Homework List

HW#1: Problems 2.8, 2.9, 2.10, 2.11, 2.12, 2.14 of the textbook. (Due: 3/25/2008)

HW#2: Problems 2.13, 2.15, 2.16, 2.17, 2.20, 2.21, 2.23, 2.24, 2.26 of the textbook. (Due: 4/1/2008)

HW#3: Problems 2.30, 2.31, 2.32, 2.33, 2.34 and 2.35 of the textbook. (Due: 4/3/2008)

HW#4: Problems 3.5, 3.6, 3.4, 3.10(b)-(d), 3.11, 3.12, 3.16 of the textbook. (Due: 4/22/2008)

HW#5: Problems 3.32, 3.33, 3.34, 3.37, 3.40 of the textbook. (Due: 4/24/2008)

HW#6: Problems 3.30, 3.46, 3.49 of the textbook. (Due: 4/29/2008)

HW#7: Problems 3.50, 3.52, 3.54, 3.57 of the textbook. (Due: 5/8/2008)

HW#8: Problems 3.59, 3.61, 3.64 of the textbook. (Due: 5/15/2008)

HW#9: Problems 4.1, 4.4, 4.6, 4.8, 4.9, 4.10 of the textbook. (Due: 5/27/2008)

HW#10: Problems 4.13, 4.15, 4.18, 4.20, 4.21 of the textbook. (Due: final exam time)

HW 1

1/2

2.8

Let x_{ij} be the bit in row i, column j. Then the ith horizontal parity check is

$$h_i = \Sigma_j x_{ij}$$

where the summation is summation modulo 2. Summing both sides of this equation (modulo 2) over the rows i, we have

$$\Sigma_i h_i = \Sigma_{i,j} x_{ij}$$

This shows that the modulo 2 sum of all the horizontal parity checks is the same as the modulo 2 sum of all the data bits. The corresponding argument on columns shows that the modulo 2 sum of the vertical parity checks is the same.

2.9

a) Any pattern of the form

--- 1 1 0 ---
--- 0 1 1 ---
--- 1 0 1 ---

will fail to be detected by horizontal and vertical parity checks. More formally, for any three rows i_1, i_2, and i_3, and any three columns j_1, j_2, and j_3, a pattern of six errors in positions $(i_1\ j_1)$, $(i_1\ j_2)$, $(i_2\ j_2)$, $(i_2\ j_3)$, $(i_3\ j_1)$, and $(i_3\ j_3)$ will fail to be detected.

b) The four errors must be confined to two rows, two errors in each, and to two columns, two errors in each; that is, geometrically, they must occur at the vertices of a rectangle within the array. Assuming that the data part of the array is J by K, then the array including the parity check bits is J+1 by K+1. There are (J+1)J/2 different possible pairs of rows (counting the row of vertical parity checks), and (K+1)K/2 possible pairs of columns (counting the column of horizontal checks). Thus there are (J+1)(K+1)JK/4 undetectable patterns of four errors.

2.10

Let $x = (x_1, x_2, \dots x_N)$ and $x' = (x'_1, x'_2, \dots x'_N)$ be any two distinct code words in a parity check code. Here N = K+L is the length of the code words (K data bits plus L check bits). Let $y = (y_1, \dots y_N)$ be any given binary string of length N. Let D(x,y) be the distance between x and y (i.e. the number of positions i for which $x_i \neq y_i$). Similarly let D(x',y) and D(x,x') be the distances between x' and y and between x and x'. We now show that

$$D(x,x') \leq D(x,y) + D(x',y)$$

To see this, visualize changing D(x,y) bits in x to obtain y, and then changing D(x',y) bits in y to obtain x'. If no bit has been changed twice in going from x to y and then to x', then it was necessary to change D(x,y) + D(x',y) bits to change x to x' and the above inequality is satisfied with equality. If some bits have been changed twice (i.e. $x_i = x'_i \neq y_i$ for some i) then strict inequality holds above.

By definition of the minimum distance d of a code, $D(x,x') \geq d$. Thus, using the above inequality, if $D(x,y) < d/2$, then $D(x',y) > d/2$. Now suppose that code word x is sent and fewer than d/2 errors occur. Then the received string y satisfies $D(x,y) < d/2$ and for every other code word x', $D(x',y) > d/2$. Thus a decoder that maps y into the closest code word must select x, showing that no decoding error can be made if fewer than d/2 channel errors occur. Note that this argument applies to any binary code rather than just parity check codes.

2.11

The first code word given, 1001011 has only the first data bit equal to 1 and has the first, third, and fourth parity checks equal to 1. Thus those parity checks must check on the first data bit. Similarly, from the second code word, we see that the first, second, and fourth parity checks must check on the second bit. From the third code word, the first, second, and third parity check each check on the third data bit. Thus

$$c_1 = s_1 + s_2 + s_3$$
$$c_2 = s_2 + s_3$$
$$c_3 = s_1 + s_3$$
$$c_4 = s_1 + s_2$$

The set of all code words is given by

0000000	0011110
1001011	1010101
0101101	0110011
1100110	1111000

The minimum distance of the code is 4, as can be seen by comparing all pairs of code words. An easier way to find the minimum distance of a parity check code is to observe that if x and x' are each code words, then x + x' (using modulo 2 componentwise addition) is also a code word. On the other hand, x + x' has a 1 in a position if and only if x and x' differ in that position. Thus the distance between x and x' is the number of ones in x + x'. It follows that the minimum distance of a parity check code is the minimum, over all non-zero code words, of the number of ones in each code word.

2.12

$$\begin{array}{r}
D^3 \\
D^4 + D^2 + D + 1 \overline{)\, D^7 + D^5 + D^4 \quad} \\
D^7 + D^5 + D^4 + D^3 \\
\hline
D^3
\end{array} = \text{Remainder}$$

2.14

Suppose g(D) contains (1+D) as a factor; thus g(D) = (1+D)h(D) for some polynomial h(D). Substituting 1 for D and evaluating with modulo 2 arithmetic, we get g(1) = 0 because of the term (1+D) = (1+1) = 0. Let e(D) be the polynomial for some arbitrary undetectable error sequence. Then e(D) = g(D)z(D) for some z(D), and hence e(1) = g(1)z(1) = 0. Now $e(D) = \Sigma_i e_i D^i$, so $e(1) = \Sigma_i e_i$. Thus e(1) = 0 implies that an even number of elements e_i are 1; i.e. that e(D) corresponds to an even number of errors. Thus all undetectable error sequences contain an even number of errors; any error sequence with an odd number of errors is detected.

2.13

Let $z(D) = D^j + z_{j-1}D^{j-1} + ... + D^i$ and assume i<j. Multiplying G(D) times Z(D) then yields

$$g(D)z(D) = D^{L+j} + (z_{j-1} + g_{L-1})D^{L+j-1} + (z_{j-2} + g_{L-1}z_{j-1} + g_{L-2})D^{L+j-2} + ... + (g_1 + z_{i+1})D^{i+1} + D^i$$

Clearly the coefficient of D^{L+j} and the coefficient of D^i are each 1, yielding the desired two non-zero terms. The above case i<j arises whenever z(D) has more than one non-zero term. For the case in which z(D) has only one non-zero term, i.e. $z(D) = D^j$ for some j, we have

$$g(D)z(D) = D^{L+j} + g_{L-1}D^{L+j-1} + ... + D^j$$

which again has at least two non-zero terms.

2.15

a) Let D^{i+L}, divided by g(D), have the quotient $z^{(i)}(D)$ and remainder $c^{(i)}(D)$ so that

$$D^{i+L} = g(D)z^{(i)}(D) + c^{(i)}(D)$$

Multiplying by s_i and summing over i,

$$s(D)D^L = \Sigma_i s_i z^{(i)}(D) + \Sigma_i s_i c^{(i)}(D)$$

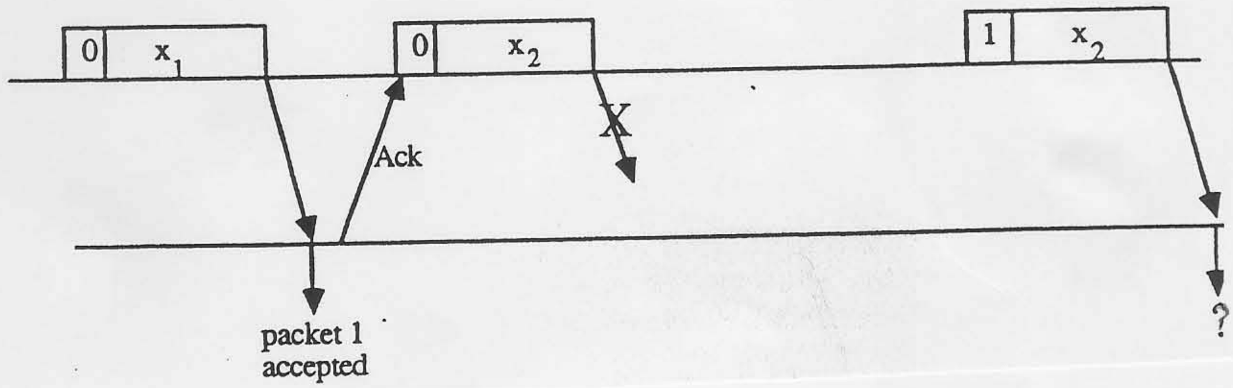
Since $\Sigma_i s_i c^{(i)}(D)$ has degree less than L, this must be the remainder (and $\Sigma_i s_i z^{(i)}(D)$ the quotient) on dividing $s(D)D^L$ by g(D). Thus $c(D) = \Sigma_i s_i c^{(i)}(D)$.

