\gamma b)\cdot\sin(\alpha a) + \cosh(\gamma b)\cdot\cos(\alpha a) = \cos k(a+b)$$

Where, $\alpha^2 = \frac{2m}{\hbar^2}E \qquad \gamma^2 = \frac{2m}{\hbar^2}(V_0 - E)$

7 (a) Kronig-Penny model ($E < V_0$)



Schrödinger equation for Regions I and II

$$(\mathrm{I})\ \frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2}E\psi = 0$$

$$(\mathrm{II})\ \frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2}(E - V_0)\psi = 0$$

$\alpha^2 = \frac{2mE}{\hbar^2}$, $\gamma^2 = \frac{2m}{\hbar^2}(V_0 - E)$ 라 두자.

Periodic potential 을 고려한 Bloch function ($\psi(x) = u(x)\cdot e^{ikx}$)을 두 식에 대입하면,

$$(\mathrm{I})\ \frac{d^2u}{dx^2} + 2ik\frac{du}{dx} - (k^2 - \alpha^2)u = 0,\quad (\mathrm{II})\ \frac{d^2u}{dx^2} + 2ik\frac{du}{dx} - (k^2 + \gamma^2)u = 0$$

각각 일반해를 구하면,

$$(\mathrm{I})\ u = e^{-ikx}(Ae^{i\alpha x} + Be^{-i\alpha x}),\quad (\mathrm{II})\ u = e^{-ikx}(Ce^{-\gamma x} + De^{\gamma x})$$

Boundary condition 을 적용하자

1) $x=0$ 에서 $\psi(x)$는 continuous $\rightarrow$ $A + B = C + D$ — ①

2) $x=0$ 에서 $d\psi/dx$는 continuous (즉 du/dx continuous)

$$(\mathrm{I})\ \frac{du}{dx} = \underbrace{-ike^{-ikx}(Ae^{i\alpha x} + Be^{-i\alpha x})}_{-ik(A+B)} + \underbrace{e^{-ikx}(i\alpha Ae^{i\alpha x} - i\alpha Be^{-i\alpha x})}_{i\alpha(A-B)}$$

$$(\mathrm{II})\ \frac{du}{dx} = \underbrace{-ike^{-ikx}(Ce^{-\gamma x} + De^{\gamma x})}_{-ik(C+D)} + e^{-ikx}(-\gamma Ce^{-\gamma x} + \gamma De^{\gamma x})$$

$$\rightarrow A(i\alpha - ik) + B(-i\alpha - ik) = C(-\gamma - ik) + D(\gamma - ik) \text{ — ②}$$

3) $x = a+b$ 에서 $\psi(x)$는 continuous ($u_{\mathrm{I}}(x=a) = u_{\mathrm{II}}(x=-b)$)

$$\rightarrow Ae^{(i\alpha - ik)a} + Be^{(-i\alpha - ik)a} = Ce^{(ik+\gamma)b} + De^{(ik-\gamma)b} \text{ — ③}$$

4) $x = a+b$ 에서 $d\psi/dx$는 continuous ($du_{\mathrm{I}}/dx\,(x=a) = du_{\mathrm{II}}/dx\,(x=-b)$)

$$\rightarrow Ai(\alpha - k)e^{ia(\alpha - k)} - Bi(\alpha + k)e^{-ia(\alpha + k)}$$

$$= -C(\gamma + ik)e^{(ik+\gamma)b} + D(\gamma - ik)e^{(ik-\gamma)b} \text{ — ④}$$

7.(a) Kronig - Penney model → four equations

① $A+B=C+D$

② $A(i\alpha-ik)+B(-i\alpha-ik)=C(-\gamma-ik)+D(\gamma-ik)$

③ $Ae^{(i\alpha-ik)a}+Be^{(-i\alpha-ik)a}=Ce^{(ik+\gamma)b}+De^{(ik-\gamma)b}$

④ $Ai(\alpha-k)e^{ia(\alpha-k)}-Bi(\alpha+k)e^{-ia(\alpha+k)}=-C(\gamma+ik)e^{(ik+\gamma)b}+D(\gamma-ik)e^{(ik-\gamma)b}$

→ 4개의 식으로 A, B, C, D 소거 → A, B, C, D 에 대한 4x4 matrix 를 구성하여 det=0 을 만족시키는 관계식을 찾으면 된다

