

[Exercises 2] Samples

2.6 Consider the system

$$\begin{cases} \dot{x}_1 = -x_1 + ax_2 - bx_1x_2 + x_2^2 \\ \dot{x}_2 = -(a+b)x_1 + bx_1^2 - x_1x_2 \end{cases}$$

Where $a > 0$ and $b \neq 0$.

(a) Find all equilibrium points of the system.

$$\begin{cases} \dot{x}_1 = -x_1 + ax_2 - bx_1x_2 + x_2^2 = 0 & \text{--- (1)} \\ \dot{x}_2 = -(a+b)x_1 + bx_1^2 - x_1x_2 = 0 & \text{--- (2)} \end{cases}$$

② 의식에서,

$$x_1(bx_1 - (a+b) - x_2) = 0$$

$$x_1 = 0 \quad \text{or} \quad x_1 = \frac{a+b+x_2}{b}$$

i) $x_1 = 0$ 일 때 ①의 식을 만족하는 x_2

$$ax_2 + x_2^2 = 0$$

$$x_2(x_2 + a) = 0$$

$$\therefore x_2 = 0, -a$$

\Rightarrow equilibrium point $(x_1, x_2) = (0, 0), (0, -a)$

$$\text{ii) } x_1 = \frac{a+b+x_2}{b} \text{ 일 때 } x_2$$

$$-\frac{a+b+x_2}{b} + a\cancel{x_2} - (a+b+\cancel{x_2})x_2 + \cancel{x_2}^2 = 0$$

$$a+b+x_2 + b^2x_2 = 0$$

$$(1+b^2)x_2 = -a-b$$

$$\therefore x_2 = \frac{-a-b}{1+b^2}$$

$$x_1 = \frac{a+b}{b} + \frac{1}{b} \frac{-a-b}{1+b^2} = \frac{(a+b)(1+b^2) - a-b}{b(1+b^2)} = \frac{(a+b)b^2}{b(1+b^2)} = \frac{b(a+b)}{1+b^2}$$

$$\Rightarrow \text{equilibrium point } (x_1, x_2) = \left(\frac{b(a+b)}{1+b^2}, \frac{-(a+b)}{1+b^2} \right)$$

$$\therefore \text{equilibrium point } (x_1, x_2) = (0, 0), (0, -a), \left(\frac{b(a+b)}{1+b^2}, \frac{-(a+b)}{1+b^2} \right)$$

Qualitative Behavior of Linear Systems - Summary

Jacobian matrix A

$$\rightarrow \text{the real Jordan form} \quad \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \quad \begin{bmatrix} \lambda & k \\ 0 & \lambda \end{bmatrix} \quad \begin{bmatrix} \alpha - \beta \\ \beta \alpha \end{bmatrix}$$

eigenvalues λ_1 and λ_2 are
real and distinct

eigenvalues are real and equal

complex eigenvalues
 $\lambda_{1,2} = \alpha \pm i\beta$

① Both eigenvalues are real. $\lambda_1 \neq 0, \lambda_2 \neq 0, \lambda_1 \neq \lambda_2$

1) both eigenvalues are negative $\lambda_2 < \lambda_1 < 0 \rightarrow \begin{cases} \lambda_2: \text{fast eigenvalue} \\ \lambda_1: \text{slow eigenvalue} \end{cases}$
 \Rightarrow stable node

2) both eigenvalues are positive $\lambda_2 > \lambda_1 > 0$

\Rightarrow unstable node

3) the eigenvalues have opposite signs. $\lambda_2 < 0 < \lambda_1 \rightarrow \begin{cases} \lambda_2: \text{stable eigenvalue} \\ \lambda_1: \text{unstable eigenvalue} \end{cases}$
 \Rightarrow saddle

② Complex eigenvalues $\lambda_{1,2} = \alpha \pm i\beta$

1) $\alpha < 0 \rightarrow$ stable focus

2) $\alpha > 0 \rightarrow$ unstable focus

3) $\alpha = 0 \rightarrow$ center

③ Nonzero multiple eigenvalues $\lambda_1 = \lambda_2 = \lambda, \lambda \neq 0$

1) $\lambda < 0 \rightarrow$ stable node

2) $\lambda > 0 \rightarrow$ unstable node

④ One or both eigenvalues are zero

(b) Determine the type of each isolated equilibrium point, for all values of $a > 0$ and $b \neq 0$.

Jacobian matrix $A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}$ || equilibrium point (x_1, x_2)

$$\begin{aligned}\frac{\partial f_1}{\partial x_1} &= -1 - b x_2 \\ \frac{\partial f_1}{\partial x_2} &= a - b x_1 + 2 x_2 \\ \frac{\partial f_2}{\partial x_1} &= -(a+b) + 2 b x_1 - x_2 \\ \frac{\partial f_2}{\partial x_2} &= -x_1\end{aligned}$$

i) Equilibrium point $(0, 0)$

Jacobian matrix $A = \begin{bmatrix} -1 & a \\ -(a+b) & 0 \end{bmatrix}$

characteristic eq., $\det(\lambda I - A) = \begin{vmatrix} \lambda + 1 & -a \\ a+b & \lambda \end{vmatrix} = \lambda(\lambda+1) + a(a+b) = 0$

$$\therefore \lambda^2 + \lambda + a^2 + ab = 0$$

$$\Rightarrow \text{eigenvalues } \lambda_{1,2} = \frac{-1 \pm \sqrt{1-4(a^2+ab)}}{2}$$

i) $1-4(a^2+ab) < 0$ 이면 eigenvalues $\lambda_{1,2} = \alpha \pm i\beta$ 와 복소수이다.

여기서 $\alpha < 0$ 이므로 equilibrium point는 stable focus이다.

ii) $1-4(a^2+ab) = 0$ 이면 eigenvalues $\lambda_{1,2} = -\frac{1}{2}$ 이므로

nonzero multiple eigenvalues이다. 여기서 $\lambda < 0$ 이므로 stable node이다.

iii) $1-4(a^2+ab) > 0$ 이면 eigenvalues는 $\lambda_{1,2}$ 는 real, distinct eigenvalues이다.

여기서, $-1 + \sqrt{1-4(a^2+ab)} < 0$ 이면 both eigenvalues는 negative이므로

stable node가 된다. 하지만 $-1 + \sqrt{1-4(a^2+ab)} > 0$ 이면 두 eigenvalues가

서로 다른 부호를 가지므로 saddle이다.

정리하면 다음과 같다.

① $1-4(a^2+ab) < 0 \rightarrow \underline{\text{stable focus}}$

② $1-4(a^2+ab) = 0 \rightarrow \underline{\text{stable node}}$

③ $1-4(a^2+ab) > 0 \rightarrow \begin{cases} \sqrt{1-4(a^2+ab)} < 1 &; \underline{\text{stable node}} \\ \sqrt{1-4(a^2+ab)} > 1 &; \underline{\text{saddle}} \end{cases}$

z) Equilibrium point $(0, -\alpha)$

$$A = \begin{bmatrix} -1-ab & -\alpha \\ -b & 0 \end{bmatrix}$$

Jacobian matrix $A = \begin{bmatrix} \cancel{-1} & \alpha-2\alpha \\ -(a+b)+a & 0 \end{bmatrix} = \begin{bmatrix} \cancel{-1} & -\alpha \\ -b & 0 \end{bmatrix}$

Characteristic eq. $\det(\lambda I - A) = \begin{vmatrix} \lambda+1 & \alpha \\ b & \lambda \end{vmatrix} = \lambda(\lambda+1) - ab = 0$.

$$\lambda^2 + \lambda - ab = 0$$

$$\rightarrow \text{eigenvalues } \lambda_{1,2} = \frac{-1 \pm \sqrt{1+4ab}}{2}$$

i) $1+4ab < 0$ 이면, eigenvalues $\lambda_{1,2} = \alpha \pm \beta i$ 의 형태의 eigenvalues이다.