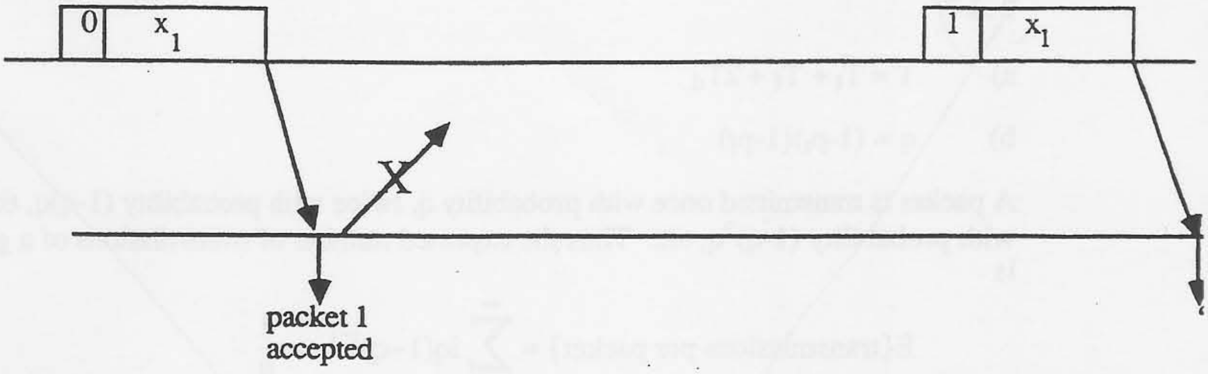
b) Two polynomials are equal if and only if their coefficients are equal, so the above polynomial equality implies

$$c_j = \Sigma_i s_i c_j^{(i)}$$

2.16

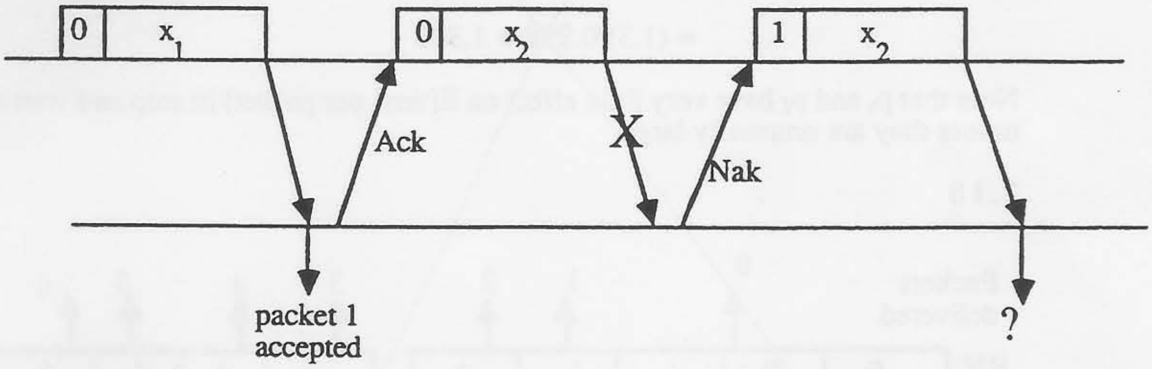
a) Consider the two scenarios below and note that these scenarios are indistinguishable to the receiver.

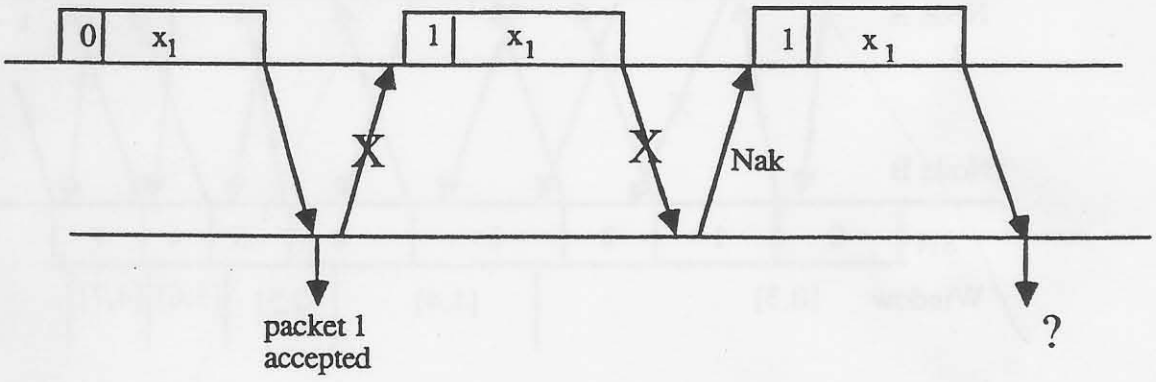




If the receiver releases the packet as x_2 in the questioned reception, then an error occurs on scenario 2. If the receiver returns an ack but doesn't release a packet (i.e. the appropriate action for scenario 2), then under scenario 1, the transmitter erroneously goes on to packet 3. Finally, if the receiver returns a nak, the problem is only postponed since the transmitter would then transmit $(2,x_2)$ in scenario 1 and $(2,x_1)$ in scenario 2. As explained on page 66, packets x_1 and x_2 might be identical bit strings, so the receiver can not resolve its ambiguity by the bit values of the packets.

b) The scenarios below demonstrate incorrect operation for the modified conditions.





2.17

a) $T = T_t + T_f + 2T_d$

b) $q = (1-p_t)(1-p_f)$

A packet is transmitted once with probability q, twice with probability (1-q)q, three times with probability $(1-q)^2q$, etc. Thus the expected number of transmissions of a given packet is

$$E\{\text{transmissions per packet}\} = \sum_{i=1}^{\infty} iq(1-q)^{i-1} = \frac{1}{q}$$

To verify the above summation, note that for any x, $0 \le x < 1$,

$$\sum_{i=1}^{\infty} ix^{i-1} = \sum_{i=1}^{\infty} \frac{dx^i}{dx} = \frac{d}{dx}\sum_{i=1}^{\infty} x^i = \frac{d}{dx}\left(\frac{x}{1-x}\right) = \frac{1}{(1-x)^2}$$

Using x for (1-q) above gives the desired result.

c) $E\{\text{time per packet}\} = (T_t + T_f + 2T_d)/q$

$= (1.3)/0.998 = 1.303$

Note that p_t and p_f have very little effect on E[time per packet] in stop and wait systems unless they are unusually large.

2.20

The simplest example is for node A to send packets 0 through n-1 in the first n frames. In case of delayed acknowledgements (i.e. no return packets in the interim), node A goes back and retransmits packet 0. If the other node has received all the packets, it is waiting for packet n, and if the modulus m equals n, this repeat of packet 0 is interpreted as packet n.

The right hand side of Eq. (2.24) is satisfied with equality if $SN = SN_{min}(t_1)+n-1$. This occurs if node A sends packets 0 through n-1 in the first n frames with no return packets from node B. The last such frame has SN = n-1, whereas SN_{min} at that time (say t_1) is 0.

Continuing this scenario, we find an example where the right hand side of Eq. (2.25) is satisfied with equality. If all the frames above are correctly received, then after the last frame, RN becomes equal to n. If another frame is sent from A (now call this time t_1) and if SN_{min} is still 0, then when it is received at B (say at t_2), we have $RN(t_2) = SN_{min}(t_1)+n$.

2.21

Let $RN(\tau)$ be the value of RN at node B at an arbitrary time τ; SN_{min} is the smallest packet number not yet acknowledged at A at time t (which is regarded as fixed) and $SN_{max}-1$ is the largest packet number sent from A at time t. Since $RN(\tau)$ is non decreasing in τ, it is sufficient to show that $RN(t+T_m+T_d) \le SN_{max}$ and to show that $RN(t-T_m-T_d) \ge SN_{min}$.

For the first inequality, note that the packet numbered SN_{max} (by definition of SN_{max}) has not entered the DLC unit at node A by time t, and thus can not have started transmission by time t. Since there is a delay of at least T_m+T_d from the time a packet transmission starts until the completion of its reception, packet SN_{max} can not have been received by time $t+T_m+T_d$. Because of the correctness of the protocol, $RN(t+T_m+T_d)$ can be no greater than the number of a packet not yet received, i.e. SN_{max}.

For the second inequality, note that for the transmitter to have a given value of SN_{min} at time t, that value must have been transmitted earlier as the request number in a frame coming back from node B. The latest time that such a frame could have been formed is $t-T_m-T_d$, so by this time RN must have been at least SN_{min}.

2.23

After a given packet is transmitted from node A, the second subsequent frame transmission termination from B carries the acknowledgement (recall that the frame transmission in progress from B when A finishes its transmission cannot carry the ack for that transmission; recall also that propagation and processing delays are negligible. Thus q is the probability of n-1 frame terminations from A before the second frame termination from B. This can be rephrased as the probability that out of the next n frame terminations from either node, either n-1 or n come from node A. Since successive frame terminations are equally likely and independently from A or B, this probability is

$$q = \sum_{i=n-1}^{n} \frac{n!}{i!(n-i)!} 2^{-n} = (n+1)2^{-n}$$

2.24

If an isolated error in the feedback direction occurs, then the ack for a given packet is held up by one frame in the feedback direction (i.e., the number RN in the feedback frame following the feedback frame in error reacknowledges the old packet as well as any new packet that might have been received in the interim). Thus q is now the probability of n-1 frame terminations from A before 3 frame terminations from B (one for the frame in progress, one for the frame in error, and one for the frame actually carrying the ack; see the solution to problem 2.23). This is the probability that n-1 or more of the next n+1 frame terminations come from A; since each termination is from A or B independently and with equal probability,

$$q = \sum_{i=n-1}^{n} \left(\frac{(n+1)!}{i!(n+1-i)!}\right) 2^{-n-1} = [n+2+(n+1)n/2]2^{-n-1}$$

2.26

We view the system from the receiver and ask for the expected number of frames, γ, arriving at the receiver starting immediately after a frame containing a packet that is accepted and running until the next frame containing a packet that is accepted. By the assumptions of the problem, if the packet in a frame is accepted, then the next frame must contain the next packet in order (if not, the transmitter must have gone back to some earlier packet, which is impossible since that earlier packet was accepted earlier and by assumption was acked in time to avoid the go back).

Since the next frame after a packet acceptance must contain the awaited packet, that packet is accepted with probability 1-p. With probability p, on the other hand, that next frame contains an error. In this case, some number of frames, say j, follow this next frame before the awaited packet is again contained in a frame. This new frame might again contain an error, but the expected number of frames until the awaited packet is accepted, starting with this new frame, is again γ. Thus, given an error in the frame after a packet acceptance, and given j further frames before the awaited packet is repeated, the expected number of frames from one acceptance to the next is $1+j+\gamma$.

