- Homework 4 -

4.2 scalar system $\dot{x} = ax^p + g(x)$.

p: positive integer

$|g(x)| \le k|x|^{p+1}$ in some nbd of $x=0$.

show that

origin is a.s. if p is odd, $a<0$,

origin is unstable if p is odd $a>0$ or p is even $a \neq 0$

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we get $x=0$ as equilibrium point

Consider the function $V(x) = \frac{1}{2}x^2$

$V(x) > 0$ and $V(0) = 0$. $\forall x \neq 0$.

$\dot{V}(x) = x(ax^p + g(x))$

$= ax^{p+1} + xg(x)$

i) if p is odd and $a<0$,

$ax^{p+1} < 0 \quad \forall x \neq 0$.

and, $|xg(x)| \le |x| \cdot k|x|^{p+1} = k|x|^{p+2}$

Since $|xg(x)|$ is bounded higher order than ax^{p+1}

we can get $\dot{V}(x) < 0$ for $|x| < |\frac{a}{k}|$.

therefore $x=0$ is asymptotically stable

ii) if p is odd and $a>0$.

then $ax^{p+1} > 0 \quad \forall x \neq 0$.

then we can find x s.t. $|x| < \frac{k}{a}$ that makes $\dot{V}(x) > 0$.

therefore $x=0$ is unstable.

iii) if p is even and $a \neq 0$.

then the sign of ax^{p+1} is not determined.

if $ax > 0$, $ax^{p+1} > 0$

and if $ax < 0$. $ax^{p+1} < 0$

Hence we can find x s.t. $ax > 0$ and $|x| < \frac{k}{a}$

that makes $\dot{V}(x) > 0$.

therefore $x=0$ is unstable.

4.3 (3) $\dot{x}_1 = x_2(1-x_1^2)$
$\dot{x}_2 = -(x_1+x_2)(1-x_1^2)$

use a quadratic Lyapunov function candidate
to show the origin is a.s.

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consider following function

$$V(x) = \frac{1}{2}x^T P x = \frac{1}{2}a x_1^2 + b x_1 x_2 + \frac{1}{2}c x_2^2 . \quad P = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

then,

$$\dot{V}(x) = a x_1 \dot{x}_1 + b \dot{x}_1 x_2 + b x_1 \dot{x}_2 + c x_2 \dot{x}_2$$
$$= (a x_1 + b x_2)(x_2 - x_2 x_1^2) + (b x_1 + c x_2)(-x_1 - x_2 + x_1^3 + x_1^2 x_2)$$
$$= (a x_1 x_2 + b x_2^2 - b x_1^2 - c x_2^2 - b x_1 x_2 - c x_1 x_2)$$
$$+ (-a x_1^3 x_2 - b x_2^2 x_1^2 + b x_1^4 + b x_1^3 x_2 + c x_2 x_1^3 + c x_1^2 x_2^2)$$
$$= (a-b-c) x_1 x_2 - b x_1^2 + (b-c) x_2^2$$
$$+ b x_1^4 + (-a+b+c) x_1^3 x_2 + (-b+c) x_1^2 x_2^2$$

to cancel cross product terms of x_1, x_2 set

$a-b-c=0$

$\therefore \dot{V}(x) = -b x_1^2 + (b-c) x_2^2 + b x_1^4 + (-b+c) x_1^2 x_2^2$

if $(b-c) < 0$, $\dot{V}(x) < 0$ for $|x_1| < 1$.

And to make $V(x)$ positive definite.

$a > 0$, $ac - b^2 > 0$.

To summarize, the constraint on P.

$a > 0$. $ac - b^2 > 0$. $0 < b < c$. $a = b + c$

We can pick such P as $P = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$.

then $V(x) = \frac{1}{2}x^T P x > 0$, $V(0) = 0$ $\forall x \neq 0$.

$\dot{V}(x) < 0$ $\forall x \neq 0$. $|x_1| < 1$

$\therefore$ the origin is asymptotically stable.

4.16 Show that the origin of

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1^3 - x_2^3$$

is g.a.s.