네 식을 행렬로 표현

1행: ①
2행: ②
3행: ③
4행: ④

$$\det\begin{vmatrix} 1 & 1 & -1 & -1 \\ i\alpha-ik & -i\alpha-ik & \gamma+ik & -\gamma+ik \\ e^{(i\alpha-ik)a} & e^{(-i\alpha-ik)a} & -e^{(ik+\gamma)b} & -e^{(ik-\gamma)b} \\ i(\alpha-k)e^{ia(\alpha-k)} & -i(\alpha+k)e^{-ia(\alpha+k)} & +(\gamma+ik)e^{(ik+\gamma)b} & -(\gamma-ik)e^{(ik-\gamma)b} \end{vmatrix}=0$$

각 행에 상수를 곱해 주어도, 혹은 한 행을 상수배하여 다른행에 더하여도 det( )=0 값은 변하지 않음
(행간 순서를 바꾸어도 det( )=0 값은 그대로)

① × ik + ② : $i\alpha$ $\quad -i\alpha$ $\quad \gamma$ $\quad -\gamma$ → 새로운 2행 (②′)

③ × ik + ④ : $i\alpha e^{ia(\alpha-k)}$ $\quad -i\alpha e^{-ia(\alpha+k)}$ $\quad \gamma\cdot e^{(ik+\gamma)b}$ $\quad -\gamma e^{(ik-\gamma)b}$ → ④′

③ × e^{ika} : $e^{i\alpha a}$ $\quad e^{-i\alpha a}$ $\quad -e^{i(a+b)k+\gamma b}$ $\quad -e^{i(a+b)k-\gamma b}$ → ③′

④′ × e^{ika} : $i\alpha e^{i\alpha a}$ $\quad -i\alpha e^{-i\alpha a}$ $\quad \gamma e^{i(a+b)k+\gamma b}$ $\quad -\gamma e^{i(a+b)k-\gamma b}$ → ④″

계산을 쉽게 하기 위해 $\begin{pmatrix} ④'' \\ ③' \\ ②' \\ ① \end{pmatrix}$ 로 배열

$$\Rightarrow \begin{pmatrix} i\alpha e^{i\alpha a} & -i\alpha e^{-i\alpha a} & \gamma e^{i(a+b)k+\gamma b} & -\gamma e^{i(a+b)k-\gamma b} \\ e^{i\alpha a} & e^{-i\alpha a} & -e^{i(a+b)k+\gamma b} & -e^{i(a+b)k-\gamma b} \\ i\alpha & -i\alpha & \gamma & -\gamma \\ 1 & 1 & -1 & -1 \end{pmatrix}$$

$i(a+b)k=t$ 로 치환

$$\det\begin{pmatrix} i\alpha e^{i\alpha a} & -i\alpha e^{-i\alpha a} & \gamma e^{t+\gamma b} & -\gamma e^{t-\gamma b} \\ e^{i\alpha a} & e^{-i\alpha a} & -e^{t+\gamma b} & -e^{t-\gamma b} \\ i\alpha & -i\alpha & \gamma & -\gamma \\ 1 & 1 & -1 & -1 \end{pmatrix}$$

$$1)\ i\alpha e^{i\alpha a} \times \det \begin{vmatrix} e^{-i\alpha a} & -e^{t+\gamma b} & -e^{t-\gamma b} \\ -i\alpha & \gamma & -\gamma \\ 1 & -1 & -1 \end{vmatrix}$$

$$\left(\frac{e^{x}+e^{-x}}{2} = \cosh x \quad \frac{e^{x}-e^{-x}}{2} = \sinh x \right)$$

$$= i\alpha e^{i\alpha a} \times \left[e^{-i\alpha a}(-\gamma-\gamma) + e^{t+\gamma b}(i\alpha+\gamma) - e^{t-\gamma b}(i\alpha-\gamma) \right]$$

$$= -2i\alpha\gamma + i\alpha e^{i\alpha a} \cdot e^{t} \left[i\alpha \cdot 2\sinh(\gamma b) + \gamma \cdot 2\cosh(\gamma b) \right]$$

$$= -2i\alpha\gamma + e^{i\alpha a} e^{t} \left[-\alpha^2 \cdot 2\sinh(\gamma b) + i\alpha\gamma \cdot 2\cosh(\gamma b) \right]$$