Note that j is the number of frames that the transmitter sends, after the above frame in error, up to and including the frame in transmission when feedback arrives concerning the frame in error. Thus the expected value of j is β. Combining the events of error and no error on the next frame after a packet acceptance, we have

$$\gamma = (1-p) + p(1+\beta+\gamma) = 1 + p(\beta+\gamma)$$

Solving for γ and for $\nu = 1/\gamma$,

$$\gamma = (1+\beta p)/(1-p) \qquad \nu = (1-p)/(1+\beta p)$$

2.30

a) The sequence below shows the stuffed bits underlined for easy readability:

0 1 1 0 1 1 1 1 1 0 0 0 1 1 1 1 1 0 1 0 1 0 1 1 1 1 1 0 1 1 1 1 1 0 0 1 1 1 1 0 1 0

b) Here the flags are shown underlined and the removed (destuffed) bits as x's:

0 1 1 1 1 1 1 0 1 1 1 1 1 x 1 1 0 0 1 1 1 1 1 x 0 1 1 1 1 1 x 1 1 1 1 1 x 1 1

0 0 0 1 1 1 1 1 1 0 1 0 1 1 1 1 1 x

2.31

The modified destuffing rule starts at the beginning of the string and destuffs bit by bit. A zero is removed from the string if the previous six bits in the already destuffed portion of the string have the value 01^5. For the given example, the destuffed string, with flags shown underlined and removed bits shown as x's, is as follows:

0 1 1 0 1 1 1 1 x 1 1 1 1 1 1 0 1 1 1 1 1 x 1 0 1 1 1 1 1 1 0

2.32

The hint shows that the data string $01^5 01x_1x_2...$ must have a zero stuffed after 01^5, thus appearing as $01^5 00x_1x_2....$ This stuffed pattern will be indistinguishable from the original string $01^5 00x_1x_2...$ unless stuffing is also used after 01^5 in the string $01^5 00x_1x_2....$ Thus stuffing must be used in this case. The general argument is then by induction. Assume that stuffing is necessary after 01^5 on all strings of the form $01^5 0^k x_1x_2....$ Then such a stuffed sequence is $01^5 0^{k+1} x_1x_2....$ It follows as before that stuffing is then necessary after 01^5 in the sequence $01^5 0^{k+1} x_1x_2....$ Thus stuffing is always necessary after 01^5.

2.33

The stuffed string is shown below with the stuffed bits underlined and a flag added at the end.

1 1 0 1 1 0 1 0 0 0 1 0 0 1 0 0 1 1 1 0 1 0 0 1 0 1

The destuffing rule is to decode (destuff) the string bit by bit starting at the beginning. A given 0 bit is then deleted from the string if the preceding three decoded bits are 010. The flag is detected when a 1 is preceded by the three decoded bits 010 and the most recently decoded bit was not deleted. The above is a general rule for detecting any type of flag sequence, rather than just 0101; for this special case, it is sufficient to look for the substring 0101 in the received string; the reason for the simplification is that if an insertion occurs within the flag, it has to occur by simply a repetition of the first flag bit.

2.34

Let γ be $\log_2 E\{K\} - j$. Since j is the integer part of $\log_2 E\{K\}$, we see that γ must lie between 0 and 1. Expressing $A = E\{K\}2^{-j} + j + 1$ in terms of γ and $E\{K\}$, we get

$$A = 2^{\gamma} + \log_2 E\{K\} - \gamma + 1$$

$$A - \log_2 E\{K\} = 2^{\gamma} - \gamma + 1$$

This function of γ is easily seen to be convex (i.e., it has a positive second derivative). It has the value 2 at $\gamma = 0$ and at $\gamma = 1$ and is less than 2 for $0 < \gamma < 1$. This establishes that

$$A \leq \log_2 E\{K\} + 2$$

Finding the minimum of $2^\gamma - \gamma + 1$ by differentiation, the minimum occurs at

$$\gamma = -\log_2(\ln 2)$$

The value of $2^\gamma - \gamma + 1$ at this minimizing point is $[\ln 2]^{-1} + \log_2(\ln 2) + 1 = 1.914...$, so

$$A \geq \log_2 E\{K\} + (\ln 2)^{-1} + \log_2(\ln 2) + 1$$

2.35

Stuffed bits are always 0's and always follow the pattern 01^5. The initial 0 in this pattern could be a bit in the unstuffed data string, or could itself be a stuffed bit. As in the analysis of subsection 2.5.2, we ignore the case where this initial 0 is a stuffed bit since it is almost negligible compared with the other case (also a well designed flag detector would not allow a stuffed bit as the first bit of a flag). If a stuffed bit (preceded by 01^5 in the data) is converted by noise into a 1, then it is taken as a flag if the next bit is 0 and is taken as an abort if the next bit is 1. Thus an error in a stuffed bit causes a flag to appear with probability 1/2 and the expected number of falsely detected flags due to errors in stuffed bits is $K2^{-7}$. If one is less crude in the approximations, one sees that there are only K-6 places in the data stream where a stuffed bit could be inserted following 01^5 in the data; thus a more refined answer is that the expected number of falsely detected flags due to errors in stuffed bits is $(K-6)2^{-7}$.

There are eight patterns of eight bits such that an error in one of the eight bits would turn the pattern into a flag. Two of these patterns, 01^7 and 1^70, cannot appear in stuffed data. Another two of the patterns, 01^500 and 001^50, can appear in stuffed data but must contain a stuffed bit (i.e. the 0 following 1^5). The first of these cases corresponds to the case in which an error in a stuffed bit causes a flag to appear, and we have already analyzed this. The second corresponds to a data string 001^5. Thus the substrings of data for which a single error in a data bit can cause a flag to appear are listed below; the position in which the error must appear is shown underlined:

$0\,\underline{0}\,1\,1\,1\,1\,1$
$0\,1\,\underline{0}\,1\,1\,1\,1\,0$
$0\,1\,1\,\underline{0}\,1\,1\,1\,0$
$0\,1\,1\,1\,\underline{0}\,1\,1\,0$
$0\,1\,1\,1\,1\,\underline{0}\,1\,0$

For any given bit position j in the K bit data string ($j \leq K-7$), the probability that one of these patterns starts on bit j is $2^{-7} + 4 \cdot 2^{-8} = 3 \cdot 2^{-7}$. Thus the probability of a false flag being detected because of an error on a data bit, starting on bit j of the data is $3p2^{-7}$. This is also the expected number of such flags, and summing over the bits of the data stream, the expected number is $(K-7)3p2^{-7}$. Approximating by replacing K-7 by K, and adding this to the expected number of false flags due to errors in stuffed bits, the overall probability of a false flag in a frame of length K is (1/32)Kp. If K-7 is not approximated by K, and if we recognize that the first pattern above can also appear starting at j=K-6, then the overall probability of a false flag is approximated more closely by (1/32)(K-6.5)p.

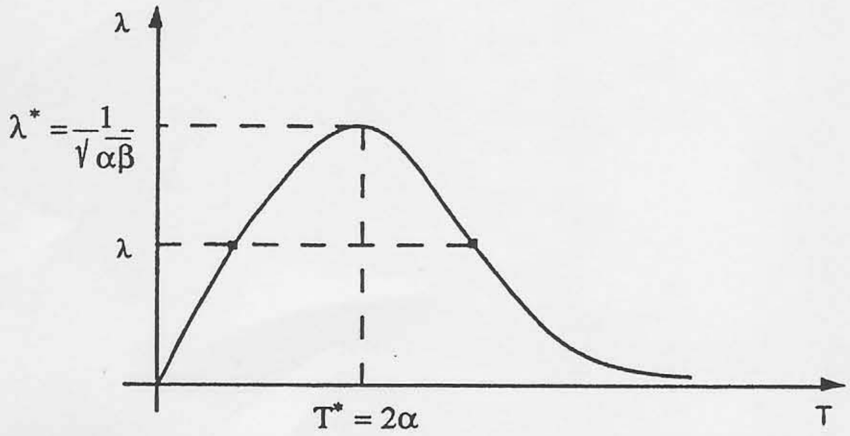
HW4 1/6

3.4

If λ is the throughput of the system, Little's theorem gives $N = \lambda T$, so from the relation $T= \alpha + \beta N^2$ we obtain $T = \alpha + \beta\lambda^2 T^2$ or

$$\lambda = \sqrt{\frac{T - \alpha}{\beta T^2}} \quad (1)$$

This relation betweeen λ ands T is plotted below.



The maximum value of λ is attained for the value T^* for which the derivative of $(T - \alpha)/\beta T^2$ is zero (or $1/(\beta T^2) - 2(T - \alpha)/(\beta T^3) = 0$). This yields $T^* = 2\alpha$ and from Eq. (1), the corresponding maximal throughput value

$$\lambda^* = \frac{1}{\sqrt{\alpha\beta}} \quad (2)$$

(b) When $\lambda < \lambda^*$, there are two corresponding values of T: a low value corresponding to an uncongested system where N is relatively low, and a high value corresponding to a congested system where N is relatively high. This assumes that the system reaches a steady-state. However, it can be argued that when the system is congested a small increase in the number of cars in the system due to statistical fluctuations will cause an increase in the time in the system, which will tend to decrease the rate of departure of cars from the system. This will cause a further increase in the number in the system and a further increase in the time in the system, etc. In other words, when we are operating on the right side of the curve of the figure, there is a tendency for *instability* in the system, whereby a steady-state is never reached: the system tends to drift towards a traffic jam where the car depature rate from the system tends towards zero and the time a car spends in the system tends towards infinity. Phenomena of this type are analyzed in the context of the Aloha multiaccess system in Chapter 4.

3.5

The expected time in question equals

$$E\{Time\} = (5 + E\{stay\ of\ 2nd\ student\})*P\{1st\ stays\ less\ or\ equal\ to\ 5\ minutes\} + (E\{stay\ of\ 1st \mid stay\ of\ 1st \geq 5\} + E\{stay\ of\ 2nd\})* P\{1st\ stays\ more\ than\ 5\ minutes\}.$$

We have E{stay of 2nd student} = 30, and, using the memoryless property of the exponential distribution,

$$E\{stay\ of\ 1st \mid stay\ of\ 1st \geq 5\} = 5 + E\{stay\ of\ 1st\} = 35.$$

Also

$$P\{1st\ student\ stays\ less\ or\ equal\ to\ 5\ minutes\} = 1 - e^{-5/30}$$
$$P\{1st\ student\ stays\ more\ than\ 5\ minutes\} = e^{-5/30}.$$

By substitution we obtain

$$E\{Time\} = (5 + 30)*(1 - e^{-5/30}) + (35 + 30)* e^{-5/30} = 35 + 30*e^{-5/30} = 60.394.$$

3.6

(a) The probability that the person will be the last to leave is 1/4 because the exponential distribution is memoryless, and all customers have identical service time distribution. In particular, at the instant the customer enters service, the remaining service time of each of the other three customers served has the same distribution as the service time of the customer.

(b) The average time in the bank is 1 (the average customer service time) plus the expected time for the first customer to finish service. The latter time is 1/4 since the departure process is statistically identical to that of a single server facility with 4 times larger service rate. More precisely we have

$$P\{no\ customer\ departs\ in\ the\ next\ t\ mins\} = P\{1st\ does\ not\ depart\ in\ next\ t\ mins\} * P\{2nd\ does\ not\ depart\ in\ next\ t\ mins\} * P\{3rd\ does\ not\ depart\ in\ next\ t\ mins\} * P\{4th\ does\ not\ depart\ in\ next\ t\ mins\} = (e^{-t})^4 = e^{-4t}.$$

Therefore

$$P\{the\ first\ departure\ occurs\ within\ the\ next\ t\ mins\} = 1 - e^{-4t},$$

and the expected time to the next depature is 1/4. So the answer is 5/4 minutes.

(c) The answer will not change because the situation at the instant when the customer begins service will be the same under the conditions for (a) and the conditions for (c).