Conside function

$$V(x) = \frac{1}{2}x_1^4 + x_2^2 \quad > 0 \quad \forall x \neq 0$$
$$V(0) = 0$$

the origin is equi. point

then.

$$\dot{V}(x) = 2x_1^3 x_2 + 2x_2(-x_1^3 - x_2^3)$$
$$= \underline{-2x_2^4 < 0} \quad \forall x \neq 0.$$

Since $|x| \to \infty$, $V(x) \to \infty$, $V(x)$ is radially unbounded.
therefore the origin is globally asymptotically stable (X)

Wrong! is Not Negative definite.
(but negative semi-definite.)

4.22 Lyapunov equation $PA + A^T P = -C^T C$

(A, C): observable.

Show that.

i) A is Hurwitz $\Leftrightarrow$ $\exists\ P = P^T > 0$ s.t. $PA + A^T P = -C^T C$

ii) A is Hurwitz $\Rightarrow$ $PA + A^T P = -C^T C$ has a unique solution.

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i) sufficiency. ($\Leftarrow$)

Consider Lyapunov candidate function

$$V(x) = x^T P x. \qquad P = P^T > 0.$$

then.

$$\dot V(x) = x^T P \dot x + \dot x^T P x = x^T (PA + A^T P) x$$

$$= -x^T (C^T C) x$$

$$= -\| Cx \| \le 0$$

$x(t)$ is the solution of $\dot x = Ax$. so

$$x(t) = \exp(At) x(0).$$

(?)

$$\dot V(x) = -\| Cx \| = 0 \quad \Rightarrow x(t) = 0 \quad \Rightarrow x(0) = 0.$$

the set E, given by 초기값?

$$E = \{ x \in \mathbb{R}^n \mid x(0) = 0 \}$$

is invariant set.

the only solution that can stay identically in S is $x(t) = 0$

thus by Lasalle's Invariant theorem,

the origin is globally asymptotically stable.

It means A is Hurwitz.

La Salle
잘못 이해.

necessity ($\Rightarrow$)

let A is Hurwitz. define P by

$$P = \int_0^\infty \exp(A^T t)(C^T C)\exp(At)\,dt.$$

Since A is Hurwitz, P exists.

$P = P^T$. P is symmetric, and for $x \neq 0$.

$$x^T P x = \int_0^\infty x^T \exp(A^T t)(C^T C)\exp(At)x\,dt$$

$$= \int_0^\infty \|C\exp(At)x\|\,dt$$

Need more from 선형시스템

Since (A, C) is observable, $C\exp(At)$ is linear independent. functions on $t \in [0, \infty)$

So, $x^T P x = 0$ iff $x = 0$. $\therefore P$ is positive definite.

And P is solution of $PA + A^T P = -C^T C$ because.

$$PA + A^T P = \int_0^\infty \exp(A^T t)(C^T C)\exp(At)A\,dt$$

$$+ \int_0^\infty A^T \exp(A^T t)(C^T C)\exp(At)\,dt$$

$$= \int_0^\infty \frac{d}{dt}\left[\exp(A^T t)(C^T C)\exp(At)\right]dt$$

$$= -C^T C.$$

※ Then Let's show that P is unique solution.

Suppose that there are another sol. $\bar{P}$ Then.

$$\begin{array}{l} PA + A^T P = -C^T C \\ \bar{P}A + A^T \bar{P} = -C^T C \end{array} \Rightarrow (P - \bar{P})A + A^T(P - \bar{P}) = 0$$

$$\exp(A^T t)\left[(P - \bar{P})A + A^T(P - \bar{P})\right]\exp(At)$$

$$= \frac{d}{dt}\left[\exp(A^T t)(P - \bar{P})\exp(At)\right] = 0.$$

$\therefore \exp(A^T t)(P - \bar{P})\exp(At) = \text{constant}$ for $\forall t$

When $t = 0$,

$$P - \bar{P} = 0.$$

So P is unique sol.