여기서, $\alpha < 0$ 이므로 stable focus이다.

ii) $1+4ab = 0$ 이면, eigenvalues $\lambda_{1,2} = -\frac{1}{2}$ 이므로 stable node이다.

iii) $1+4ab > 0$ 이면, eigenvalues는 $\lambda_{1,2}$ 는 real and distinct eigenvalues이다.

여기서, $-1 + \sqrt{1+4ab} < 0$ 이면 $\lambda_{1,2}$ 는 모두 negative이므로 stable node이다.

$-1 + \sqrt{1+4ab} > 0$ 이면 $\lambda_{1,2}$ 는 서로 다른 부호를 가지므로 saddle이다.

위를 정리하면 다음과 같다.

① $1+4ab < 0 \rightarrow \text{stable focus}$

② $1+4ab = 0 \rightarrow \text{stable node}$

③ $1+4ab > 0 \rightarrow \sqrt{1+4ab} < 1 ; \text{stable node}$

$\sqrt{1+4ab} > 1 ; \text{saddle}$

$$3) \text{ Equilibrium point } \left(\frac{b(a+b)}{1+b^2}, \frac{-(a+b)}{1+b^2} \right)$$

$$\text{식을 간단화 전개하기 위해 } \frac{a+b}{1+b^2} = M \text{ 으로 하고 풀어, } \Rightarrow (bM, -M)$$

$$\text{Jacobian matrix } A = \begin{bmatrix} -1+bM & a-b^2M-2M \\ -(a+b)+2b^2M+M & -bM \end{bmatrix}$$

$$\text{characteristic eq. } \det(\lambda I - A) = \begin{vmatrix} \lambda + 1 - bM & -a + b^2M + 2M \\ a + b - 2b^2M - M & \lambda + bM \end{vmatrix}$$

$$(\lambda + 1 - bM)(\lambda + bM) - (a + b - 2b^2M - M)(-a + b^2M + 2M) = 0$$

$$\lambda^2 + (1 - bM + bM)\lambda + bM(1 - bM)$$

$$- (-a^2 + ab^2M + 2aM - ab + b^3M + 2bM + 2ab^2M - 2b^4M^2 - 4b^2M^2 + aM - \sqrt{M^2 - 2M})$$

$$= \lambda^2 + \lambda + a^2 + ab - 2ab^2M - aM - ab^2M - b^3M + 2b^4M^2 - 2aM - 2bM + 4b^2M^2 + 2M^2$$

$$= \lambda^2 + \lambda + 2M^2(b^4 + 2b^2 + 1) + M(-b - 3a - b^3 - 3ab^2) + a^2 + ab$$

$$= \lambda^2 + \lambda + 2M^2(b^2 + 1)^2 - bM(1 + b^2) - 3aM(1 + b^2) + a^2 + ab$$

$$= \lambda^2 + \lambda + 2M^2(b^2 + 1)^2 - (1 + b^2)(b + 3a)M + a^2 + ab = 0$$

"Q"

$$\Rightarrow \text{eigenvalues } \lambda_1, \lambda_2 = \frac{-1 \pm \sqrt{1 - 4Q}}{2}$$

$$Q = 2M^2(b^2 + 1)^2 - (1 + b^2)(b + 3a)M + a^2 + ab$$

$$= 2 \cdot \frac{(a+b)^2}{(1+b^2)^2} \cdot (b^2 + 1)^2 - (1 + b^2)(b + 3a) \frac{a+b}{(1+b^2)} + a^2 + ab$$

$$= 2(a^2 + 2ab + b^2) - (3a + b)(a + b) + a^2 + ab$$

$$= 2a^2 + 4ab + 2b^2 - 3a^2 - 4ab - b^2 + a^2 + ab$$

$$= b^2 + ab$$

$$\Rightarrow \text{eigenvalues } \lambda_{1,2} = \frac{-1 \pm \sqrt{1 - 4(b^2 + ab)}}{2}$$

i) $(-4(b^2 + ab)) < 0$ 이면 eigenvalues $\lambda_{1,2}$ 가 complex eigenvalues이다. 즉 $\alpha \pm i\beta$.

여기서 $\alpha < 0$ 이므로 stable focus이다.

ii) $(-4(b^2 + ab)) = 0$ 이면 eigenvalues $\lambda_{1,2} = -\frac{1}{2}$ 이므로 stable node이다.

iii) $(-4(b^2 + ab)) > 0$ 이면 $\lambda_{1,2}$ 는 real and distinct eigenvalues이다.

$-1 + \sqrt{1 - 4(b^2 + ab)} < 0$ 이면 두 eigenvalues가 negative 이므로 stable node.

이제 $a = 1, b = 1$ 인 경우를 살펴보자.

7장 2차 항면 다음과 같다

- ① $1 - 4(b^2 + ab) < 0 \rightarrow \text{stable focus}$
- ② $1 - 4(b^2 + ab) = 0 \rightarrow \text{stable node}$
- ③ $1 - 4(b^2 + ab) > 0 \rightarrow \begin{cases} \sqrt{1 - 4(b^2 + ab)} < 1 &; \text{stable node} \\ \sqrt{1 - 4(b^2 + ab)} > 1 &; \text{saddle} \end{cases}$

(c) For each of the following cases, construct the phase portrait and discuss the qualitative behavior of the system.

i. $a = b = 1$

ii. $a = 1, b = -\frac{1}{2}$

iii. $a = 1, b = -2$

i) $a = b = 1$

equilibrium point $(x_1, x_2) = (0, 0), (0, -1), (1, -1)$

① equilibrium point $(0, 0)$ 일정, (b) 윤곽선에 구한 경계에 대입

$$1 - 4(a^2 + ab) = 1 - 4(1+1) = -7 < 0 \text{ 이므로 stable focus}$$

② equilibrium point $(0, -1)$ 일정,

$$1 + 4ab = 5 > 0$$

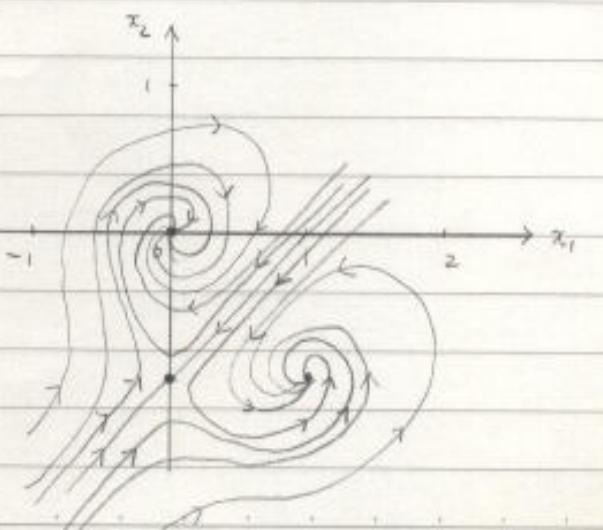
$$\sqrt{1 + 4ab} = \sqrt{5} > 1 \text{ 이므로 saddle}$$