3.10

(b) Let N_1, N_2 be the number of arrivals in two disjoint intervals of lengths τ_1 and τ_2. Then

$$P\{N_1+N_2 = n\} = \Sigma^n_{k=0}P\{N_1 = k, N_2 = n-k\} = \Sigma^n_{k=0}P\{N_1 = k\}P\{N_2 = n-k\}$$
$$= \Sigma^n_{k=0}e^{-\lambda\tau 1}[(\lambda\tau_1)^k/k!]e^{-\lambda\tau 2}[(\lambda\tau_2)^{(n-k)}/(n-k)!]$$
$$= e^{-\lambda(\tau 1 + \tau 2)}\Sigma^n_{k=0}[(\lambda\tau_1)^k(\lambda\tau_2)^{(n-k)}]/[k!(n-k)!]$$
$$= e^{-\lambda(\tau 1 + \tau 2)}[(\lambda\tau_1 + \lambda\tau_2)^n/n!]$$

(The identity

$$\Sigma^n_{k=0}[a^k b^{(n-k)}]/[k!(n-k)!] = (a + b)^n/n!$$

can be shown by induction.)

(c) The number of arrivals of the combined process in disjoint intervals is clearly independent, so we need to show that the number of arrivals in an interval is Poisson distributed, i.e.

$$P\{A_1(t + \tau) + \ldots + A_k(t + \tau) - A_1(t) - \ldots - A_k(t) = n\}$$
$$= e^{-(\lambda 1 + \ldots + \lambda k)\tau}[(\lambda_1 + \ldots + \lambda_k)\tau]^n/n!$$

For simplicity let k=2; a similar proof applies for k > 2. Then

$$P\{A_1(t + \tau) + A_2(t + \tau) - A_1(t) - A_2(t) = n\}$$
$$= \Sigma^n_{m=0}P\{A_1(t + \tau) - A_1(t) = m, A_2(t + \tau) - A_2(t) = n-m\}$$
$$= \Sigma^n_{m=0}P\{A_1(t + \tau) - A_1(t) = m\}P\{A_2(t + \tau) - A_2(t) = n-m\}$$

and the calculation continues as in part (b). Also

$$P\{1 \text{ arrival from } A_1 \text{ prior to } t \mid 1 \text{ occured}\}$$
$$= P\{1 \text{ arrival from } A_1, 0 \text{ from } A_2\}/P\{1 \text{ occured}\}$$
$$= (\lambda_1 t e^{-\lambda 1 t}e^{-\lambda 2 t})/(\lambda t e^{-\lambda t}) = \lambda_1/\lambda$$

(d) Let t be the time of arrival. We have

$$P\{t < s \mid 1 \text{ arrival occured}\} = P\{t < s, 1 \text{ arrival occured}\}/P\{1 \text{ arrival occured}\}$$
$$= P\{1 \text{ arrival occured in } [t_1, s), 0 \text{ arrivals occured in } [s, t_2]\}/P\{1 \text{ arrival occured}\}$$
$$= (\lambda(s - t_1)e^{-\lambda(s - t1)}e^{-\lambda(s - t2)})/ (\lambda(t_2 - t_1)e^{-\lambda(t2 - t1)}) = (s - t_1)/(t_2 - t_1)$$

This shows that the arrival time t is uniformly distributed in $[t_1, t_2]$.

3.11

(a) Let us call the two transmission lines 1 and 2, and let $N_1(t)$ and $N_2(t)$ denote the respective numbers of packet arrivals in the interval [0,t]. Let also $N(t) = N_1(t) + N_2(t)$. We calculate the joint probability $P\{N_1(t) = n, N_2(t) = m\}$. To do this we first condition on N(t) to obtain

$$P\{N_1(t) = n, N_2(t) = m\} = \Sigma_{k=0}^{\infty} P\{N_1(t) = n, N_2(t) = m \mid N(t) = k\}P\{N(t) = k\}.$$

Since

$$P\{N_1(t) = n, N_2(t) = m \mid N(t) = k\} = 0 \qquad \text{when} \quad k \neq n+m$$

we obtain

$$P\{N_1(t) = n, N_2(t) = m\} = P\{N_1(t) = n, N_2(t) = m \mid N(t) = n + m\}P\{N(t) = n + m\}$$
$$= P\{N_1(t) = n, N_2(t) = m \mid N(t) = n + m\}e^{-\lambda t}[(\lambda t)^{n+m}/(n + m)!]$$

However, given that n+m arrivals occurred, since each arrival has probability p of being a line 1 arrival and probability 1-p of being a line 2 arrival, it follows that the probability that n of them will be line 1 and m of them will be line 2 arrivals is the binomial probability

$$\binom{n+m}{n}p^n(1-p)^m$$

Thus

$$P\{N_1(t) = n, N_2(t) = m\} = \binom{n+m}{n}p^n(1-p)^m e^{-\lambda t}\frac{(\lambda t)^{n+m}}{(n+m)!}$$

$$= e^{-\lambda tp}\frac{(\lambda tp)^n}{n\,!} \; e^{-\lambda t(1-p)}\frac{(\lambda t(1-p))^m}{m!} \qquad (1)$$

Hence

$$P\{N_1(t) = n\} = \sum_{m=0}^{\infty} P\{N_1(t) = n, N_2(t) = m\}$$

$$= e^{-\lambda tp}\frac{(\lambda tp)^n}{(n)!}\sum_{m=0}^{\infty} e^{-\lambda t(1-p)}\frac{(\lambda t(1-p))^m}{m!}$$

$$= e^{-\lambda tp}\frac{(\lambda tp)^n}{(n)!}$$

That is, $\{N_1(t), t \geq 0\}$ is a Poisson process having rate λp. Similarly we argue that $\{N_2(t), t \geq 0\}$ is a Poisson process having rate $\lambda(1 - p)$. Finally from Eq. (1) it follows that the two processes are independent since the joint distribution factors into the marginal distributions.

5/6

(b) Let A, A_1, and A_2 be as in the hint. Let I be an interarrival interval of A_2 and consider the number of arrivals of A_1 that lie in I. The probability that this number is n is the probability of n successive arrivals of A_1 followed by an arrival of A_2, which is $\rho^n(1 - \rho)$. This is also the probability that a customer finds upon arrival n other customers waiting in an M/M/1 queue. The service time of each of these customers is exponentially distributed with parameter μ, just like the interarrival times of process A. Therefore the waiting time of the customer in the M/M/1 system has the same distribution as the interarrival time of process A_2. Since by part (a), the process A_2 is Poisson with rate $\mu - \lambda$, it follows that the waiting time of the customer in the M/M/1 system is exponentially distributed with parameter $\mu - \lambda$.

3.12

For any scalar s we have using also the independence of τ_1 and τ_2

$$P(\min\{\tau_1,\tau_2\} \geq s) = P(\tau_1 \geq s, \tau_2 \geq s) = P(\tau_1 \geq s)\, P(\tau_2 \geq s)$$

$$= e^{-\lambda_1 s} e^{-\lambda_2 s} = e^{-(\lambda_1 + \lambda_2)s}$$

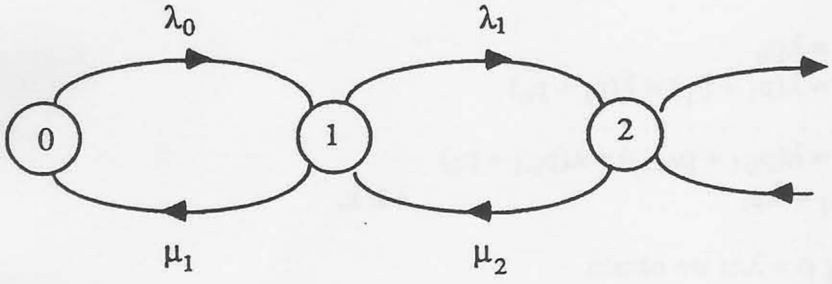
Therefore the distribution of $\min\{\tau_1,\tau_2\}$ is exponential with mean $1/(\lambda_1 + \lambda_2)$.

By viewing τ_1 and τ_2 as the arrival times of the first arrivals from two independent Poisson processes fwith rates λ_1 and λ_2, we see that the equation $P(\tau_1 < \tau_2) = \lambda_1/(\lambda_1 + \lambda_2)$ follows from Problem 3.10(c).

Consider the M/M/1 queue and the amount of time spent in a state k>0 between transition into the state and transition out of the state. This time is $\min\{\tau_1,\tau_2\}$, where τ_1 is the time between entry to the state k and the next customer arrival and τ_2 is the time between entry to the state k and the next service completion. Because of the memoryless property of the exponential distribution, τ_1 and τ_2 are exponentially distributed with means $1/\lambda$ and $1/\mu$, respectively. It follows using the fact shown above that the time between entry and exit from stae k is exponentially distributed with mean $1/(\lambda+\mu)$. The probability that the transition will be from k to k+1 is $\lambda/(\lambda+\mu)$ and that the transition will be from k to k-1 is $\mu/(\lambda+\mu)$. For state 0 the amount of time spent is exponentially distributed with mean $1/\lambda$ and the probability of a transition to state 1 is 1. Because of this it can be seen that M/M/1 queue can be described as a continuous Markov chain with the given properties.

6/6

3.16



The figure shows the Markov chain corresponding to the given system. The local balance equation for it can be written down as :

$$\rho_0 p_0 = p_1$$

$$\rho_1 p_1 = p_2$$

$$\dots \qquad \dots$$

$$\Rightarrow p_{n+1} = \rho_n p_n = \rho_{n-1}\rho_n p_{n-1} = \dots = (\rho_0 \rho_1 \cdots \rho_n) p_0$$

but,

$$\sum_{i=0}^{\infty} p_i = p_0(1 + \rho_0 + \rho_0 \rho_1 + \dots.) = 1$$

$$\Rightarrow p_0 = \left[1 + \sum_{k=0}^{\infty} \left(\rho_0 \cdots \rho_k \right) \right]^{-1}$$

3.32

Following the hint we write for the ith packet.

$$U_i = R_i + \sum_{j=1}^{N_i} X_{i-j}$$

where

U_i : Unfinished work at the time of arrival of the ith customer
R_i : Residual service time of the ith customer
N_i : Number found in queue by the ith customer
X_j : Service time of the jth customer

Hence

$$E\{U_i\} = E\{R_i\} + E\left\{\sum_{j=1}^{N_i} E\{X_{i-j} \mid N_i\}\right\}$$

Since X_{i-j} and N_i are independent

$$E\{U_i\} = E\{R_i\} + E\{X\}E\{N_i\}$$

and by taking limit as $i \to \infty$ we obtain $U = R + (1/\mu)N_Q = R + (\lambda/\mu)W = R + \rho W$, so

$$W = (U - R)/\rho.$$

Now the result follows by noting that both U and R are independent of the order of customer service (the unfinished work is independent of the order of customer service, and the steady state mean residual time is also independent of the customer service since the graphical argument of Fig. 3.16 does not depend on the order of customer service).

