[Exercises 3] Samples

[Problem 3.6] Let $f(t,x)$ be piecewise continouous in t

<Solution> (a)

From given condition

$$\dot{x} = f(t,x), \qquad x(t_0) = x_0$$

The solution $x(t)$ is given by

$\mu = K_1$, $L \to K_2$

$\gamma = \|x_0\|$

$$x(t) = x_0 + \int_{t_0}^{t} f(s,x(s))ds$$

$$\Rightarrow \|x(t)\| \le \|x_0\| + \int_{t_0}^{t} \|f(s,x(s))\| ds \le \gamma + \mu(t-t_0) + \int_{t_0}^{t} L\,\|x(s)\|\,ds$$

Using Gronwall-Bellman Inequality,

$$\|x(t)\| \le \gamma + \mu(t-t_0) - \gamma - \mu(t-t_0) + \int_{t_0}^{t} L[\gamma + \mu(s-t_0)]\exp L(t-s)ds$$

Integrating the right terms

$$\|x(t)\| \le \gamma + \mu(t-t_0) - \gamma - \mu(t-t_0) + \gamma\exp[L(t-t_0)] + \int_{t_0}^{t} \mu\exp L(t-s)ds$$

$$= \gamma\exp[L(t-t_0)] + \frac{\mu}{L}\{\exp[L(t-t_0)] - 1\}$$

From the definition,

$$f(t,x) \le k_1 + k_2\,\|x\|.$$

Then,

$$\|f(t,x)\| \le \|x_0\| + k_1(t-t_0) - \|x_0\| - k_1(t-t_0) + \|x_0\|\exp[k_2(t-t_0)] + \int_{t_0}^{t} k_1\exp[k_2(t-s)]ds$$

$$= \|x_0\|\exp[k_2(t-t_0)] + \frac{k_1}{k_2}\{\exp[\underset{K_2}{\cancel{L}}(t-t_0)] - 1\}$$

■

(b) Can the solution have a finite escape time?

<Solution>

The finite escape time is used to describe the phenomenon that a trajectory escapes to infinity at a finite time. So, if $x(t) = 1/(t-1)$ then, a finite escape time = 1.

Therefore,

$$x(t) \le \|x_0\|e^{k_2(t-t_0)} + \frac{k_1}{k_2}\left\{e^{k_2(t-t_0)} - 1\right\},$$

has no finite escape time when $\|x_0\|$ has finite value.

■

[Problem 3.23]

<Solution>

From given condition,

$$f(x) = \int_0^1 \frac{\partial f}{\partial x}(\sigma x)d\sigma x, \quad \forall x \in D \; ?$$

내꺼 빼먹는 것.
문제 순서로 풀 것.

And from the given hint,

$$g(\sigma) = f(\sigma x), \quad 0 \le \sigma \le 1$$

Then,

$$\dot{g} = \frac{\partial f}{\partial x}(\sigma x)\frac{\partial \sigma x}{\partial x} = \frac{\partial f}{\partial x}(\sigma x)x$$

Therefore,

$$f(x) = f(x) - f(0) = g(1) - g(0)$$

and

$$g(1) - g(0) = \int_0^1 \dot{g}(\sigma)d\sigma = \int_0^1 \frac{\partial f}{\partial x}(\sigma x)d\sigma x$$

■

[Problem 3.24]

Let $V : R \times R^n \to R$ be continuously differentiable. Suppose that $V(t, 0) = 0$ for all $t \ge 0$ and

$$V(t,x) \ge c_1 \|x\|^2 ; \left\| \frac{\partial V}{\partial x}(t,x) \right\| \le c_4 \|x\|, \ \forall \ (t,x) \in [0,\infty) \times D$$

where c_1 and c_4 are positive constant and $D \subset R^n$ is a convex domain that contains the origin $x = 0$.

(a) Show that $V(t,x) \le \frac{1}{2} c_4 \|x\|^2$ for all $x \in D$.

Hint: Use the representation $V(t,x) = \int_0^1 \frac{\partial V}{\partial x}(t, \sigma x)\, d\sigma\, x$.