③ equilibrium point $(1, -1)$ 일정,

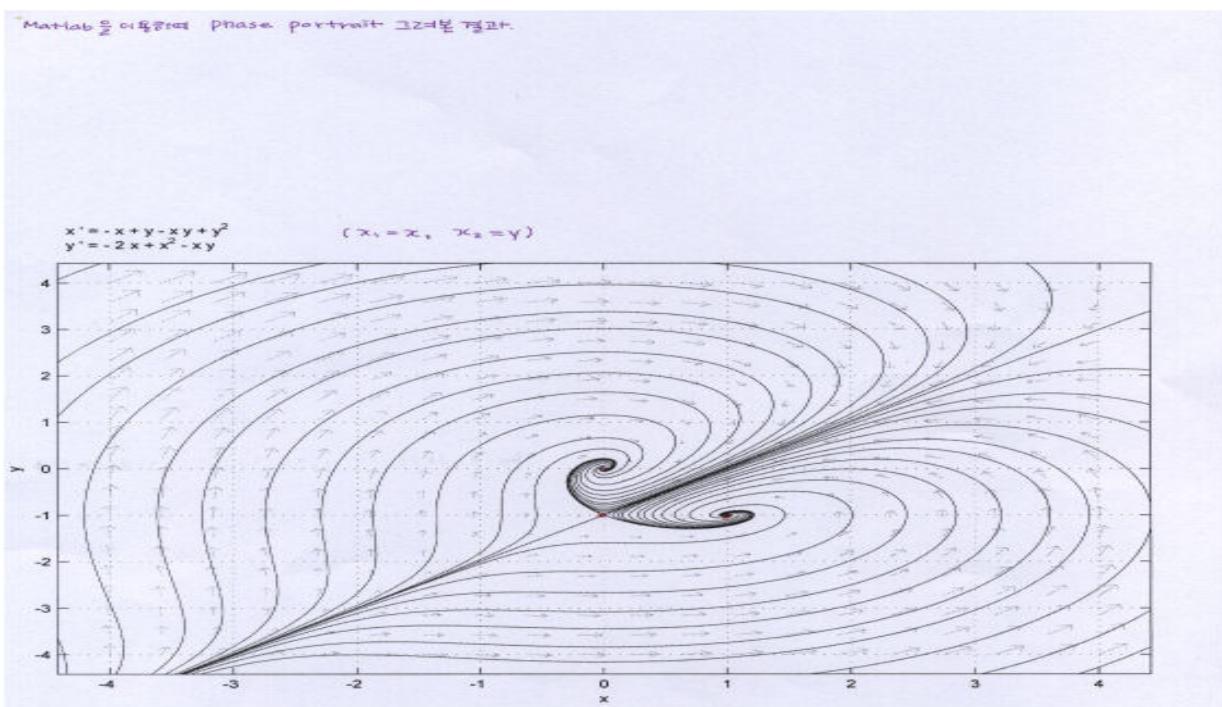
$$1 - 4(b^2 + ab) = 1 - 4(1+1) = -7 < 0 \text{ 이므로 stable focus}$$

$$\begin{aligned}\dot{x}_1 &= -x_1 + x_2 - x_1 x_2 + x_2^2 \\ \dot{x}_2 &= -2x_1 + x_1^2 - x_1 x_2\end{aligned}$$

phase portrait \rightarrow



Matlab을 이용하여 phase portrait 그려보기



2) $a=1, b=-\frac{1}{2}$

equilibrium point $(x_1, x_2) = (0, 0), (0, -1), \left(-\frac{1}{5}, -\frac{2}{5}\right)$

① equilibrium point $(0, 0)$ 일 때, (b) 문제에서 구한 결과에 대입

$$-4(a^2+ab) = -4(1-\frac{1}{2}) = -1 < 0 \text{ 이므로 stable focus}$$

② equilibrium point $(0, -1)$ 일 때,

$$1+4ab = 1+4(-\frac{1}{2}) = -1 < 0 \text{ 이므로 stable focus}$$

③ equilibrium point $\left(-\frac{1}{5}, -\frac{2}{5}\right)$ 일 때,

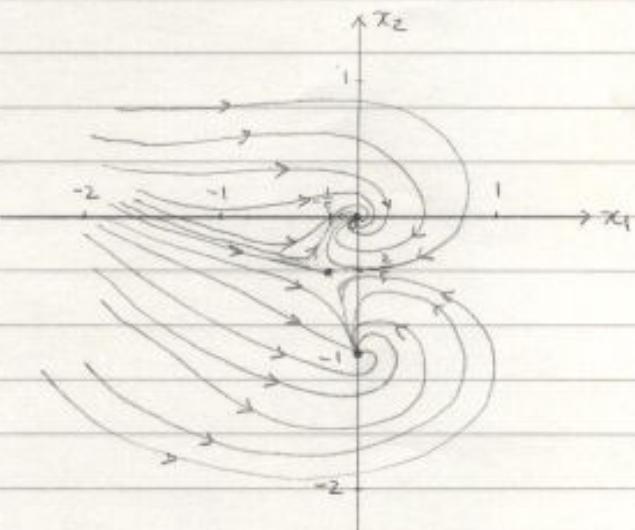
$$1-4(b^2+ab) = 1-4(\frac{1}{4}-\frac{1}{2}) = 1-4(-\frac{1}{4}) = 2 > 0.$$

$$\sqrt{1-4(b^2+ab)} = \sqrt{2} > 1 \text{ 이므로 saddle.}$$

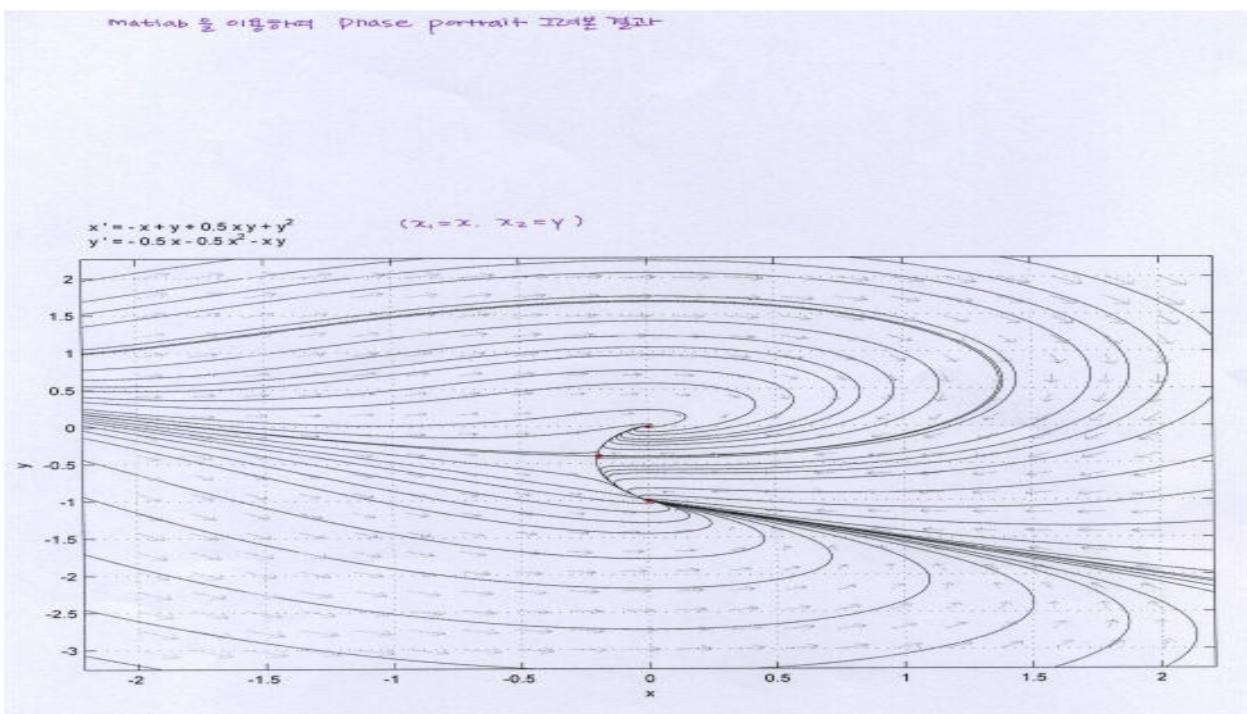
$$\dot{x}_1 = -x_1 + x_2 + \frac{1}{2}x_1x_2 + x_2^2$$

$$\dot{x}_2 = -\frac{1}{2}x_1 - \frac{1}{2}x_1^2 - x_1x_2$$

phase portrait \rightarrow



matlab 을 이용하여 Phase portrait 그리기 결과



$$3) a=1, b=-2$$

$$\text{equilibrium point } (x_1, x_2) = (0, 0), (0, -1), \left(\frac{2}{5}, \frac{1}{5}\right)$$