3.33

Consider the limited service gated system with zero packet length and reservation interval equal to a slot. We have

T_{TDM} = Waiting time in the gated system

For $E\{X^2\} = 0$, $E\{V\} = 1$, $\sigma_V^2 = 0$, $\rho = 0$ we have from the gated system formula (3.77)

Waiting time in the gated system $= (m + 2 - 2\lambda)/(2(1 - \lambda)) = m/(2(1 - \lambda)) + 1$

which is the formula for T_{TDM} given by Eq. (3.59) .

3.34

(a) The system utilization is ρ, so the fraction of time the system transmits data is ρ. Therefore the portion of time occupied by reservation intervals is $1 - \rho$.

(b) If

p: Fraction of time a reservation interval is followed by an empty data interval

and M(t) is the number of reservation intervals up to time t, then the number of packets transmitted up to time t is $\approx (1 - p)M(t)$. The time used for reservation intervals is $\approx M(t)E\{V\}$, and for data intervals $\approx (1 - p)M(t)E\{X\}$. Since the ratio of these times must be $(1 - \rho)/\rho$ we obtain

$$(1 - \rho)/\rho = (M(t)E\{V\})/((1 - p)M(t)E\{X\}) = E\{V\}/((1 - p)E\{X\})$$

or

$$1 - p = (\rho E\{V\})/((1 - \rho)E\{X\})$$

which using $\lambda = \rho/E\{X\}$, yields $p = (1 - \rho - \lambda E\{V\})/(1 - \rho)$

3.37

(a)

$\lambda = 1/60$ per second
$E(X) = 16.5$ seconds
$E(X^2) = 346.5$ seconds
$T = E(X) + \lambda E(X^2)/2(1-\lambda E(X))$
$= 16.5 + (346.5/60)/2(1- 16.5/60) = 20.48$ seconds

(b) Non-Preemptive Priority

In the following calculation, subscript 1 will imply the quantities for the priority 1 customers and 2 for priority 2 customers. Unsubscripted quantities will refer to the overall system.

$$\lambda = \frac{1}{60}, \ \lambda_1 = \frac{1}{300}, \ \lambda_2 = \frac{1}{75}$$

$$E(X) = 16.5, \ E(X_1) = 4.5, \ E(X_2) = 19.5$$

$$E(X^2) = 346.5$$

$$R = \frac{1}{2}\lambda E(X^2) = 2.8875$$

$$\rho_1 = \lambda_1 E(X_1) = 0.015$$

$$\rho_2 = \lambda_2 E(X_2) = 0.26$$

$$W_1 = \frac{R}{1-\rho_1} = 2.931$$

$$W_2 = \frac{R}{1-\rho_2} = 4.043$$

$$T_1 = 7.4315, \ T_2 = 23.543$$

$$T = \frac{\lambda_1 T_1 + \lambda_2 T_2}{\lambda} = 20.217$$

3/5

(c) Preemptive Queueing

The arrival rates and service rates for the two priorities are the same for preemptive system as the non-preemptive system solved above.

$$E(X_1^2) = 22.5,\ \ E(X_2^2) = 427.5$$

$$R_1 = \frac{1}{2}\lambda_1 E(X_1^2) = 0.0075$$

$$R_2 = R_1 + \frac{1}{2}\lambda_2 E(X_2^2) = 2.8575$$

$$T_1 = \frac{E(X_1)(1-\rho_1) + R_1}{1-\rho_1}$$

$$T_2 = \frac{E(X_2)(1-\rho_1-\rho_2) + R_2}{(1-\rho_1)(1-\rho_1-\rho_2)}$$

$$T = (\lambda_1 T_1 + \lambda_2 T_2)/\lambda = 19.94$$

3.40

(a) The algebraic verification using Eq. (3.79) listed below

$$W_k = R/(1 - \rho_1 - \ldots - \rho_{k-1})(1 - \rho_1 - \ldots - \rho_k)$$

is straightforward. In particular by induction we show that

$$\rho_1 W_1 + \cdots + \rho_k W_k = \frac{R(\rho_1 + \quad + \rho_k)}{1 - \rho_1 - \cdots - \rho_k}$$

The induction step is carried out by verifying the identity

$$\rho_1 W_1 + \cdots + \rho_k W_k + \rho_{k+1} W_{k+1} = \frac{R(\rho_1 + \cdots + \rho_k)}{1 - \rho_1 - \cdots - \rho_k} + \frac{\rho_{k+1} R}{(1 - \rho_1 - \cdots \rho_k)(1 - \rho_1 - \cdots - \rho_{k+1})}$$

The alternate argument suggested in the hint is straightforward.

(b) Cost

$$C = \sum_{k=1}^{n} c_k N_Q^k = \sum_{k=1}^{n} c_k \lambda_k W_k = \sum_{k=1}^{n} \left(\frac{c_k}{\overline{X}_k}\right) \rho_k W_k$$

We know that $W_1 \leq W_2 \leq \ldots.. \leq W_n$. Now exchange the priority of two neighboring classes i and j=i+1 and compare C with the new cost

$$C' = \sum_{k=1}^{n} \left(\frac{c_k}{\overline{X}_k}\right) \rho_k W'_k$$

In C' all the terms except k = i and j will be the same as in C because the interchange does not affect the waiting time for other priority class customers. Therefore

$$C'-C = \frac{c_j}{\overline{\overline{X}}_j}\rho_j W'_j + \frac{c_i}{\overline{\overline{X}}_i}\rho_i W'_i - \frac{c_i}{\overline{\overline{X}}_i}\rho_i W_i - \frac{c_j}{\overline{\overline{X}}_j}\rho_j W_j.$$

We know from part (a) that

$$\sum_{k=1}^{n} \rho_k W_k = \text{constant.}$$

Since W_k is unchanged for all k except k = i and j (= i+1) we have

$$\rho_i W_i + \rho_j W_j = \rho_i W'_i + \rho_j W'_j.$$

Denote

$$B = \rho_i W'_i - \rho_i W_i = \rho_j W_j - \rho_j W'_j$$

Clearly we have $B \geq 0$ since the average waiting time of customer class i will be increased if class i is given lower priority. Now let us assume that

$$\frac{c_i}{\overline{\overline{X}}_i} \leq \frac{c_j}{\overline{\overline{X}}_j}$$

Then

$$C'-C = \frac{c_i}{\overline{\overline{X}}_i}(\rho_i W'_i - \rho_i W_i) - \frac{c_j}{\overline{\overline{X}}_j}(\rho_j W_j - \rho_j W'_j) = B\left(\frac{c_i}{\overline{\overline{X}}_i} - \frac{c_j}{\overline{\overline{X}}_j}\right)$$

Therefore, only if $\frac{c_i}{\overline{\overline{X}}_i} < \frac{c_{i+1}}{\overline{\overline{X}}_{i+1}}$ can we reduce the cost by exchanging the priority order of i and i+1. Thus, if (1,2,3,...,n) is an optimal order we must have

$$\frac{c_1}{\overline{\overline{X}}_1} \geq \frac{c_2}{\overline{\overline{X}}_2} \geq \frac{c_3}{\overline{\overline{X}}_3} \geq \ldots \geq \frac{c_n}{\overline{\overline{X}}_n}$$

HW 6

1/2

3.30

From Little's Theorem (Example 1) we have that P{the system is busy} = $\lambda E\{X\}$. Therefore P{the system is empty} = $1 - \lambda E\{X\}$.

The length of an idle period is the interarrival time between two typical customer arrivals. Therefore it has an exponential distribution with parameter λ, and its average length is $1/\lambda$.

Let B be the average length of a busy period and let I be the average length of an idle period. By expressing the proportion of time the system is busy as $B/(I + B)$ and also as $\lambda E\{X\}$ we obtain

$$B = E\{X\}/(1 - \lambda E\{X\}).$$

From this the expression $1/(1 - \lambda E\{X\})$ for the average number of customers served in a busy period is evident.

3.46

We have

$$W = R/(1 - \rho)$$

where

$$R = \lim_{t\to\infty}\left\{\frac{1}{t}\sum_{i=1}^{M(t)}\frac{1}{2}X_i^2 + \frac{1}{t}\sum_{i=1}^{L(t)}\frac{V_i^2}{2}\right\}$$

where L(t) is the number of vacations (or busy periods) up to time t. The average length of an idle period is

$$I = \int_0^\infty p(v)\left[\int_0^V v\lambda e^{-\lambda\tau}d\tau + \int_V^\infty \tau\lambda e^{-\lambda\tau}d\tau\right]dv$$

and it can be seen that the steady-state time average number of vacations per unit time

$$\lim_{t\to\infty}\frac{L(t)}{t} = \frac{1-\rho}{I}$$

We have

$$\lim_{t\to\infty}\frac{1}{t}\sum_{i=1}^{L(t)}\frac{V_i^2}{2} = \lim_{t\to\infty}\frac{L(t)}{t}\frac{\sum_{i=1}^{L(t)}\frac{V_i^2}{2}}{L(t)} = \lim_{t\to\infty}\frac{L(t)}{t}\frac{\overline{V^2}}{2I} = \frac{\overline{V^2}(1-\rho)}{2I}$$

Therefore

$$R = \frac{\lambda\overline{X^2}}{2} + \frac{\overline{V^2}(1-\rho)}{2I}$$

and

$$W = \frac{\lambda\overline{X^2}}{2(1-\rho)} + \frac{\overline{V^2}}{2I}$$

3.49

(a) Since the arrivals are Poisson with rate λ, the mean time until the next arrival starting from any given time (such as the time when the system becomes empty) is $1/\lambda$. The time average fraction of busy time is $\lambda E[X]$. This can be seen by Little's theorem applied to the service facility (the time average number of customers in the server is just the time average fraction of busy time), or it can be seen by letting $\sum_{i=1}^{n} X_i$ represent the time the server is busy with the first n customers, dividing by the arrival time of the n^{th} customer, and going to the limit.

Let E[B] be the mean duration of a busy period and $E[I] = 1/\lambda$ be the mean duration of an idle period. The time average fraction of busy time must be E[B]/(E[B]+E[I]). Thus

$$\lambda E[X] = E[B]/(E[B]+1/\lambda)\ ; \qquad E[B] = \frac{E[X]}{1 - \lambda E[X]}$$

This is the same as for the FCFS M/G/1 system (Problem 3.30).

(b) If a second customer arrives while the first customer in a busy period is being served, that customer (and all subsequent customers that arrive while the second customer is in the system) are served before the first customer resumes service. The same thing happens for any subsequent customer that arrives while the first customer is actually in service. Thus when the first customer leaves, the system is empty. One can view the queue here as a stack, and the first customer is at the bottom of the stack. It follows that E[B] is the expected system time given a customer arriving to an empty system.