$$2)\ (-1) \times -i\alpha e^{-i\alpha a} \times \det \begin{vmatrix} e^{i\alpha a} & -e^{t+\gamma b} & -e^{t-\gamma b} \\ i\alpha & \gamma & -\gamma \\ 1 & -1 & -1 \end{vmatrix}$$

$$= i\alpha e^{-i\alpha a} \times \left[e^{i\alpha a} \cdot (-\gamma-\gamma) + e^{t+\gamma b}(-i\alpha+\gamma) - e^{t-\gamma b}(-i\alpha-\gamma) \right]$$

$$= -2i\alpha\gamma + i\alpha e^{-i\alpha a} \cdot e^{t} \left[-i\alpha \cdot 2\sinh(\gamma b) + \gamma \cdot 2\cosh(\gamma b) \right]$$

$$= -2i\alpha\gamma + e^{-i\alpha a} \cdot e^{t} \left[\alpha^2 \cdot 2\sinh(\gamma b) + i\alpha\gamma \cdot 2\cosh(\gamma b) \right]$$

$$\begin{cases} \cosh(ix) = \dfrac{e^{ix}+e^{-ix}}{2} \\ \quad = \dfrac{\cos x + \cancel{i\sin x} + \cos x - \cancel{i\sin x}}{2} \\ \quad = \cos x \\ \sinh(ix) = \dfrac{e^{ix}-e^{-ix}}{2} \\ \quad = \dfrac{\cancel{\cos x} + i\sin x - \cancel{\cos x} + i\sin x}{2} \\ \quad = i\sin x \end{cases}$$

$$1) + 2) : \quad -4i\alpha\gamma + e^{t} \left\{ -\alpha^2 \cdot 2\sinh(\gamma b) \cdot 2\sinh(i\alpha a) + i\alpha\gamma \cdot 2\cosh(\gamma b) \cdot 2\cosh(i\alpha a) \right\}$$

$$= -4i\alpha\gamma + e^{t} \left\{ -4\alpha^2 i \sinh(\gamma b) \sin(\alpha a) + 4i\alpha\gamma \cosh(\gamma b) \cos(\alpha a) \right\}$$

$$3)\ \gamma e^{t+\gamma b} \times \det \begin{vmatrix} e^{i\alpha a} & e^{-i\alpha a} & -e^{t-\gamma b} \\ i\alpha & -i\alpha & -\gamma \\ 1 & 1 & -1 \end{vmatrix}$$

$$= \gamma e^{t+\gamma b} \times \left[e^{i\alpha a}(i\alpha+\gamma) - e^{-i\alpha a}(-i\alpha+\gamma) - e^{t-\gamma b}(i\alpha+i\alpha) \right]$$

$$= \gamma e^{t+\gamma b} \times \left[i\alpha \cdot 2\cosh(i\alpha a) + \gamma \cdot 2\sinh(i\alpha a) \right] - 2i\alpha\gamma e^{2t}$$

$$= \gamma e^{t+\gamma b} \times \left\{ i\alpha \cdot 2\cos(\alpha a) + i\gamma \cdot 2\sinh(\alpha a) \right\} - 2i\alpha\gamma \cdot e^{2t}$$

$$4)\ (-1) \times -\gamma e^{t-\gamma b} \times \det \begin{vmatrix} e^{i\alpha a} & e^{-i\alpha a} & -e^{t+\gamma b} \\ i\alpha & -i\alpha & \gamma \\ 1 & 1 & -1 \end{vmatrix}$$

$$= \gamma e^{t-\gamma b} \times \left[e^{i\alpha a}(i\alpha-\gamma) - e^{-i\alpha a}(-i\alpha-\gamma) - e^{t+\gamma b}(i\alpha+i\alpha) \right]$$

$$= \gamma e^{t-\gamma b} \times \left[i\alpha \cdot 2\cosh(i\alpha a) - \gamma \cdot 2\sinh(i\alpha a) \right] - 2i\alpha\gamma e^{2t}$$

$$= \gamma e^{t-\gamma b} \times \left[i\alpha \cdot 2\cos(\alpha a) - i\gamma\, 2\sinh(\alpha a) \right] - 2i\alpha\gamma e^{2t}$$

3)+4) $\quad -4i\alpha\gamma e^{2t} + \gamma e^{t}\left[i\alpha \cdot 2\cos(\alpha a) \cdot 2\cosh(\gamma b) + i\gamma \cdot 2\sin(\alpha a) \cdot 2\sinh(\gamma b) \right]$