① equilibrium point $(0, 0)$ 일 때. (b) 문제에서 구한 조건에 대입

$$-4(a^2+ab) = -4(1-2) = 1+4 = 5 > 0$$

$$\sqrt{-4(a^2+ab)} = \sqrt{5} > 1 \quad \text{이므로 saddle.}$$

② equilibrium point $(0, -1)$ 일 때.

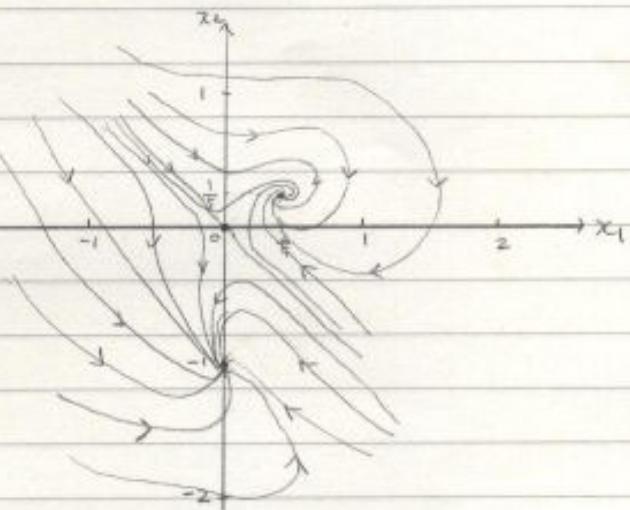
$$1+4ab = 1+4(-2) = -7 < 0 \quad \text{이므로 stable focus}$$

③ equilibrium point $\left(\frac{2}{5}, \frac{1}{5}\right)$ 일 때.

$$1-4(b^2+ab) = 1-4(4-2) = -7 < 0 \quad \text{이므로 stable focus.}$$

$$\begin{cases} \dot{x}_1 = -x_1 + x_2 + 2x_1x_2 + x_2^2 \\ \dot{x}_2 = x_1 - 2x_1^2 - x_1x_2 \end{cases}$$

phase portrait →

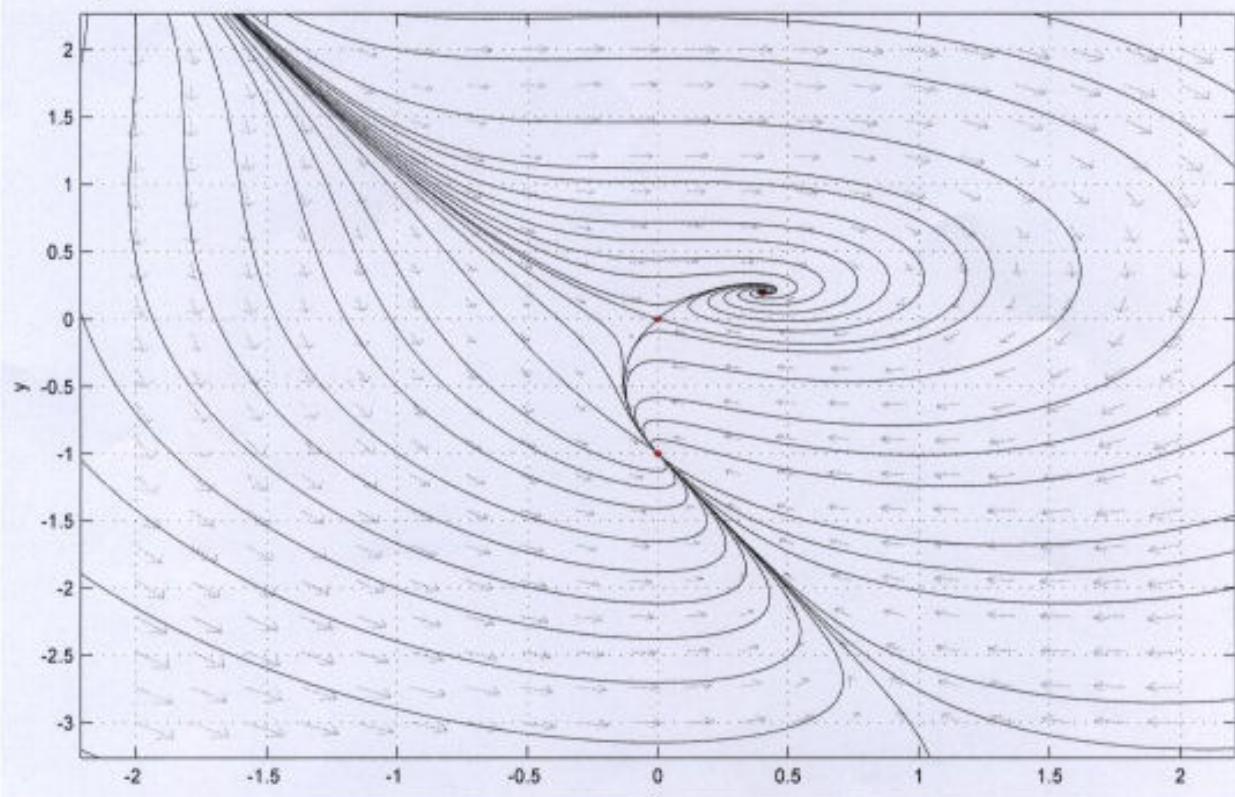


matlab 을 이용하여 그려보는 phase portrait 결과

phase

$$\begin{aligned}x' &= -x + y + 2xy + y^2 \\y' &= x - 2x^2 - xy\end{aligned}$$

$$(x_1=x, x_2=y)$$



2.15 Consider the system

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = u$$

where the control input u can take the values ± 1 .

(a) Sketch the phase portrait when $u=1$.

$$\begin{cases} \dot{x}_1 = x_2 = f_1(x_1, x_2) \\ \dot{x}_2 = 1 = f_2(x_1, x_2) \end{cases}$$

이 system의 equilibrium point는 존재하지 않는다

그대문로 initial state가 어떻게 변화하는 trajectory를 통해 phase portrait을 구한다.

