

Homework4. Solution

$$1. \phi = \operatorname{Re} \{ \varphi e^{i\omega t} \} + \operatorname{Re} \{ \psi e^{i\omega t} \}$$

(free wave component + local wave component)

◆Boundary Value Problem

$$\textcircled{1} \nabla \phi^2 = 0 , \quad \nabla \psi^2 = 0 \quad (\text{Laplace equation})$$

$$\textcircled{2} \frac{\partial^2 \phi}{\partial t^2} + g \frac{\partial \phi}{\partial z} = 0 , \quad \left(-\omega^2 + g \frac{\partial}{\partial z} \right) \psi = 0 \quad \text{on } z=0 \quad (\text{F.S.B.C})$$

$$\textcircled{3} \frac{\partial \phi}{\partial z} = 0 \Rightarrow \frac{\partial \psi}{\partial z} = 0 \quad \text{on } z=-h \quad (\text{Bottom condition})$$

$$\textcircled{4} \frac{\partial \phi}{\partial x} = \frac{\partial \xi}{\partial t} \Rightarrow \frac{\partial \psi}{\partial x} = \frac{\partial \xi}{\partial t} = \operatorname{Re} \{ i\omega \xi_0 e^{i\omega t} \} \quad \text{on } x=0 \quad (\text{Body boundary condition})$$

$$\textcircled{5} \psi \rightarrow 0 \quad \text{as } x \rightarrow \infty \quad (\text{Radiation condition})$$

$$\text{Put, } \psi = X(x) \cdot Z(z)$$

$$\frac{X_{xx}}{X} = -\frac{Z_{zz}}{Z} = -k^2 \quad |k| = \sigma \quad (\textcircled{1} \circ \textcircled{2})$$

$$X = Ae^{\sigma x} + Be^{-\sigma x} \quad (\textcircled{5} \circ \textcircled{2})$$

$$Z_{zz} + \sigma^2 Z = 0$$

$$Z = Ce^{-kz} + De^{kz}$$

$$\frac{\partial \psi}{\partial z} = i\sigma(Ce^{-i\sigma h} - De^{i\sigma h}) = 0$$

$$Z = De^{2i\sigma h} \sigma^{i\sigma z} + De^{i\sigma z} = 2De^{i\sigma h} \cos[\sigma(z+h)] + \operatorname{Re} [\varphi e^{i\omega t}]$$

$$\psi = \psi_0 e^{i\sigma h} \cos[\sigma(z+h)] e^{-\sigma x} \quad (\textcircled{3} \circ \textcircled{4})$$

여기에 ②조건을 이용하면 $\omega^2 = -g\sigma \tan(\sigma h)$ 을 얻는다.

(dispersion relation of local wave)

$$\psi = \sum \tilde{\psi}_n e^{i\omega t} \quad (\tilde{\psi}_n = \psi_0 e^{i\sigma_n x} \cos[\sigma_n(z+h)] e^{-\sigma_n x})$$

$$\frac{\partial \phi}{\partial x} = \operatorname{Re} [i\omega \xi_0 e^{i\omega t}] \quad \begin{cases} \xi_0 = \text{constant} & \text{at } -d < z < 0 \\ \xi_0 = 0 & \text{at } -h < z < -d \end{cases}$$