The customers already in the system when a given customer arrives receive no service until the given customer departs. Thus the system time of the given customer depends only on its own service time and the new customers that arrive while the given customer is in the system. Because of the memoryless property of the Poisson arrivals and the independence of service times, the system time of the given customer is independent of the number of customers (and their remaining service times) in the system when the given customer arrives. Since the expected system time of a given customer is independent of the number of customers it sees upon arrival in the system, the expected time is equal to the expected system time when the given customer sees an empty system; this is E[B] as shown above.

(c) Given that a customer requires 2 units of service time, look first at the expected system time until 1 unit of service is completed. This is the same as the expected system time of a customer requiring one unit of service (i.e., it is one unit of time plus the service time of all customers who arrive during that unit and during the service of other such customers). When one unit of service is completed for the given customer, the given customer is in service with one unit of service still required, which is the same as if a new customer arrived requiring one unit of service. Thus the given customer requiring 2 units of service has an expected system time of 2C. Extending the argument to a customer requiring n units of service, the expected system time is nC. Doing the argument backwards for a customer requiring 1/n of service, the expected system time is C/n. We thus conclude that E[system time | X=x] = Cx.

(d) We have

$$E[B] = \int_0^\infty Cx\, dF(x) = CE[X]; \quad C = \frac{1}{1-\lambda E[X]}$$

HW#7

3.50

(a) Since $\{p_j\}$ is the stationary distribution, we have for all $j \in S$

$$p_j\left(\sum_{i \in \bar{S}} q_{ji} + \sum_{i \notin \bar{S}} q_{ji}\right) = \sum_{i \in \bar{S}} p_i q_{ij} + \sum_{i \notin \bar{S}} p_i q_{ij}$$

Using the given relation, we obtain for all $j \in \bar{S}$

$$p_j \sum_{i \in \bar{S}} q_{ji} = \sum_{i \in \bar{S}} p_i q_{ij}$$

Dividing by $\sum_{i \in \bar{S}} p_i$, it follows that

$$\bar{p}_j \sum_{i \in \bar{S}} q_{ji} = \sum_{i \in \bar{S}} \bar{p}_i q_{ij}$$

for all $j \in \bar{S}$, showing that $\{\bar{p}_j\}$ is the stationary distribution of the truncated chain.

(b) If the original chain is time reversible, we have $p_j q_{ji} = p_i q_{ij}$ for all i and j, so the condition of part (a) holds. Therefore, we have $\bar{p}_j q_{ji} = \bar{p}_i q_{ij}$ for all states i and j of the truncated chain.

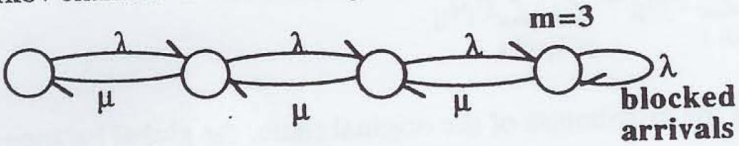
(c) The finite capacity system is a truncation of the two independent M/M/1 queues system, which is time reversible. Therefore, by part (b), the truncated chain is also time reversible. The formula for the steady state probabilities is a special case of Eq. (3.39) of Section 3.4.

3.52

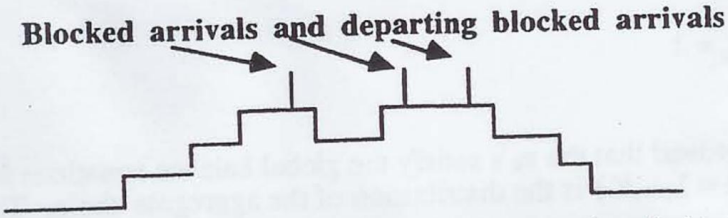
Consider a customer arriving at time t_1 and departing at time t_2. In reversed system terms, the arrival process is independent Poisson, so the arrival process to the left of t_2 is independent of the times spent in the system of customers that arrived at or to the right of t_2. In particular, $t_2 - t_1$ is independent of the (reversed system) arrival process to the left of t_2. In forward system terms, this means that $t_2 - t_1$ is independent of the departure process to the left of t_2.

3.54

The Markov chain for an M/M/1/m system is



Since this is a birth/death process, the chain is reversible. If we include the arrivals that are blocked from the system, then the arrival process is Poisson (by definition of M/M/1/m). If we include the blocked arrivals also as departures, then the departure process is also Poisson (by reversibility).



If we omit the blocked arrivals from consideration, the admitted arrival process has rate $\lambda(1-p_m)$, but this process is definitely not Poisson. One could, if one were truly masochistic, calculate things like the interarrival density, but the only sensible way to characterize the process is to characterize it jointly with the state process, in which case it is simply the process of arrivals during intervals when the state is less than m. The departure process, omitting the departures of blocked arrivals, is the same as the process of admitted arrivals, as can be seen by reversibility.

3.57

The session numbers and their rates are shown below:

Session	Session number p	Session rate x_p
ACE	1	100/60 = 5/3
ADE	2	200/60 = 10/3
BCEF	3	500/60 = 25/3
BDEF	4	600/60 = 30/3

The link numbers and the total link rates calculated as the sum of the rates of the sessions crossing the links are shown below:

Link	Total link rate
AC	$x_1 = 5/3$
CE	$x_1 + x_3 = 30/3$
AD	$x_2 = 10/3$
BD	$x_4 = 10$
DE	$x_2 + x_4 = 40/3$
BC	$x_3 = 25/3$
EF	$x_3 + x_4 = 55/3$

For each link (i,j) the service rate is

$$\mu_{ij} = 50000/1000 = 50 \text{ packets/sec,}$$

and the propagation delay is $D_{ij} = 2 \times 10^{-3}$ secs. The total arrival rate to the system is

$$\gamma = \Sigma_i x_i = 5/3 + 10/3 + 25/3 + 30/3 = 70/3$$

The average number on each link (i, j) (based on the Kleinrock approximation formula) is:

$$N_{ij} = \frac{\lambda_{ij}}{\mu_{ij} - \lambda_{ij}} + \lambda_{ij} D_{ij}$$

From this we obtain:

Link	Average Number of Packets on the Link
AC	(5/3)/(150/3 - 5/3) + (5/3)(2/1000) = 5/145 + 1/300
CE	1/4 + 1/50
AD	1/14 + 1/150
BD	1/4 + 1/50

DE	4/11 + 2/75
BC	1/5 + 1/60
EF	11/19 + 11/300

The average total number in the system is $N = \Sigma_{(i,j)} N_{ij} \equiv 1.84$ packet. The average delay over all sessions is $T = N/\gamma = 1.84 \times (3/70) = 0.0789$ secs. The average delay of the packets of an individual session are obtained from the formula

$$T_p = \sum_{(i,j) \text{ on } p} \left[\frac{\lambda_{ij}}{\mu_{ij}(\mu_{ij} - \lambda_{ij})} + \frac{1}{\mu_{ij}} + D_{ij} \right]$$

For the given sessions we obtain applying this formula

Session p	Average Delay T_p
1	0.050
2	0.053
3	0.087
4	0.090

HW#8

3.59

If we insert a very fast M/M/1 queue ($\mu \to \infty$) between a pair of queues, then the probability distribution for the packets in the rest of the queues is not affected. If we condition on a single customer being in the fast queue, since this customer will remain in this queue for $1/\mu$ ($\to 0$) time on the average, it is equivalent to conditioning on a customer moving from one queue to the other in the original system.

If $P(n_1,...,n_k)$ is the stationary distribution of the original system of k queues and $P'(n_1,...,n_k,n_{k+1})$ *is the corresponding probability distribution after the insertion of the fast* queue k+1, then

$$P(n_1,...,n_k \mid \text{arrival}) = P'(n_1,...,n_k,\ n_{k+1} = 1 \mid n_{k+1} = 1),$$

which by independence of $n_1, ..., n_k, n_{k+1}$, is equal to $P(n_1,...,n_k)$.

3.61

We have $\sum_{i=0}^{m} p_i = 1$.

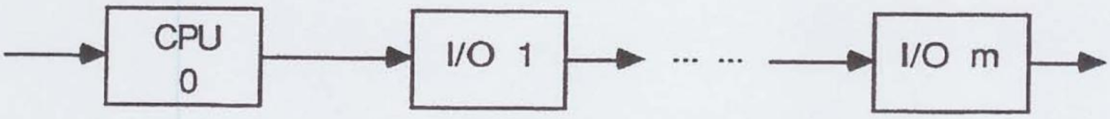
The arrival rate at the CPU is λ/p_0 and the arrival rate at the i^{th} I/O port is $\lambda p_i/p_0$. By Jackson's Theorem, we have

$$P(n_0, n_1, ... ,n_m) = \prod_{i=0}^{m} \rho_i^{n_i} (1 - \rho_i)$$

$$\text{where } \rho = \frac{\lambda}{\mu_0 p_0}$$

$$\text{and } \rho_i = \frac{\lambda p_i}{\mu_i p_0} \qquad \text{for } i > 0$$

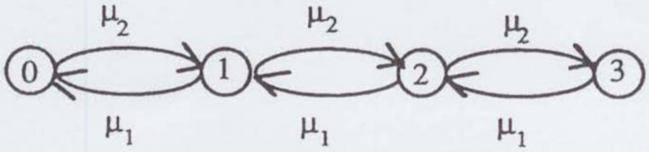
The equivalent tandem system is as follows:



The arrival rate is λ. The service rate for queue 0 is $\mu_0 p_0$ and for queue i (i > 0) is $\mu_i p_0/p_i$.

3.64

(a) The state is determined by the number of customers at node 1 (one could use node 2 just as easily). When there are customers at node 1 (which is the case for states 1, 2, and 3), the departure rate from node 1 is μ_1; each such departure causes the state to decrease as shown below. When there are customers in node 2 (which is the case for states 0, 1, and 2), the departure rate from node 2 is μ_2; each such departure causes the state to increase.



(b) Letting p_i be the steady state probability of state i, we have $p_i = p_{i-1}\,\rho$, where $\rho = \mu_2/\mu_1$. Thus $p_i = p_0\rho^i$. Solving for p_0,

$$p_0 = [1 + \rho + \rho^2 + \rho^3]^{-1}, \quad p_i = p_0\,\rho^i; \quad i=1,2,3.$$

(c) Customers leave node 1 at rate μ_1 for all states other than state 0. Thus the time average rate r at which customers leave node 1 is $\mu_1(1-P_0)$, which is

$$r = \frac{\rho+\rho^2+\rho^3}{1+\rho+\rho^2+\rho^3}\,\mu_1$$

(d) Since there are three customers in the system, each customer cycles at one third the rate at which departures occur from node 1. Thus a customer cycles at rate r/3.