<Solution>

$$\begin{aligned} V(t,x) &= \int_0^1 \frac{\partial V}{\partial x}(t,\sigma x)d\sigma\, x \\ &\le \int_0^1 \left\| \frac{\partial V}{\partial x}(t,\sigma x) \right\| \|x\| d\sigma \le \int_0^1 \sigma c_4 \|x\|^2 d\sigma \\ &\le \frac{1}{2} c_4 \|x\|^2 \end{aligned}$$

■

(b) Show that the constants c_1 and c_4 must satisfy $2c_1 \le c_4$.

<**Solution**>

$$\begin{aligned} V(t,x) &\ge c_1 \|x\|^2 \\ V(t,x) &\le \frac{1}{2} c_4 \|x\|^2 \end{aligned} \quad \Rightarrow c_1 \le \frac{1}{2} c_4$$

■

(c) Show that $W(t,x) = \sqrt{V(t,x)}$ satisfies the Lipschitz condition

$$\left| W(t,x_2) - W(t,x_1) \right| \le \frac{c_4}{2\sqrt{c_1}} \|x_2 - x_1\|, \ \forall\, t \ge 0, \forall\, x_1, x_2 \in D$$

<**Solution**>

Consider,

$$W(t,x) = \int_0^1 \frac{\partial W}{\partial x}(t,\sigma x) d\sigma\, x$$

and

$$\frac{\partial W}{\partial x} = \frac{1}{2\sqrt{V(t,x)}} \frac{\partial V}{\partial x}$$

$$\Rightarrow \left| W(t,x_1) - W(t,x_2) \right| \le \int_0^1 \left| \frac{\partial W}{\partial x}(t,\sigma x_1) d\sigma x_1 - \frac{\partial W}{\partial x}(t,\sigma x_2) d\sigma x_2 \right| \quad ?$$

$$\le \int_0^1 \left| \frac{1}{2\sqrt{V(t,\sigma x_1)}} \frac{\partial V(t,\sigma x_1)}{\partial x_1} - \frac{1}{2\sqrt{V(t,\sigma x_1)}} \frac{\partial V(t,\sigma x_2)}{\partial x_2} \right| \quad ?$$

Therefore,

$$\sqrt{V(t,\sigma x_1)} \ge c_1 \sigma \|x_1\|, \ \left\| \frac{\partial V(t,\sigma x_1)}{\partial x_1} \right\| \le c_4 \|x_1\| \Rightarrow \int_0^1 \frac{c_4}{2\sqrt{c_1}} \|x_1 - x_2\| d\sigma \le \frac{c_4}{2\sqrt{c_1}} \|x_1 - x_2\| \quad ?$$

■

3.6 Let $f(t,x)$ be piecewise continuous in t, locally Lipschitz in x, and

$$\|f(t,x)\| \le k_1 + k_2\|x\|, \quad -(1) \quad \forall (t,x) \in [t_0, \infty) \times R^n$$

(a) Show that the solution of (3.1) satisfies

$$\|x(t)\| \le \|x_0\| \exp[k_2(t-t_0)] + \frac{k_1}{k_2}\{\exp[k_2(t-t_0)-1]\}$$

for all $t \ge t_0$ for which the solution exists.

$$\dot{x} = f(t,x) \qquad x(t_0) = x_0$$

$$\rightarrow x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\,ds$$

$$\|x(t)\| \le \|x_0\| + \int_{t_0}^{t} \|f(s, x(s))\|\,ds$$

(1)

$$\le \|x_0\| + \int_{t_0}^{t} [k_1 + k_2\|x(s)\|]\,ds$$

$$= \|x_0\| + \int_{t_0}^{t} k_1\,ds + \int_{t_0}^{t} k_2\|x(s)\|\,ds$$

$$= \|x_0\| + k_1(t-t_0) + \int_{t_0}^{t} k_2\|x(s)\|\,ds$$

Gronwall-Bellman's Lemma

$$\le \|x_0\| + k_1(t-t_0) + \int_{t_0}^{t} [\|x_0\| + k_1(s-t_0)]\,k_2 \exp\left[\int_s^t k_2\,d\tau\right] ds$$