$$\begin{aligned}
\frac{\partial \phi}{\partial x} &= \operatorname{Re} \left\{ -ik\varphi e^{i\omega t} \right\} + \operatorname{Re} \left\{ \frac{\partial \psi}{\partial x} e^{i\omega t} \right\} \\
&= \operatorname{Re} \left\{ i\omega \xi_0 e^{i\omega t} \right\} \\
\Rightarrow -ik\varphi + \sum_{n=1}^{\infty} (-\sigma_n) \psi_n &= i\omega \xi_0 \\
\int_{-h}^0 \varphi \tilde{\psi}_n dz = 0 &\quad \text{if } n \neq m, \quad \int_{-h}^0 \tilde{\psi}_n \psi_m dz = 0 \quad (\text{orthogonality}) \\
\int_{-h}^0 (-ik\varphi) \varphi dz + \sum \cancel{\int_{-h}^0 (-\sigma_n) \tilde{\psi}_n \psi_m dz} &= \int_{-h}^0 (i\omega \xi_0) \psi dz \\
\int_{-h}^0 -k\psi^2 dz &= \int_{-h}^0 (\omega \xi_0) \psi dz \\
A &= -\frac{\omega}{k} \int_{-h}^0 \tilde{\varphi} \xi_0 dz \quad \left(\varphi = -\frac{igA}{\omega} K(z) \text{ 라 두면} \right) \\
A &= -\frac{\omega}{k} \int_{-d}^0 \frac{\sqrt{2} \cosh k(z+h)}{\sqrt{h + \frac{g}{\omega^2} \sinh^2(kh)}} \xi_0 dz \\
&= -\frac{\omega}{k^2} \frac{\sqrt{2} \sinh(kh) - \sinh(k(-d+h))}{\sqrt{h + \frac{g}{\omega^2} \sinh^2(kh)}} \xi_0
\end{aligned}$$

$$2.(1) \quad \omega = \frac{2\pi}{T} = \frac{2\pi}{2} = \pi \quad (\text{rad/sec})$$

$$(2) \text{ first mode } \psi_1 = \psi_0 e^{-\sigma_1 x} \cos \{ \sigma_1 (z+h) \}$$

Local dispersion relationship $\omega^2 = -g\sigma \tan(\sigma h)$

$$\frac{\omega^2}{g\sigma_1} = -\tan(\sigma_1 h) \rightarrow \frac{\pi^2}{(9.81)\sigma_1} = -\tan(3.5\sigma_1)$$

$$\therefore \sigma_1 = 0.603$$

$$\text{At } x=0, \quad \psi_1 = \psi_0 \cos \{ \sigma_1 (z+h) \}$$

초기 wave component 의 50%이 되는 지점의 크기는 $\frac{1}{2}\psi_0 \cos \{ \sigma_1 (z+h) \}$

$$\frac{1}{2} \psi_0 \cos \overbrace{\{\sigma_1(z+h)\}} = \psi_0 e^{-\sigma_1 x} \cos \overbrace{\{\sigma_1(z+h)\}}$$

$$\frac{1}{2} = e^{-\sigma_1 x} \rightarrow x = 1.149 \text{m}$$

$$u = \frac{\partial \psi_1}{\partial x} = \sigma = 0.603$$

$$(3) \omega^2 = gk \tanh kh \quad \text{으로 } k=1.01$$

$$A = \frac{-\omega}{k^2} \cdot \frac{\sqrt{2} \cdot [\sinh(kh) - \sinh(k(-d+h))]}{\sqrt{h + \frac{g}{\omega^2} \sinh^2(kh)}} \xi_0$$

$$\therefore A = 2.77 \xi_0$$

3.(1)

$$V_p = U$$

$$V_p = \frac{\omega}{k} = \sqrt{\frac{g}{k}} = \frac{g}{\omega} \quad \therefore \omega = \frac{g}{U} \quad , \quad k = \frac{g}{U^2} \quad (\text{Deep sea})$$

(2) Source at $x=+L/2$

Sink at $x=-L/2$

$$\eta_{source} = a \cos \left\{ k \left(x + \frac{L}{2} \right) \right\}$$

$$\eta_{sink} = -a \cos \left\{ k \left(x - \frac{L}{2} \right) \right\}$$

$$\eta = a \cos \left\{ k \left(x + \frac{L}{2} \right) \right\} - a \cos \left\{ k \left(x - \frac{L}{2} \right) \right\}$$

$$-2a \sin \left\{ \frac{kL}{2} \right\} \sin \{kx\}$$

$$\eta = 0 \Rightarrow \text{위해서는 } \frac{kL}{2} = n\pi \Rightarrow \text{된다. } (n=1, 2, 3, \dots)$$

$$L = \frac{2n\pi}{k} = \frac{2n\pi U^2}{g} \quad , \quad (n=1, 2, 3, \dots) \quad , \quad \therefore k = \frac{g}{U^2}$$