$$x = (x_1, x_2) \rightarrow x + f(x) = (x_1 + f_1(x_1, x_2), x_2 + f_2(x_1, x_2))$$

i) initial state $x_0 = (0, -1)$ 일 때

$$x_0 + f(x) = (-1, 0)$$

$$(-1, 0) + f(x) = (-1, 1)$$

$$(-1, 1) + f(x) = (0, 2)$$

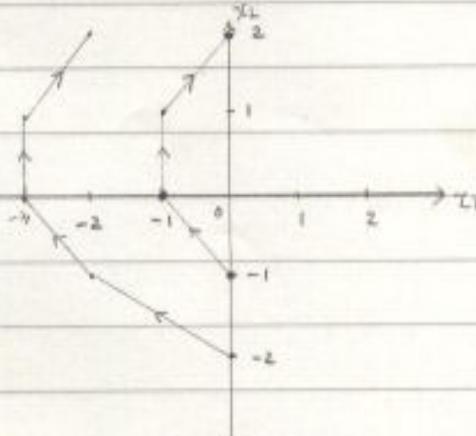
ii) initial state $x_0 = (0, -2)$ 일 때

$$x_0 + f(x) = (-2, -1)$$

$$(-2, -1) + f(x) = (-3, 0)$$

$$(-3, 0) + f(x) = (-3, 1)$$

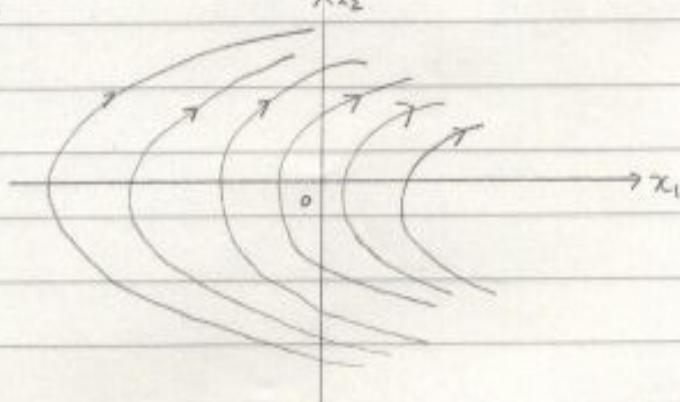
$$(-3, 1) + f(x) = (-2, 2)$$



<vector field>

위와 같이 여러 initial value에 대응한 trajectory를 나타내어 phase portrait을

구하면 다음과 같다.



(b) Sketch the phase portrait when $u = -1$.

$$\begin{cases} \dot{x}_1 = x_2 = f_1(x_1, x_2) \\ \dot{x}_2 = -1 \end{cases}$$

Initial state 초기 x_0 를 trajectory를 알아보자. (a)와 같은 방법으로 trajectory를 알아보자.

i) Initial state $x_0 = (0, 1)$ 일 때,

$$x_0 + f(x) = (1, 0)$$

$$(1, 0) + f(x) = (1, -1)$$

$$(1, -1) + f(x) = (0, -2)$$

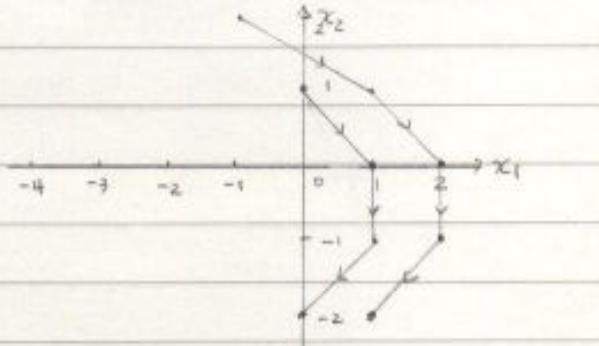
ii) Initial state $x_0 = (-1, 2)$ 일 때

$$x_0 + f(x) = (1, 1)$$

$$(1, 1) + f(x) = (2, 0)$$

$$(2, 0) + f(x) = (2, -1)$$

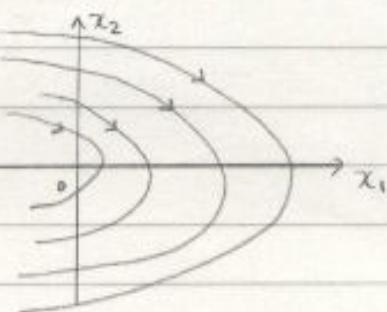
$$(2, -1) + f(x) = (1, -2)$$



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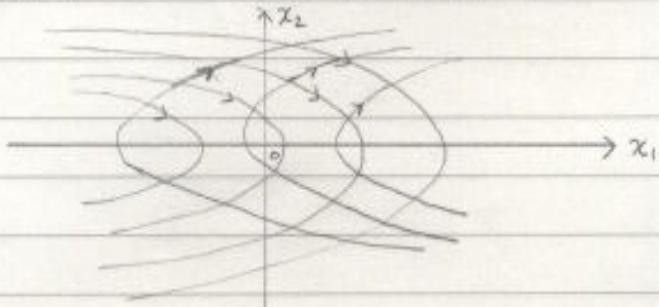
우리가 초기 initial states 초기 x_0 의 trajectory를 알아보고, phase portrait를 구하면