$$= \|x_0\| + k_1(t-t_0) + \underbrace{\int_{t_0}^{t} [\|x_0\| + k_1(s-t_0)]\,k_2 e^{k_2(t-s)}\,ds}_{} \quad -(2)$$

$$\underbrace{\int_{t_0}^{t} \|x_0\| k_2 e^{k_2(t-s)}\,ds}_{(3)} + \underbrace{k_1\int_{t_0}^{t} (s-t_0)k_2 e^{k_2(t-s)}\,ds}_{(4)}$$

$$(3);\quad \|x_0\|\int_{t_0}^{t} k_2 e^{k_2(t-s)}\,ds = \|x_0\|\left[-e^{k_2(t-s)}\right]_{t_0}^{t}$$

$$= \|x_0\|\{e^{k_2(t-t_0)} - 1\}$$

$$= \|x_0\|e^{k_2(t-t_0)} - \|x_0\| \quad \cdots\cdots (3)'$$

$$(4);\quad k_1 k_2\int_{t_0}^{t} s e^{k_2(t-s)}\,ds - k_1 t_0\int_{t_0}^{t} k_2 e^{k_2(t-s)}\,ds$$

$$= k_1 k_2 e^{k_2 t}\int_{t_0}^{t} s e^{-k_2 s}\,ds - k_1 t_0 e^{k_2 t}\int_{t_0}^{t} k_2 e^{-k_2 s}\,ds$$

$$= k_1 k_2 e^{k_2 t}\left[\frac{e^{-k_2 s}(-k_2 s - 1)}{k_2^2}\right]^{t} - k_1 t_0 e^{k_2 t}\left[-e^{-k_2 s}\right]_{t_0}^{t}$$

$$= k_1 \cancel{k_2} e^{k_2 t} \cdot \frac{1}{k_2^{2}} \left(e^{-k_2 t}(-k_2 t - 1) - e^{-k_2 t_0}(-k_2 t_0 - 1) \right) + k_1 t_0 e^{k_2 t} \left(e^{-k_2 t} - e^{-k_2 t_0} \right)$$

$$= \frac{k_1}{\cancel{k_2}} e^{k_2 t} e^{-k_2 t} (-\cancel{k_2} t) - \frac{k_1}{k_2} e^{k_2 t} e^{-k_2 t} + \frac{k_1}{\cancel{k_2}} \cancel{e^{k_2 t} e^{-k_2 t_0}} \cancel{k_2} t_0 + \frac{k_1}{k_2} e^{k_2 t} e^{-k_2 t_0}$$

$$+ k_1 t_0 e^{k_2 t} e^{-k_2 t} - \cancel{k_1 t_0 e^{k_2 t} e^{-k_2 t_0}}$$

$$= \frac{k_1}{k_2} e^{k_2 (t - t_0)} - \frac{k_1}{k_2} e^{k_2 (t - t)} + k_1 (t_0 - t)$$

$$= \frac{k_1}{k_2} \left(e^{k_2 (t - t_0)} - 1 \right) - k_1 (t - t_0) \quad \cdots (4)'$$

(3)' 과 (4)' 의 결과를 (2) 식에 대입하면 다음과 같은 결과를 얻을 수 있다.

$$\|x(t)\| \le \cancel{\|x_0\|} + \cancel{k_1 (t - t_0)} + \|x_0\| e^{k_2 (t - t_0)} - \cancel{\|x_0\|} - \cancel{k_1 (t - t_0)} + \frac{k_1}{k_2} \left(e^{k_2 (t - t_0)} - 1 \right)$$

$$\therefore \ \|x(t)\| \le \|x_0\| e^{k_2 (t - t_0)} + \frac{k_1}{k_2} \left\{ e^{k_2 (t - t_0)} - 1 \right\} \qquad \forall\, t \ge t_0$$

(b) Can the solution have a finite escape time?

finite escape time : describe the pheomenon that a trajectory escape to infinite at a finite time.