(e) The Markov process is a birth-death process and thus reversible. What appears as a departure from node i in the forward process appears as an arrival to node i in the backward process. If we order the customers 1, 2, and 3 in the order in which they depart a node, and note that this order never changes (because of the FCFS service at each node), then we see that in the backward process, the customers keep their identity, but the order is reversed with backward departures from node i in the order 3, 2, 1, 3, 2, 1, ...

HW#9

4.1

a) State n can only be reached from states 0 to n+1, so, using Eq. (3A.1),

$$p_n = \sum_{i=0}^{n+1} p_i P_{in} \; ; \qquad 0 \le n < m$$

$$\sum_{n=0}^{m} p_n = 1$$

b) Solving for the final term, p_{n+1}, in the first sum above,

$$p_{n+1} = \frac{p_n(1-P_{nn}) - \sum_{i=0}^{n-1} p_i P_{in}}{P_{n+1,n}}$$

c) $\quad p_1 = p_0(1-P_{00})/P_{10}$

$$p_2 = \frac{p_1(1-P_{11}) - p_0 P_{02}}{P_{21}} = p_0\left[\frac{(1-P_{00})(1-P_{11})}{P_{10}P_{21}} - \frac{P_{02}}{P_{21}}\right]$$

d) Combining the above equations with $p_0+p_1+p_2 = 1$, we get

$$p_0 = \frac{P_{10}P_{21}}{P_{10}P_{21}+(1-P_{00})P_{21}+(1-P_{00})(1-P_{11})-P_{02}P_{10}}$$

4.4

a) Let $E\{n\}$ be the expected number of backlogged nodes, averaged over time. Since $m-E\{n\}$ is the expected number of nodes that can accept packets, and q_a is the probability that each receives a packet in a slot, the expected number of accepted arrivals per slot is

$$E\{N_a\} = q_a(m-E\{n\})$$

b) Since a limited number (i.e., m) arrivals can be in the system at any time, the time average accepted arrival rate must equal the time average departure rate, which is the time average success rate, $E\{P_{succ}\}$. Thus

$$E\{P_{succ}\} = E\{N_a\} = q_a(m-E\{n\})$$

c) The expected number of packets in the system, $E\{N_{sys}\}$ immediately after the beginning of a slot is the expected backlog, $E\{n\}$, plus the expected number of arrivals accepted during the previous slot, $E\{N_a\}$. Thus,

$$E\{N_{sys}\} = E\{n\} + E\{N_a\} = E\{n\}(1-q_a) + q_a m$$

d) From Little's theorem, the expected delay T is $E\{N_{sys}\}$ divided by the accepted arrival rate $E\{N_a\}\}$ Note that we are only counting the delay of the packets accepted into the system and note also that we are regarding accepted arrivals as arriving discretely at the slot boundaries.

$$T = E\{N_{sys}\}/E\{N_a\} = 1 + E\{n\}/[q_a(m-E\{n\})]$$

e) The above equations express the relevant quantities in terms of $E\{n\}$ and make clear that $E\{N_a\}$ and $E\{P_{succ}\}$ decrease and $E\{N_{sys}\}$ and T increase as n is decreased. Thus it makes no difference which of these quantities is optimized; improving one improves the others.

4.6

a) Substituting Eqs. (4.1) and (4.2) into (4.5),

$$P_{succ} = (m-n)q_a(1-q_a)^{m-n-1}(1-q_r)^n + nq_r(1-q_a)^{m-n}(1-q_r)^{n-1}$$

Differentiating this with respect to q_r (for $n > 1$) and consolidating terms, we get

$$\frac{\partial P_{succ}}{\partial q_r} = n(1-q_a)^{m-n}(1-q_r)^{n-1}\left[\frac{1}{1-q_r} - \frac{q_a(m-n)}{1-q_a} - \frac{q_r n}{1-q_r}\right]$$

The quantity inside brackets is decreasing in q_r; it is positive for $q_r = 0$ and negative as q_r approaches 1. Thus there is a point at which this quantity is 0 and that point maximizes P_{succ}.

b) If we set q_r equal to q_a in the bracketed quantity above, it becomes $(1-q_a m)/(1-q_r)$. This is positive under the assumption that $q_a < 1/m$. Thus, since the quantity in brackets is decreasing in q_r, it is zero for $q_r > q_a$.

c)

$$\frac{dP_{succ}}{dq_a} = \frac{\partial P_{succ}}{\partial q_a} + \frac{dq_r(q_a)}{dq_a}\frac{\partial P_{succ}}{\partial q_r} = \frac{\partial P_{succ}}{\partial q_a}$$

The above relation follows because $\partial P_{succ}/\partial q_r = 0$ at $q_r(q_a)$. We then have

$$\frac{\partial P_{succ}}{\partial q_a} = n(1-q_a)^{m-n-1}(1-q_r)^{n}\left[\frac{1}{1-q_a} - \frac{q_a(m-n)}{1-q_a} - \frac{q_r n}{1-q_r}\right]$$

Note that the bracketed term here differs from the bracketed term in part a) only in the first term. Since the bracketed term in part a) is 0 at $q_r(q_a)$ and $q_r(q_a) > q_a$, it follows that the bracketed term here is negative. Thus the total derivative of P_{succ} with respect to q_a is negative.

d) If arrivals are immediately regarded as backlogged, then an unbacklogged node generates a transmission with probability $q_a q_r$. Thus the probability of success is modified by replacing q_a with $q_a q_r$. This reduces the value of q_a in P_{succ} and therefore, from part c), increases the value of P_{succ} at the optimum choice of q_r.

4.8

a) Let X be the time in slots from the beginning of a backlogged slot until the completion of the first success at a given node. Let $q = q_r p$ and note that q is the probability that the node will be successful at any given slot given that it is still backlogged. Thus

$$P\{X=i\} = q(1-q)^{i-1}\ ;\ i \geq 1$$

$$E\{X\} = \sum_{i=1}^{\infty} iq(1-q)^{i-1} = \frac{1}{q}$$

The above summation uses the identity

$$\sum_{i=1}^{\infty} iz^{i-1} = \sum_{i=1}^{\infty} \frac{dz^i}{dz} = \frac{d\sum z^i}{dz} = \frac{d[z/(1-z)]}{dz} = \frac{1}{(1-z)^2}$$

Taking q = 1-z gives the desired result. A similar identity needed for the second moment is

$$\sum_{i=1}^{\infty} i^2 z^{i-1} = \sum_{i=1}^{\infty} \frac{d^2 z^{i+1}}{dz^2} - \sum_{i=1}^{\infty} \frac{dz^i}{dz} = \frac{d^2[z^2/(1-z)]}{dz^2} - \frac{1}{(1-z)^2} = \frac{1+z}{(1-z)^3}$$

Using this identity with q=1-z, we have

$$E\{X^2\} = \sum_{i=1}^{\infty} i^2 q(1-q)^{i-1} = \frac{2-q}{q^2} = \frac{2-pq_r}{(pq_r)^2}$$

b) For an individual node, we have an M/G/1 queue with vacations. The vacations are deterministic with a duration of 1 slot, and the service time has the first and second moments found in part a). Thus, using Eq. (3.55) for the queueing delay and adding an extra service time to get the system delay,

$$T = \frac{(\lambda/m)E\{X^2\}}{2(1-\rho)} + \frac{1}{2} + \frac{1}{q} = \frac{\lambda(2-\rho)}{2q^2(1-\rho)m} + \frac{1}{2} + \frac{1}{q}$$

Since the arrival rate is λ/m and the service rate is q, we have $\rho = \lambda/(mq)$. Substituting this into the above expression for T and simplifying,

$$T = \frac{1}{q_r p(1-\rho)} + \frac{1-2\rho}{2(1-\rho)}$$

c) For p=1 and q_r=1/m, we have $\rho = \lambda$, so that

$$T = \frac{m}{1-\lambda} + \frac{1-2\lambda}{2(1-\lambda)}$$

In the limit of large m, this is twice the delay of TDM as given in Eq. (3.59).

4.9

a) Let ν be the mean number of packets in the system. Given n packets in the system, with each packet independently transmitted in a slot with probability ν^{-1}, the probability of an idle slot, $P\{I|n\}$ is $(1-\nu^{-1})^n$. The joint probability of an idle slot and n packets in the system is then

$$P\{n,I\} = P\{n\}P\{I|n\} = \frac{\exp(-\nu)\nu^n}{n!}(1-\nu^{-1})^n$$

$$P\{I\} = \sum_{n=0}^{\infty} P\{n,I\} = \sum_{n=0}^{\infty} \frac{\exp(-\nu)\,(\nu-1)^n}{n!} = \frac{1}{e}$$

b) Using the results above, we can find $P\{n|I\}$

$$P\{n|I\} = \frac{P\{n,I\}}{P\{I\}} = \frac{\exp(-\nu+1)\,(\nu-1)^n}{n!}$$

Thus, this probability is Poisson with mean ν-1.

c) We can find the joint probability of success and n in the system similarly

$$P\{n,S\} = P\{n\}P\{S|n\} = \frac{\exp(-\nu)\,\nu^n}{n!}\,n(1-\nu^{-1})^{n-1}\nu^{-1} = \frac{\exp(-\nu)\,(\nu-1)^{n-1}}{(n-1)!}$$

$$P\{S\} = \sum_{n=0}^{\infty} \frac{\exp(-\nu)\,(\nu-1)^{n-1}}{(n-1)!} = \frac{1}{e}$$

d) From this, the probability that there were n packets in the system given a success is

$$P\{n|S\} = \frac{P\{n,S\}}{P\{S\}} = \frac{\exp(-\nu+1)\,(\nu-1)^{n-1}}{(n-1)!}$$

Note that n-1 is the number of remaining packets in the system with the successful packet removed, and it is seen from above that this remaining number is Poisson with mean ν-1.