4.24 Consider

$$\dot{x} = f(x) - k\, G(x) R^{-1}(x) G^T(x) \left(\frac{\partial V}{\partial x}\right)^T$$

$V(x)$: C^1, $V > 0$ s.t

$$\frac{\partial V}{\partial x} f(x) + q(x) - \frac{1}{4} \frac{\partial V}{\partial x} G(x) R^{-1}(x) G^T(x) \left(\frac{\partial V}{\partial x}\right)^T = 0 \quad \cdots (*)$$

$q(x) \geq 0$,

$R(x)$ ~~nonsingular~~ (positive definite for all x.)

$k > 0$ constant

Show that the origin is a.s.

i) $q(x) > 0$, and $k \geq \frac{1}{4}$

ii) $q(x) \geq 0$, $k > \frac{1}{4}$ and the only solution of $\dot{x} = f(x)$ that can stay in $\{q(x) = 0\}$ is $x(t) = 0$.

when the origin $\rightarrow$ G.A.S ?

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i) consider $V(x(t))$.

$$\dot{V} = \frac{\partial V}{\partial x}\dot{x} = \frac{\partial V}{\partial x}\left[f(x) - kG(x)R^{-1}(x)G^T(x)\left(\frac{\partial V}{\partial x}\right)^T\right] \leftarrow (*)$$

$$= \left(\frac{1}{4} - k\right)\frac{\partial V}{\partial x} G(x) R^{-1}(x) G^T(x) \left(\frac{\partial V}{\partial x}\right)^T - q(x)$$

let $z = G^T(x)\left(\frac{\partial V}{\partial x}\right)^T$ then

$$\dot{V} = \left(\frac{1}{4} - k\right) z^T R^{-1} z - q(x).$$

therefore if $q(x) > 0$, $k \geq \frac{1}{4}$ and $R^{-1} \geq 0$.

the origin is a.s.

I think this condition is omitted at the question.

iii) If $q(x) \geq 0$ and $k > \frac{1}{4}$ (assume that $R^{-1} > 0$) then

$$\dot{V} = (\tfrac{1}{4} - k) z^T R^{-1} z - q(x) \geq 0$$

$\dot{V} = 0$ when $z^T R^{-1} z = 0$ and $q(x) = 0$.

From the assumption $R^{-1} > 0$,

$$z = G^T(x) \left(\frac{\partial V}{\partial x}\right)^T = 0 \text{ and } q(x) = 0.$$

$$\therefore S = \{x \mid \dot{V}(x) = 0\}$$
$$= \{x \mid G^T(x) \left(\frac{\partial V}{\partial x}\right)^T = 0 \text{ and } q(x) = 0\}$$

then we get $\dot{x} = f(x)$.

If the only solution of $\dot{x} = f(x)$ that can stay in the set $S = \{q(x) = 0\}$ is $x(t) = 0$, by Lasalle's Invariant theorem the origin is a.s.

If $V(x)$ is defined on $\mathbb{R}^n$ and radially unbounded, the origin is G.A.S.

4.2 $\dot{x} = ax^p + g(x)$, $p \in \mathbb{N}$. $|g(x)| \le k|x|^{p+1}$ in some $B(0)$

i) show that $x=0$ is asymptotically stable if p is odd and $a<0$

pf) 가정에서 $|g(x)| \le k|x|^{p+1}$ in some neighborhood of $x=0$

→ 이 neighborhood를 $B_\varepsilon(0)$ 이라 잡자.

그러면, $x \in B_\varepsilon(0)$ 에서 $-k|x|^{p+1} \le g(x) \le k|x|^{p+1}$ ($\because$ scalar)

let $V(x) = x^2 \longrightarrow$ PD.