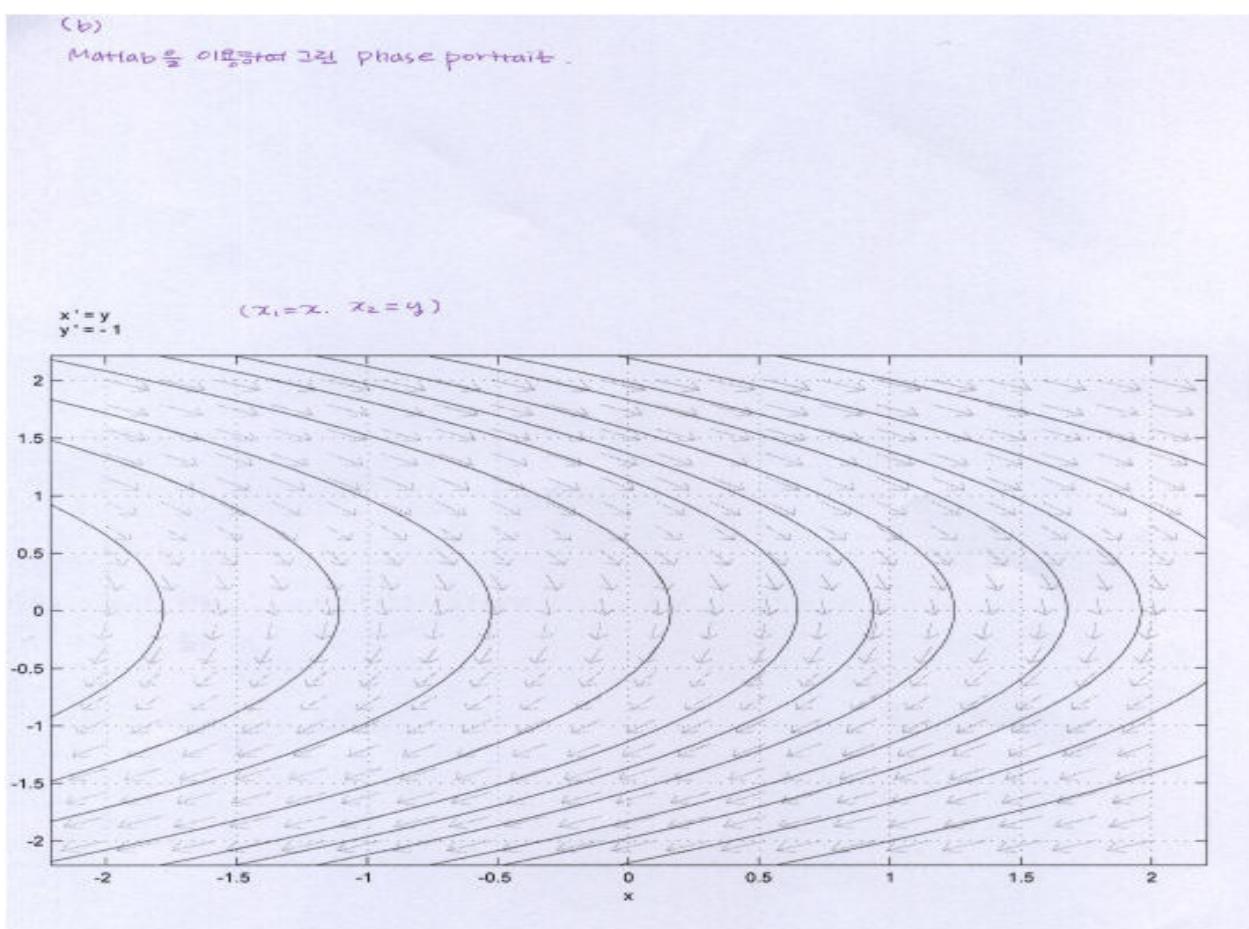
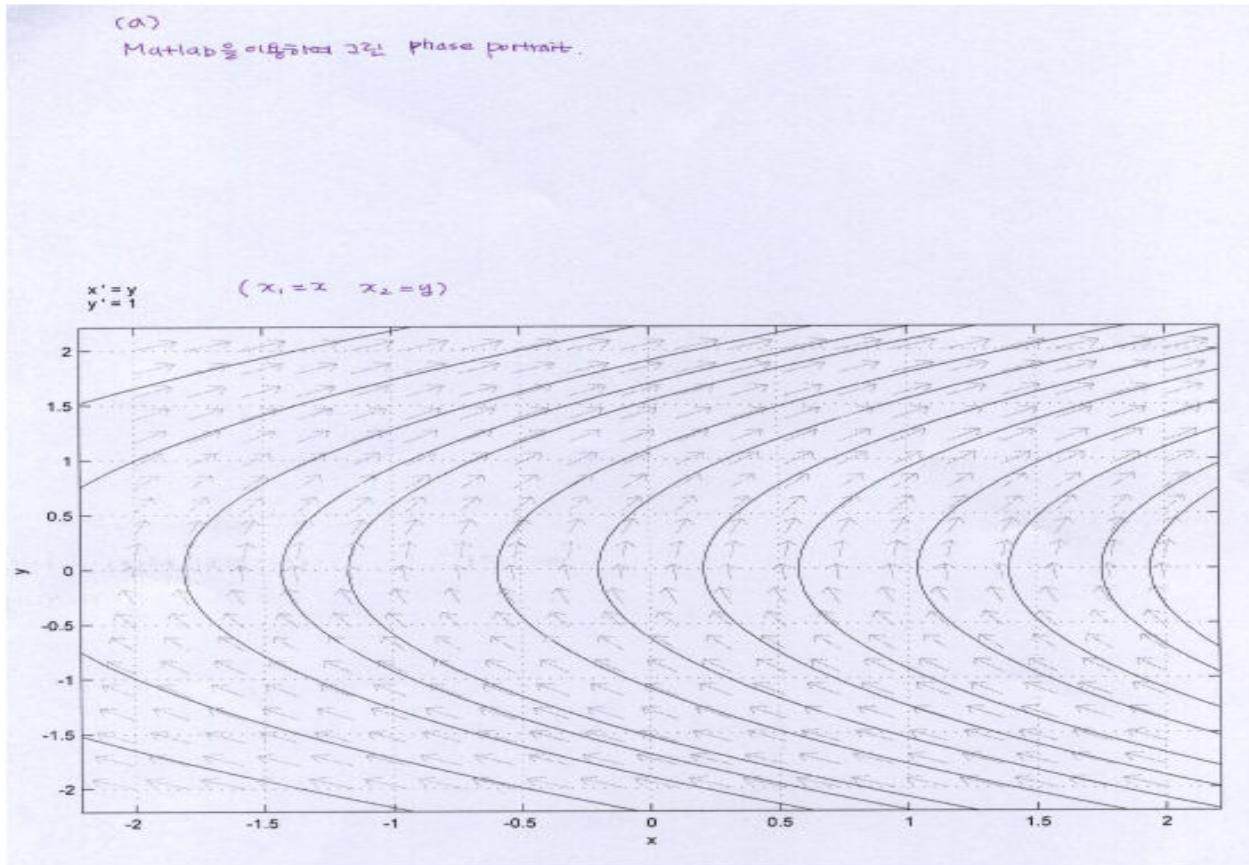
다음과 같다.



(c) By Superimposing the two phase portraits, develop a strategy for switching the control between ± 1 so that any point in the state plane can be moved to the origin in finite time.

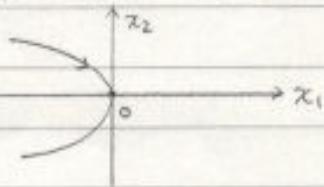
(a)와 (b)의 phase portraits를 합치면 다음과 같다.



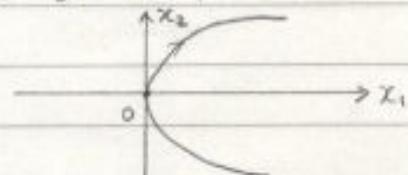


finite time 동안에 어떤 point에서 origin으로 되게 하는 control을 찾아보자

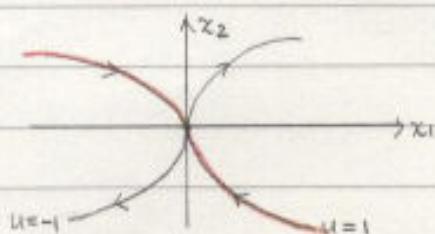
$u=-1$ 일 때 origin을 지나는 phase portrait는 다음과 같다.



$u=+1$ 일 때 origin을 지나는 phase portrait는 다음과 같다.

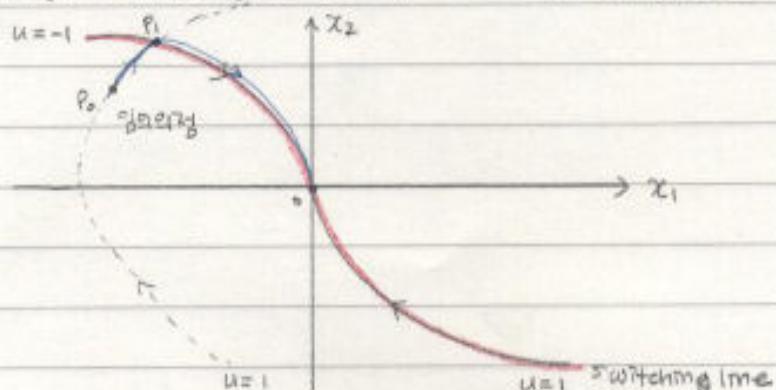


그리므로 어떤 point가 origin으로 오려면 위의 두 phase portrait 위에 존재해야 한다.



그중에서 위의 red line 위에 state가 존재하면, $u=1$ or $u=-1$ 로 origin으로 오게 할 수 있으므로, state가 red line 위로 오도록 해야 한다. (이런 red line을 switching line이라고 한다.)

따라서 임의의 점에서 switching line으로 오도록 control input을 준다. switching line에서 origin으로 오도록 control input을 주면 된다.



위와 같이 임의의 점 P_0 가 있다면, P_0 는 control input $u=1$ 로 P_1 으로 이동, P_1 에서 $u=-1$ 로 바꾸어 주면 origin으로 올 수 있다.

(이와 같은 control을 bang-bang control이라고 한다.)

2.6

$$\begin{cases} \dot{x}_1 = -x_1 + ax_2 - bx_1x_2 + x_2^2 & \text{if } \\ \dot{x}_2 = -(a+b)x_1 + bx_1^2 - x_1x_2 & (a>0, b \neq 0) \end{cases}$$

a) find equilibrium points.