1)+2)+3)+4) $\quad -4i\alpha\gamma - 4i\alpha\gamma e^{2t}$

$$+e^{t}\left\{ \begin{array}{l} -4\alpha^2 i \sinh(\gamma b)\sin(\alpha a) + 4i\alpha\gamma\cosh(\gamma b)\cos(\alpha a) \\ + 4i\gamma^2 \sinh(\gamma b)\sin(\alpha a) + 4i\alpha\gamma\cosh(\gamma b)\cos(\alpha a) \end{array} \right\} = 0$$

정리하여 양변을 $-4i\alpha\gamma$ 로 나누어 주면,

$$e^{2t} - e^{t}\left\{ \frac{\gamma^2-\alpha^2}{\alpha\gamma}\sinh(\gamma b)\sin(\alpha a) + 2\cosh(\gamma b)\cos(\alpha a) \right\} + 1 = 0 \quad \therefore\ e^{t}\text{에 관한 이차방정식}$$

$$\therefore\ e^{t} = \left[\frac{\gamma^2-\alpha^2}{2\alpha\gamma}\sinh(\gamma b)\sin(\alpha a) + \cosh(\gamma b)\cos(\alpha a) \right] \pm \sqrt{[\quad]^2 - 1}$$

$$e^{t} = e^{i(a+b)k} \quad \therefore\ e^{t} = \cos k(a+b) + i\sin k(a+b)$$

등식이 성립하기 위해 각각 실수 부분과 허수부분이 같아야 한다

$$\therefore\ \cos k(a+b) + i\sin k(a+b) = [\qquad] \pm \sqrt{[\quad]^2 - 1}$$

$$= [\qquad] \pm i\sqrt{1-[\quad]^2}$$

$\sin\theta = \pm\sqrt{1-\cos^2\theta}$ 이므로, 본 식은 $\cos k(a+b) = [\qquad]$ 만 성립하면 등호를 만족한다.

$$\therefore\ \frac{\gamma^2-\alpha^2}{2\alpha\gamma}\sinh(\gamma b)\sin(\alpha a) + \cosh(\gamma b)\cos(\alpha a) = \cos k(a+b)$$

(b) Describe in what condition, the above relation becomes

$$P\frac{\sin\alpha a}{\alpha a} + \cos\alpha a = \cos ka \quad \text{where} \quad P = \frac{maV_0 b}{\hbar^2}$$

7. (b) $\quad \frac{\gamma^2-\alpha^2}{2\alpha\gamma}\sinh(\gamma b)\cdot\sin(\alpha a) + \cosh(\gamma b)\cdot\cos(\alpha a) = \cos k(a+b)$

b는 매우 작고, V_0는 매우 큰 값이라 가정하자 (but $V_0 \cdot b$ remains finite)

이 때 $\quad \gamma = \sqrt{\frac{2m}{\hbar^2}(V_0-E)} \approx \sqrt{\frac{2m}{\hbar^2}}\cdot\sqrt{V_0}$

$$\therefore\ \gamma b = \sqrt{\frac{2m}{\hbar^2}}\sqrt{(V_0 b)\cdot b} \qquad \left(\begin{array}{l} \cosh x = \frac{e^{x}+e^{-x}}{2} \\ \sinh x = \frac{e^{x}-e^{-x}}{2} \end{array} \right) \text{taylor expansion}$$

$b \sim 0$ 이므로 γb 역시 0에 근사한 값이며, 따라서 $\cosh(\gamma b) \approx 1$, $\sinh(\gamma b) \approx \gamma b$ 이다

또한 $\gamma \gg \alpha$ 이므로 $\frac{\gamma^2-\alpha^2}{2\alpha\gamma} \approx \frac{\gamma^2}{2\alpha\gamma} = \frac{\gamma}{2\alpha}$

$$\therefore\ \frac{\gamma}{2\alpha}\cdot\gamma b\,\sin(\alpha a) + \cos(\alpha a) = \cos k(a+b) \qquad (b\sim 0 \text{ 이므로 } a+b \approx a)$$

$$\therefore\ \frac{m}{\alpha\hbar^2}V_0 b\,\sin(\alpha a) + \cos(\alpha a) = \cos(ka)$$

$P = \frac{maV_0 b}{\hbar^2}$ 라 두면, $\quad P\frac{\sin(\alpha a)}{\alpha a} + \cos(\alpha a) = \cos(ka)$