$\dot{V}(x) = 2x\dot{x} = 2ax^{p+1} + 2x\,g(x) \le 2a|x|^{p+1} + 2k|x||x|^{p+1}$ ~~$|x|^{p+2}$~~

지금 가정에서 p : odd → $p+1$: even ⇒ $x^{p+1} = |x|^{p+1}$ ($\because$ scalar)

⇒ $\dot{V}(x) \le 2|x|^{p+1}(a + k|x|)$ ~~$\le 2|x|^{p+1}(a+k|x|)$~~.

at $|x| < \frac{-a}{k}$ ⇒ $a + k|x| < 0$ ⇒ $\dot{V}(x) < 0$.

let $\varepsilon_2 = \min(-\frac{a}{k}, \varepsilon)$ ⇒ $x \in B_{\varepsilon_2}(0)$ 에서는

$|g(x)| \le k|x|^{p+1}$, $\dot{V}(x) < 0$ 모두 만족됨

$\therefore$ $\dot{x} = ax^p + g(x)$ 는 $a<0$, p : odd 일 때

$x=0$ 에서 asymptotically stable.

ii) show that $x=0$ is unstable if p is odd and $a>0$ or p is even and $a \ne 0$.

pf) 아까 이용했던 $V(x) = x^2$ 를 그대로 이용.

$V(x) = x^2 \to$ PD → 임의의 $B(0)$ 안에서 $V(x) > 0$.

$\dot{V}(x) > 0$ 인 원점주변의 neighborhood의 존재를 보이면 o.k.

$\dot{V} = 2x\dot{x} = 2ax^{p+1} + 2x\,g(x) \ge 2ax^{p+1} - 2k\cancel{x}|x|^{p+2}$

ⓐ 처음 조건인 $a>0$ & p is odd

p : odd → $p+1$: even ($\because x^{p+1} = |x|^{p+1}$)

→ $\dot{V} \ge 2|x|^{p+1}(a - k|x|)$ ~~$\ge 2|x|^{p+1}(a-k|x|)$~~

⇒ $|x| < \frac{a}{k}$ 이면 $\dot{V} > 0$

⇒ $x=0$ 주변의 임의의 neighborhood ~~에는~~ 의 모든 점에서

$V(x) > 0$ & $\dot{V}(x) > 0$ ~~인 점이~~ 존재 이다. $\therefore$ unstable.

ⓑ $a \neq 0$ & p is even

Let $a>0$, $\dot V \geq 2ax^{p+1} - 2k|x|^{p+2}$ For $x>0$, $\dot V \geq 2ax^{p+1}(a-kx)$ For $0<x<\frac{a}{k}$, $\dot V > 0$ $\Rightarrow$ unstable. Let $a<0$ 생략.

$\dot V \geq 2ax^{p+1} - 2kx|x|^{p+1} = 2|x|^p x(a-k|x|)$?

$\Rightarrow \frac{a}{k} > x > 0$ 이고, $|g| < k|x|^{p+1}$ 이 만족하는 영역에서 항상 $\dot V > 0$

$\Rightarrow$ 임의의 원점 근방에서 $\dot V > 0$ 인 점이 항상 존재 $\therefore$ unstable.

4.3 (3) $\dot x_1 = x_2(1-x_1^2)$ $\quad \dot x_2 = -(x_1+x_2)(1-x_1^2)$

Good.

let $V(x) = x^T P x$, P: symmetric, PD matrix.

$\dot V(x) = f^T P x + x^T P f = 2x^T P f$

let $P = \begin{pmatrix}1&0\\0&1\end{pmatrix} \rightarrow$ PD & symmetric fn.

$\Rightarrow V(x) = x^T P x$ is PD.

$$\dot V(x) = 2\begin{pmatrix}x_1 & x_2\end{pmatrix}\begin{pmatrix}1&0\\0&1\end{pmatrix}\begin{pmatrix}x_2(1-x_1^2)\\-(x_1+x_2)(1-x_1^2)\end{pmatrix}$$

$$= 2\begin{pmatrix}x_1 & x_2\end{pmatrix}\begin{pmatrix}x_2(1-x_1^2)\\-(x_1+x_2)(1-x_1^2)\end{pmatrix}$$

$$= -2x_2^2(1-x_1^2)$$

$|x_1| < 1$ 이면 $\dot V(x)$: NSD $\left(\begin{array}{l} x_2 \neq 0 \text{ 이면 } \dot V < 0 \\ x_2 = 0 \text{ 에서 } 0. \end{array} \right.$

$\boxed{\text{La Salle thm 이용}}$ $\rightarrow$ $|x_1|<1$ 인 곳에서 $\dot V = 0 \Leftrightarrow x_2 = 0$.