→ $\|x(t)\|$ 는 finite time t 동안 finite 하다.

t 가 infinite 가 되면 $\|x(t)\|$ 도 infinite 가 되므로 ($t \to \infty$ → $\|x(t)\| \to \infty$)

finite escape time 을 갖지 않는다.

3.23 Let $f(x)$ be a continuously differentiable function that maps a convex domain $D \subset \mathbb{R}^n$ into $\mathbb{R}^n$. Suppose D contains the origin $x=0$ and $f(0)=0$. Show that

$$f(x) = \int_0^1 \frac{\partial f}{\partial x}(\sigma x)\, d\sigma\, x \qquad \forall x \in D$$

$\forall x \in D$. D는 convex domain 이므로,

$\sigma x \in D$, $0 \le \sigma \le 1$, 이다.

$g(\sigma) = f(\sigma x) \qquad 0 \le \sigma \le 1$

(1) — $g(1) - g(0) = \int_0^1 g'(\sigma)\, d\sigma$ 를 이용하여 위의 식을 보이자.

$$g'(\sigma) = \frac{dg(\sigma)}{d\sigma} = \frac{df(\sigma x)}{d\sigma} = \frac{\partial f(\sigma x)}{\partial x} \cdot \frac{\partial(\sigma x)}{\partial \sigma}$$

$$= \frac{\partial f(\sigma x)}{\partial x} \cdot x \qquad \text{—(2)}$$

$$\left.\begin{array}{l} g(1) = f(x) \\ g(0) = f(0) = 0 \end{array}\right\} \rightarrow g(1) - g(0) = f(x) \qquad \text{—(3)}$$

(2) 와 (3) 식을 (1)에 대입하면 다음을 구할 수 있다.

$$\therefore f(x) = \int_0^1 \frac{\partial f(\sigma x)}{\partial x} \cdot x\, d\sigma$$

3.24 Let $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ be continuously differentiable.

Suppose that $V(t,0) = 0$ for all $t \geq 0$ and

$$V(t,x) \geq c_1 \|x\|^2 \quad \text{— (1)}$$

$$\left\| \frac{\partial V}{\partial x}(t,x) \right\| \leq c_4 \|x\| \quad \text{— (2)} \qquad \forall (t,x) \in [0,\infty) \times D$$

Where c_1 and c_4 are positive constants and $D \subset \mathbb{R}^n$ is a convex domain that contains the origin $x = 0$.

(a) Show that $V(t,x) \leq \frac{1}{2} c_4 \|x\|^2$ for all $x \in D$.

$$V(t,x) = \int_0^1 \frac{\partial V}{\partial x}(t, \sigma x) d\sigma \, x$$

$$V(t,x) \leq \int_0^1 \left\| \frac{\partial V}{\partial x}(t, \sigma x) \right\| \|x\| \, d\sigma$$

(2)

$$\leq \int_0^1 c_4 \|\sigma x\| \cdot \|x\| \, d\sigma$$

$$\leq \int_0^1 c_4 \cdot \sigma \|x\|^2 d\sigma = c_4 \|x\|^2 \int_0^1 \sigma \, d\sigma$$

$$= c_4 \|x\|^2 \left(\frac{1}{2}\right)$$

$$\therefore V(t,x) \leq \frac{1}{2} c_4 \|x\|^2 \quad \text{— (3)}$$

(b) Show that the constants c_1 and c_4 must satisfy $2c_1 \leq c_4$.

(1)에 의해, $c_1 \|x\|^2 \leq V(t,x)$

(3)에 의해, $V(t,x) \leq \frac{1}{2} c_4 \|x\|^2$

$$\Rightarrow c_1 \|x\|^2 \leq \frac{1}{2} c_4 \|x\|^2$$

$\|x\|^2 \geq 0$ 이므로,

$$c_1 \leq \frac{1}{2} c_4$$

$$\therefore 2c_1 \leq c_4 \quad \text{— (4)}$$

(c) Show that $W(t,x) = \sqrt{V(t,x)}$ satisfies the Lipschitz condition

$$|W(t,x_2) - W(t,x_1)| \le \frac{c_4}{2\sqrt{c_1}} \|x_2 - x_1\| \quad \forall t \ge 0, \quad \forall x_1, x_2 \in D$$

$$W(t,x) = \sqrt{V(t,x)}$$

앞의 조건들을 $W(t,x) = \sqrt{V(t,x)}$ 에 대해 정리해 보자.