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ when } \text{Df}(H2734) = 0$$

$$\begin{cases} -x_1 + ax_2 - bx_1x_2 + x_2^2 = 0 \rightsquigarrow x_1 = \frac{ax_2 + x_2^2}{1+bx_2} \\ -(a+b)x_1 + bx_1^2 - x_1x_2 = 0 \quad \leftarrow \text{equation } x_2 \text{ from } x_1. \end{cases}$$

pair 2
of saddle pt.

$$\rightarrow x_2(x_2+a)(x_2+\frac{a+b}{1+b})=0 \quad \therefore x_2 = 0, -a, -\frac{a+b}{1+b}$$

$\because x_2 \neq 0 \Rightarrow x_1 = 0, 0, \frac{b(a+b)}{1+b}$

b) Determined the type of each equilibrium point.

\rightarrow 3 equilibrium points using linearization by $\dot{x} = Ax$

계수의 행렬은 대각화 $A =$ eigen value λ 와 type ν 를 갖는다.

(ν : hyperbolic equilibrium points ν_1, ν_2)

0. $(x_1, x_2) = (0, 0)$ 0

$$A = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix} \Bigg|_{(x_1, x_2) = (0, 0)} = \begin{pmatrix} -1 - bx_2 & a - bx_1 + 2x_2 \\ -(a+b) + 2bx_1 - x_2 & -x_1 \end{pmatrix} \Bigg|_{(0,0)} = \begin{pmatrix} -1 & a \\ -(a+b) & 0 \end{pmatrix}$$

$$\lambda = -1 \pm \sqrt{1 - 4a(a+b)}$$

$$1 - 4a(a+b) < 0 \quad \downarrow \quad \begin{matrix} 1 - 4a(a+b) \\ > 0 \end{matrix}$$

$$\begin{cases} \text{i)} 1 - 4a(a+b) > 0 \rightsquigarrow \text{实根 두 개} \rightsquigarrow \text{stable node or saddle} \\ \text{ii)} 1 - 4a(a+b) = 0 \rightsquigarrow \text{实根 한 개} \rightsquigarrow \text{stable node (纯虚根)} \\ \text{iii)} " " < 0 \rightsquigarrow \text{虚根 두 개} \rightsquigarrow \text{stable focus}, \\ \quad \quad \quad \therefore \text{pt가 순회} \end{cases}$$

$$\textcircled{1} \quad (x_1, x_2) = (0, -a) \text{ is a eq. pt.}$$

$$A = \begin{pmatrix} -1-bx_1 & a-bx_1+2x_2 \\ -(a+b)x_1+2bx_2-x_2 & -x_1 \end{pmatrix} \Bigg|_{(0, -a)} = \begin{pmatrix} -(a+b) & -a \\ -b & 0 \end{pmatrix}$$

$$\rightarrow \lambda = \frac{1}{2} \left(-1 + ab \pm \sqrt{(1+ab)^2 - 4ab} \right) = -1, ab,$$

i) $b < 0$: $\lambda = -1 \neq 0$: stable node.

ii) $b = \frac{-1}{ab}$: $\lambda = -1 \neq 0$: .. (unstable).

iii) $b > 0$: $\lambda = -1 \pm \sqrt{ab}$: saddle

$$\textcircled{2} \quad (x_1, x_2) = \left(\frac{b(a+b)}{1+b^2}, -\frac{(a+b)}{1+b^2} \right) \text{ is a eq. pt.}$$

$$A = \begin{pmatrix} -\frac{b(a+b)}{1+b^2} & \frac{a+b^2(a+b)}{1+b^2} + 2 \cdot \frac{ab}{1+b^2} \\ -\frac{(a+b)}{1+b^2} - \frac{2b^2(a+b)}{1+b^2} & \frac{b(a+b)}{1+b^2} \end{pmatrix}$$

$$\lambda = \frac{-1 \pm \sqrt{-4b^2 - 4ab + 1}}{2} \quad -4b^2 - 4ab + 1 < 1 \quad -4b^2 - 4ab + 1 > 1$$

ii) $-4b^2 - 4ab + 1 > 0 \Rightarrow b^2 + ab \neq 0 \Rightarrow$ stable node or saddle.

iii) $" = 0 \Rightarrow b^2 = ab \Rightarrow$ stable node (unstable).

iv) $" < 0 \Rightarrow b^2 < ab \Rightarrow$ stable focus (unstable).

c) for each of the following cases, construct the phase portrait and discuss the qualitative behavior of the system.

i) $a = b = 1$.

$$\rightarrow \text{equilibrium points: } (0, 0), (0, -1), (1, -1)$$

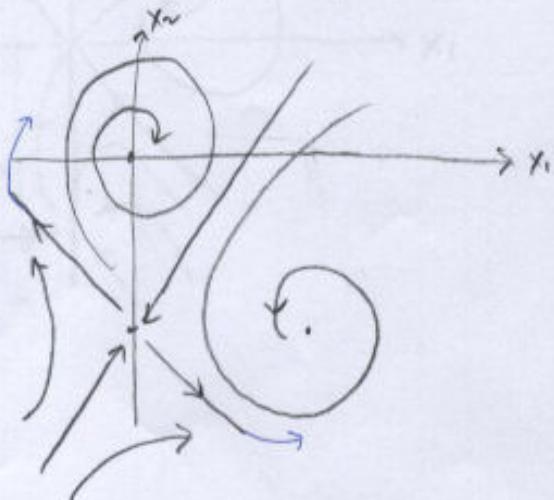
7) equilibrium point with $\lambda_1 \neq \lambda_2$.

$$(0,0) : \lambda = \frac{-1 \pm \sqrt{1-\delta}}{\nu} \rightarrow \text{stable focus}.$$

$$(0,-1) : \lambda = -1, 1 \rightarrow \text{saddle}.$$

$$(-1,0) : \lambda = \frac{-1 \pm \sqrt{-4-\nu+1}}{\nu} \rightarrow \text{stable focus}.$$

7) nonlinear linearization near the equilibrium point type of point
near the phase portrait will be.

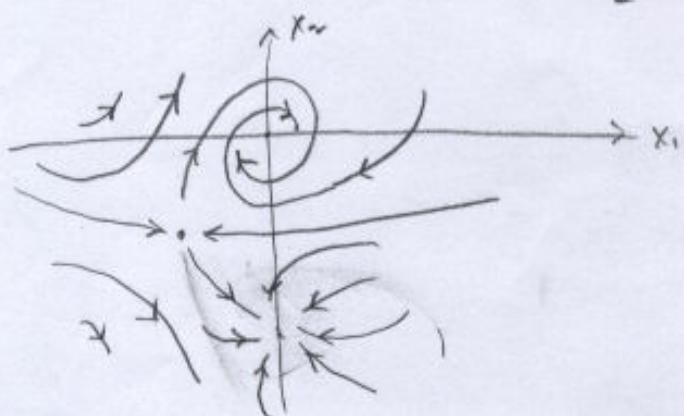


$$\Rightarrow a=1, b=-\frac{1}{\nu}$$

$$(0,0) : \lambda = \frac{-1 \pm \sqrt{1-4(1-\frac{1}{\nu})}}{\nu} \rightarrow \text{stable focus}.$$

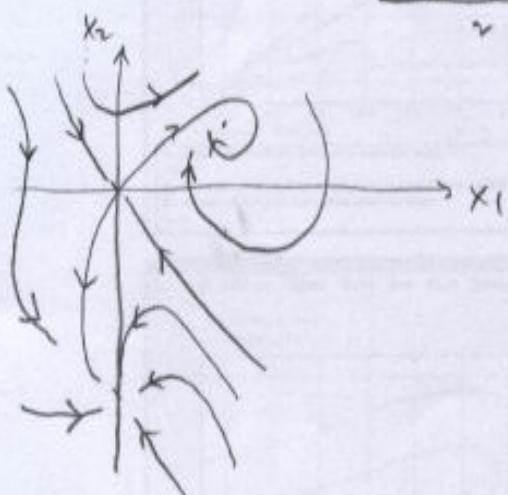
$$(0,-1) : \lambda = -1, -\frac{1}{\nu} \rightarrow \text{stable node}.$$