$x_2 = 0$, $|x_1|<1$ 인 곳에서 invariant set 이 원점으로만 되어 있는 것을 보이면 o.k.

$\dot x_1 = x_2(1-x_1^2) = 0$ $\quad \dot x_2 = -x_1(1-x_1^2)$

$\Rightarrow |x_1|<1$ 이므로 $x_1 = 0$ 이어야만 계속 $x_2 = 0$.

$\therefore$ origin is largest invariant set.

Origin is asymptotically stable.

4.16 $\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1^3 - x_2^3 \end{cases}$

let $V(x) = x_1^4 + 2x_2^2$ → PD fn, radially unbounded.

$\dot{V}(x) = 4x_1^3 \dot{x}_1 + 4x_2 \dot{x}_2 = 4x_1^3 x_2 - 4x_1^3 x_2 - 2x_2^4$

$= -2x_2^4$ → negative semi definite.

$\dot{V}(x) = 0$ only when $x_2 = 0$.

→ 원점만이 $x_2 = 0$ 인 곳의 invariant set 임을 보이면 o.k.,

at $x_2 = 0 \rightarrow \dot{x}_1 = 0$, $\dot{x}_2 = -x_1^3$

⇒ 계속 $x_2 = 0$ 인 set 에 머무르기 위해서는 $x_1 = 0$ 일 수 밖에 없다.

∴ origin is largest invariant set,

∴ system is globally asymptotically stable.

4.22 i) show A is Hurwitz $\Longleftrightarrow \exists P = P^T > 0$ s.t. $PA + A^T P = -C^T C$

($\Leftarrow$) let $V(x) = x^T P x > 0$ at $x \neq 0$ ($\because P > 0$).

$\dot{x} = Ax \Rightarrow x(t) = e^{At} x(0)$

$\dot{V}(x) = x^T A^T P x + x^T P A x = x^T (A^T P + PA) x$

$= -x^T C^T C x = -\|Cx\|_2^2$

$\dot{V}(x) \leq 0$, $\dot{V}(x) = 0$ only when $Cx(t) = 0$.

LaSalle's thm을 통해 $Cx(t) = 0$ 이 되는 invariant set이 원점인 것을 보이면 o.k. →

(A, C) observable 의 성질.
Details in the "linear system theory"

$Cx(t) = Ce^{At} x(0)$, (A, C) → observable.

⇒ $Cx(t) = 0$ only when $x(0) = 0$.

⇒ 원점이 largest invariant set

⇒ 원점은 asymptotically stable

LTI system 에서 원점이 asy. stable 과 A is Hurwitz는 동치 ∴ A is Hurwitz.

(⇒) assume A is Hurwitz.

let $P = \int_0^\infty e^{A^T\tau} C^T C e^{A\tau} d\tau \Rightarrow P = P^T$

A is Hurwitz ⇒ P 가 존재. (∵ 적분 가능하므로)

$$PA + A^TP = \int_0^\infty e^{A^T\tau} C^T C e^{A\tau} A \, d\tau + \int_0^\infty A^T e^{A^T\tau} C^T C e^{A\tau} d\tau$$
$$= \int_0^\infty \frac{d}{d\tau} e^{A^T\tau} C^T C e^{A\tau} d\tau$$
$$= e^{A^T\tau} C^T C e^{A\tau} \Big|_0^\infty = -C^TC.$$

P 가 PD인 것을 보이면 ok.

$$x^TPx = \int_0^\infty x^T e^{A^T\tau} C^T C e^{A\tau} x \, d\tau$$
$$= \int_0^\infty \| C e^{A\tau} x \|^2 d\tau > 0 \text{ when } Ce^{A\tau}x=0$$

(A,C) 가 observable 하기 때문에 $x=0$ 인 경우에만 $Ce^{A\tau}x = 0$.

∴ P is positive definite and symmetric.

ii) show if A is Hurwitz, P is unique

pf) assume P is not unique.