$$(1) \rightarrow \sqrt{V(t,x)} \ge \sqrt{c_1}\, \|x\| \quad — (1)'$$

$$(3) \rightarrow \sqrt{V(t,x)} \le \sqrt{\frac{c_4}{2}}\, \|x\| \quad — (3)'$$

$V(t,x)$는 continuosly differentiable 하므로 Mean value theorem을 이용하자.
$V(t,0)=0$ 이므로 x_1와 x_2 사이에 $x=0$가 존재할 경우와 그렇지 않을 경우로 나누어 생각하자.

① x_1와 x_2 사이에 $x=0$ 존재할 경우, $x=x_1$에서 $x=0$까지, $x=0$에서 $x=x_2$까지로 나누어 생각할 수 있다.

$$|W(t,x_1) - W(t,0)| = |W(t,x_1)| = \sqrt{V(t,x_1)} \le \sqrt{\frac{c_4}{2}}\, \|x_1\| \quad (3)'$$

$$|W(t,x_2) - W(t,0)| = |W(t,x_2)| = \sqrt{V(t,x_2)} \le \sqrt{\frac{c_4}{2}}\, \|x_2\| \quad (3)'$$

$$\rightarrow |W(t,x_2) - W(t,x_1)| = |W(t,x_2) - W(t,0) + W(t,0) - W(t,x_1)|$$

$$\le \sqrt{\frac{c_4}{2}}\, \|x_2\| - \sqrt{\frac{c_4}{2}}\, \|x_1\|$$

$$= \sqrt{\frac{c_4}{2}}\, (\|x_2\| - \|x_1\|)$$

$$= \sqrt{\frac{c_4}{2}}\, \|x_2 - x_1\|$$

(4)에 의해, $1 \le \frac{c_4}{2c_1} \rightarrow 1 \le \sqrt{\frac{c_4}{2c_1}}$ 이므로,

$$|W(t,x_2) - W(t,x_1)| \le \frac{c_4}{2\sqrt{c_1}} \|x_2 - x_1\| \quad — (5)$$

② x_1 와 x_2 사이에 $x=0$ 존재하지 않을 경우,

Mean value theorem 을 이용한다. $x_1 < y < x_2$

$$\frac{W(t,x_2)-W(t,x_1)}{x_2-x_1} = \frac{\partial W(t,y)}{\partial x}$$

$$W(t,x_2)-W(t,x_1) = \frac{\partial W(t,y)}{\partial x}(x_2-x_1)$$

$$W(t,x) = \sqrt{V(t,x)} \rightarrow \frac{\partial W(t,x)}{\partial x} = \frac{1}{2\sqrt{V(t,x)}}\frac{\partial V(t,x)}{\partial x}$$

$$\Rightarrow W(t,x_2)-W(t,x_1) = \frac{1}{2\sqrt{V(t,y)}}\frac{\partial V(t,y)}{\partial x}(x_2-x_1)$$

$$|W(t,x_2)-W(t,x_1)| \leq \frac{1}{2\sqrt{c_1}\,\cancel{\|y\|}}\cdot c_4\cancel{\|y\|}\,\|x_2-x_1\|$$

$$= \frac{c_4}{2\sqrt{c_1}}\|x_2-x_1\| \quad \text{— (6)}$$

x_1 와 x_2 사이에 $x=0$ 이 존재할 때와 존재하지 않을 때 두 경우에 대해 (5)와 (6)의 조건을 얻었다. (5)와 (6)은 같은 결과이므로 $x=0$ 이 x_1와 x_2 사이에 존재하거나, 그렇지 않거나 두 경우 모두 성립함을 알 수 있다.

$$\therefore\ |W(t,x_2)-W(t,x_1)| \leq \frac{c_4}{2\sqrt{c_1}}\|x_2-x_1\|$$