4.10

a) All nodes are initially in mode 2, so when the first success occurs, the successful node moves to mode 1. While that node is in mode 1, it transmits in every slot, preventing any other node from entering mode 1. When that node eventually transmits all its packets and moves back to mode 2, we return to the initial situation of all nodes in mode 2. Thus at most one node at a time can be in mode 1.

b) The probability of successful transmission, p_1, is the probability that no other node is transmitting. Thus $p_1 = (1-q_r)^{m-1}$. The first and second moment of the time between successful transmissions is the same computation as in problem 4.8a. We have

$$\overline{X} = \frac{1}{p_1} \qquad \overline{X^2} = \frac{2-p_1}{p_1^2}$$

c) The probability of some successful dummy transmission in a given slot when all nodes are in mode 2 is $p_2 = mq_r(1-q_r)^{m-1}$. The first two moments of the time to such a success is the same problem as above, with p_2 in place of p_1. Thus

$$\overline{v} = \frac{1}{p_2} \qquad \overline{v^2} = \frac{2-p_2}{p_2^2}$$

d) The system is the same as the exhaustive multiuser system of subsection 3.5.2 except for the random choice of a new node to be serviced at the end of each reservation interval. Thus for the ith packet arrival to the system as a whole, the expected queueing delay before the given packet first attempts transmission is

$$E\{W_i\} = E\{R_i\} + E\{N_i\}\overline{X} + E\{Y_i\}$$

where R_i is the residual time to completion of the current packet service or reservation interval and Y_i is the duration of all the whole reservation intervals during which packet i must wait before its node enters mode 1. Since the order of serving packets is independent of their service time, $E\{N_i\} = \lambda E\{W\}$ in the limit as i approaches infinity. Also, since the length of each reservation interval is independent of the number of whole reservation intervals that the packet must wait, $E\{Y_i\}$ is the expected number of whole reservation intervals times the expected length of each. Thus

$$W = \frac{R + E\{S\}\overline{v}}{1-\rho}, \quad \rho = \lambda\overline{X}$$

e) As in Eq. (3.64),

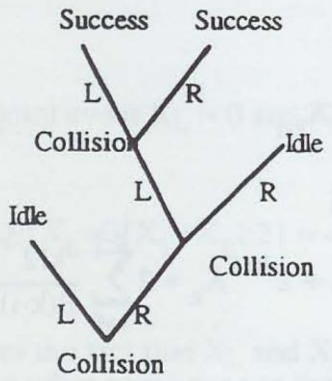
$$R = \frac{\lambda\overline{X^2}}{2} + \frac{(1-\rho)\overline{v^2}}{2\overline{v}}$$

Finally, the number of whole reservation intervals that the packet must wait is zero with probability 1/m, one with probability (1-1/m)/m, and in general i with probability $(1-1/m)^i/m$. Thus $E\{S\} = m-1$. Substituting these results and those of parts b) and c) into the above expression for W, we get the desired expression.

HW#10

4.13

a) The tree and the corresponding operations for each slot are shown below



Slot	Transmit Set	Waiting Sets	Feedback
1	S	-	e
2	L	R	0
3	R	-	e
4	RL	RR	e
5	RLL	RLR,RR	1
6	RLR	RR	1
7	RR	-	0

b) The second collision (i.e., that on slot 3) would have been avoided by the first improvement to the tree algorithm.

c) e,0,e,1,1; the final set, RR, would have been incorporated into the next collision resolution period in the second improvement.

4.15

a) The probability of i packets joining the left subset, given k packets in the original set, is given by the binomial distribution

$$\frac{k!\, 2^{-k}}{i!(k-i)!}$$

b) Assuming $k \geq 2$, the CRP starts with an initial collision that takes one slot. Given that i packets go into the left subset, A_i is the expected number of additional slots required to transmit the left subset and A_{k-i} is the expected number on the right. Taking the expectation over the number i of packets in the left subset, we get the desired result,

$$A_k = 1 + \sum_{i=0}^{k} \frac{k!2^{-k}}{i!(k-i)!}(A_i + A_{k-i})$$

c) Note that

$$\sum_{i=0}^{k} \frac{k!2^{-k}}{i!(k-i)!} A_i = \sum_{i=0}^{k} \frac{k!2^{-k}}{i!(k-i)!} A_{k-i}$$

Thus

$$A_k = 1 + 2\sum_{i=0}^{k} \frac{k!2^{-k}}{i!(k-i)!} A_i = 1 + 2^{-k+1} A_k + 2\sum_{i=0}^{k-1} \frac{k!2^{-k}}{i!(k-i)!} A_i$$

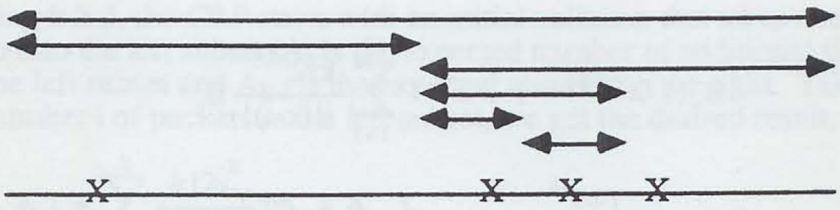
Taking the A_k term to the left side of the equation, we have

$$c_{ik} = \frac{k!2^{-k+1}}{i!(k-i)!(1-2^{-k+1})};\ i<k; \quad c_{kk} = \frac{1}{1-2^{-k+1}}$$

Evaluating this numerically, $A_2 = 5$ and $A_3 = 23/3$.

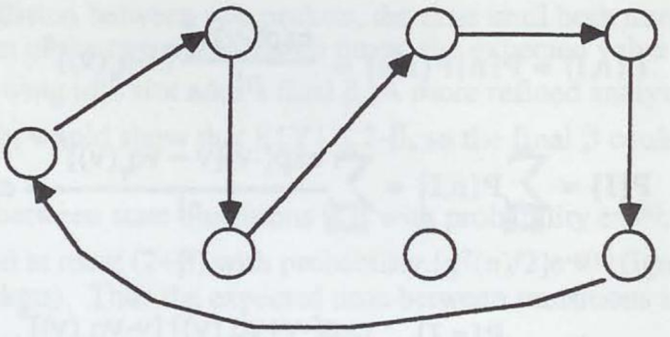
4.18

a) and b)



left boundary	interval size	right boundary	feedback at end	rule for next
T_k	α_0	$T_k+\alpha_0$	e	4.15
T_k	$\alpha_0/2$	$T_k+\alpha_0/2$	1	4.16
$T_k+\alpha_0/2$	$\alpha_0/2$	$T_k+\alpha_0$	e	4.15
$T_k+\alpha_0/2$	$\alpha_0/4$	$T_k+3\alpha_0/4$	e	4.15
$T_k+\alpha_0/2$	$\alpha_0/8$	$T_k+5\alpha_0/8$	1	4.16
$T_k+5\alpha_0/8$	$\alpha_0/8$	$T_k+3\alpha_0/4$	1	4.18

c) The diagram below shows the path through the Markov chain for the given sequence of events:



4.20

Let n_k (or n, suppressing the subscript for slot time) be the backlog at slot k and assume n is Poisson with known mean ν. Each of the n packets is independently transmitted in slot k with probability $q_r(\nu)$, so the probability that the kth slot is idle, given n, is $P\{I|n\} = [1-q_r(\nu)]^n$. Thus the joint probability of n backlogged packets and an idle slot is

$$P\{n,I\} = P\{n\}P\{I|n\} = \frac{\exp(-\nu)\nu^n}{n!}[1-q_r(\nu)]^n$$

$$P\{I\} = \sum_{n=0}^{\infty} P\{n,I\} = \sum_{n=0}^{\infty} \frac{\exp(-\nu)[\nu - \nu q_r(\nu)]^n}{n!} = \exp[-\nu q_r(\nu)]$$

$$P\{n|I\} = \frac{P\{n,I\}}{P\{I\}} = \frac{\exp[-\nu+\nu q_r(\nu)]\,[\nu-\nu q_r(\nu)]^n}{n!}$$

Thus, this probability is Poisson with mean $\nu-\nu q_r(\nu)$. Next consider a success

$$P\{n,S\} = P\{n\}P\{S|n\} = \frac{\exp(-\nu)\,\nu^n}{n!}n[1-q_r(\nu)]^{n-1}q_r(\nu)$$

$$= \frac{\exp(-\nu)\,[\nu-\nu q_r(\nu)]^{n-1}\nu q_r(\nu)}{(n-1)!}$$

$$P\{S\} = \sum_{n=1}^{\infty} P\{n,S\} = \nu q_r(\nu)\exp(-\nu)\sum_{n=1}^{\infty}\frac{[\nu-\nu q_r(\nu)]^{n-1}}{(n-1)!} = \nu q_r(\nu)\exp[-\nu q_r(\nu)]$$

$$P\{n|S\} = \frac{\exp[-\nu+\nu q_r(\nu)]\,[\nu-\nu q_r(\nu)]^{n-1}}{(n-1)!}$$

This says that the aposteriori distribution of n-1, given S, is Poisson with mean $\nu-\nu q_r(\nu)$.

4.21

a) Since X_1 and X_2 are non-negative random variables, $\max(X_1, X_2) \le X_1+X_2$ for all sample values. Taking expectations,

$$\overline{Y} \le 2\overline{X} = 2$$

Suppose X takes values β with probability $1-\varepsilon$ and $k\beta$ with probability ε. Since

$$\overline{X} = \beta(1-\varepsilon) + k\beta\varepsilon = 1, \text{ we have } \varepsilon = \frac{1-\beta}{\beta(k-1)}$$

Y takes on the value β with probability $(1-\varepsilon)^2$ and the value $k\beta$ with probability $2\varepsilon-\varepsilon^2$, so

$$\overline{Y} = \beta[1+(k-1)(2\varepsilon-\varepsilon^2)] = \beta + (1-\beta)(2-\varepsilon)$$

As k gets large, ε gets small and the final ε in the above expression is negligible. Thus, for small β, $E\{Y\} \approx 2$.

b) With a collision between two packets, the time until both transmissions are finished is the maximum of the two transmission times; the expected value of this is at most 2 from a), and the following idle slot adds a final β (A more refined analysis, using β as the minimum packet length, would show that $E\{Y\} \le 2-\beta$, so the final β could be omitted).

c) The time between state transitions is β with probability $e^{-g(n)}$, $(1+\beta)$ with probability $g(n)e^{-g(n)}$, and at most $(2+\beta)$ with probability $[g^2(n)/2]e^{-g(n)}$ (ignoring collisions of more than two packets). Thus the expected time between transitions is at most

$$\beta e^{-g(n)} + (1+\beta)g(n)e^{-g(n)} + (1+\beta/2)g^2(n)e^{-g(n)}$$

d) The success probability in state n is $g(n)e^{-g(n)}$, so the expected number of departures per unit time is the ratio of this to expected time between transitions (this can be justified rigorously by renewal theory). Thus the expected number of departures per unit time is at least

$$\frac{g(n)e^{-g(n)}}{\beta e^{-g(n)} + (1+\beta)g(n)e^{-g(n)} + (1+\beta/2)g^2(n)e^{-g(n)}} = \frac{g(n)}{\beta+(1+\beta)g(n)+(1+\beta/2)g^2(n)}$$

e) Taking the derivative of this with respect to g(n), we find a maximum where $g^2(n) = \beta/(1+\beta/2)$. Thus for small β, g(n) is approximately the square root of β. Substituting this back into the expression in d), the maximum throughput (i.e., departures per unit time), is approximately $1-2\sqrt{\beta}$.