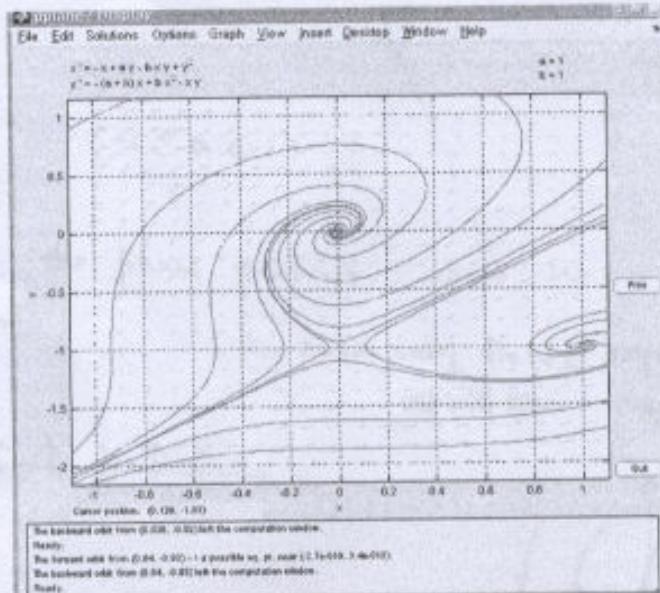
$$(-0.2, -0.4) : \lambda = \frac{-1 \pm \sqrt{1+2+\nu}}{\nu} \rightarrow \text{saddle}$$



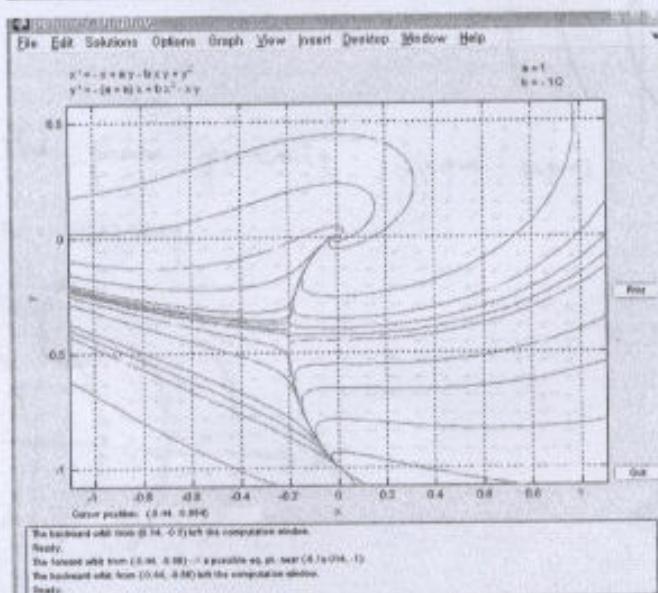
III) $a=1$, $b=-\gamma$

$$\begin{cases} (0, 0) & : \lambda = \frac{-1 \pm \sqrt{1 - 4(1-\gamma)}}{\gamma} \rightsquigarrow \text{saddle.} \\ (0, -1) & : \lambda = -1, -\gamma \rightsquigarrow \text{stable node.} \\ (0.4, 0.2) & : \lambda = \frac{-1 \pm \sqrt{-16 + \gamma + 1}}{\gamma} \rightsquigarrow \text{stable focus.} \end{cases}$$

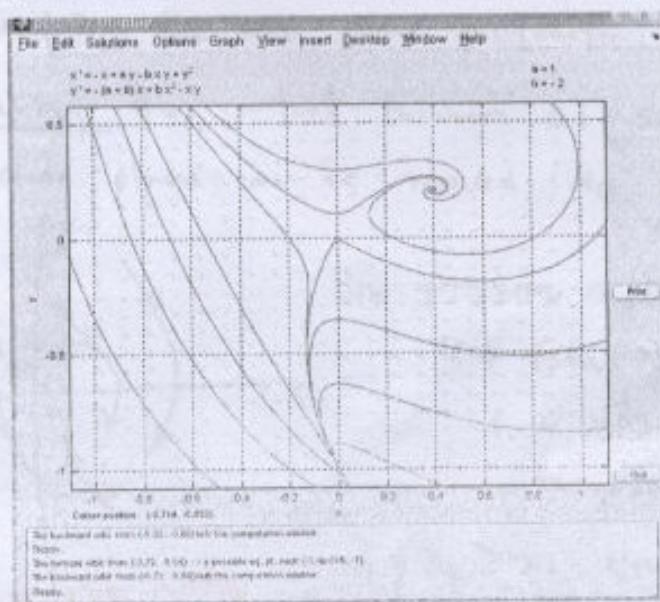




i)



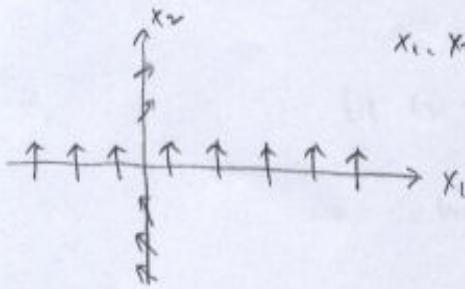
ii)



iii)

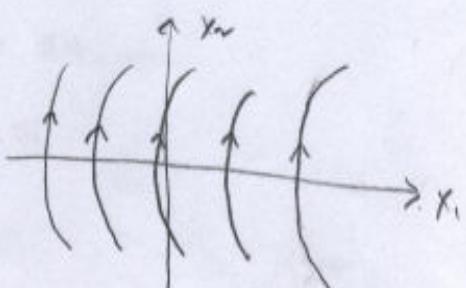
$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = u \end{cases} \quad (-1 \leq u \leq 1)$$

a). Sketch the phase portrait when $u=1$



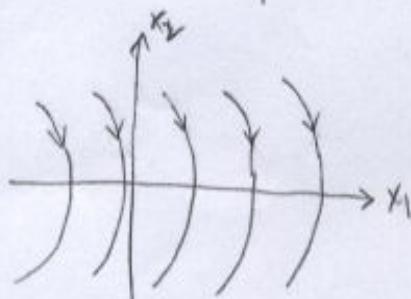
x_1, x_2 일 때 field는 어떤 경로인가? 그에 따른 phase portrait는?

경로는 직선이다.

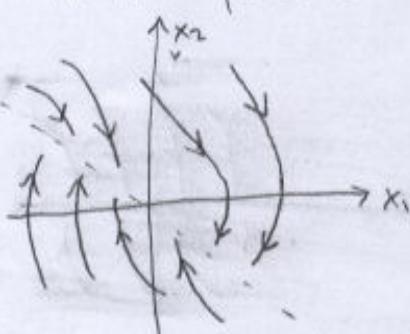


b). sketch the phase portrait when $u=-1$

$u=-1$ 일 때 경로는?



✓ By superimposing the two phase portraits, develop a strategy for switching the control between ± 1 so that any point in the state plane can be moved to the origin in infinite time.



2번 경로를 2, 4번 경로로

각각 다른 경로가 가능하다.

1, 3번 경로 2번 경로 4, 2번 경로

trajectory를 통해 가능하다.

✓ 경로를 주고서 switching

하는 경우.

✓ 경로를 주고서 경로를 찾는 경우.

$\frac{d}{dt} \begin{cases} x_2 \geq x_1 \text{ 일 때 } u = -1 \\ x_2 < x_1 \text{ 일 때 } u = +1 \end{cases}$