⇒ ∃ P와 다른 P^* 도 $P^*A + A^TP = -C^TC$ 를 만족한다고 가정.

$$\Rightarrow P^*A + A^TP = -C^TC$$
$$-)\ (PA + A^TP = -C^TC)\ (?)$$
$$A^T(P^* - P)A = 0, \quad (A)$$

A is Hurwitz ⇒ 모든 eigenvalue의 실수값이 음수 ⇒ eigenvalue 중엔 0이 없다

∴ A is nonsingular ⇒ $\exists A^{-1}$

$$(A^T)^{-1}\cancel{A^T}(P^*-P)A\cancel{(A)^{-1}} = (A^T)^{-1}0(A)^{-1} = 0$$

∴ $P^* - P = 0$

∴ P is unique

4.24 $\dot{x} = f(x) - kG(x)R^{-1}(x)G^{T}(x)\left(\frac{\partial V}{\partial x}\right)^T$

$V \rightarrow$ cont. diff & PD fn

$\frac{\partial V}{\partial x}f(x) + g(x) - \frac{1}{4}\frac{\partial V}{\partial x}G(x)R^{-1}(x)G^T(x)\left(\frac{\partial V}{\partial x}\right)^T = 0$

$\dot{V}(x) = \frac{\partial V}{\partial x}\dot{x} = \frac{\partial V}{\partial x}f(x) - \frac{\partial V}{\partial x}kG(x)R^{-1}(x)G^T(x)\left(\frac{\partial V}{\partial x}\right)^T$

$= \left(\frac{1}{4} - k\right)\frac{\partial V}{\partial x}G(x)R^{-1}G^T(x)\left(\frac{\partial V}{\partial x}\right)^T - g(x)$

(a) 위에서 V : PD fn 이라 했으므로

$\dot{V} < 0$ 인 것만 보이면 원점은 asy. stable.

internet 에서 khalil 책의 errata를 찾아보니

R 의 조건이 nonsingular 가 아니라 positive

definite 으로 나와있습니다. 이 조건을 갖고

문제를 풀겠습니다.

$\dot{V} = \left(\frac{1}{4} - k\right)\left[G^T\frac{\partial V}{\partial x}^T\right]^T R^{-1}\left[G^T\frac{\partial V}{\partial x}^T\right] - g(x)$

R is PD $\Rightarrow R^{-1}$ is also PD

($\because$ eigen value 의 부호는 동일하기에)

$\Rightarrow \left[G^T\frac{\partial V}{\partial x}^T\right]^T R^{-1}\left[G^T\frac{\partial V}{\partial x}^T\right] \geq 0$, $-g(x) < 0$

$k \geq \frac{1}{4} \Rightarrow \dot{V} < 0$ $\therefore$ asy. stable

(b) 조건에서 $g(x) \rightarrow$ PSD, $k > \frac{1}{4}$.

$\Rightarrow \dot{V} = 0$ at $\frac{\partial V}{\partial x}G = 0$ and $g(x) = 0$.

그 이외에선 $\dot{V} < 0$. $\Rightarrow \frac{\partial V}{\partial x}G = g(x) = 0$ 인

largest invariant set 이 원점인 것을 보이면 됨.

$\dot{x} = f(x) - kG(x)R^{-1}(x)\left[\frac{\partial V}{\partial x}G\right]^T$

$\frac{\partial V}{\partial x}G = 0$ 을 만족하는 invariant set에서

$\dot{x} = f(x)$ ($\because \frac{\partial V}{\partial x}G = 0$), 조건에서

$\dot{x} = f(x)$ 을 해가 $g(x) = 0$ 을 만족 시키는

invariant set $\rightarrow$ 원점만.

$\therefore$ 원점만이 $\dot{V} = 0$ 을 만족시키는 invariant set.

$\therefore$ origin is asy. stable

global 하기 위해선 위의 두 조건중 하나가 성립하면서 V 가 radially unbounded 해야 함

3.17

Consider the initial-value problem

$x' = f(t, x)$ $\quad x(0) = x(t=0)$.

$D \subset \mathbb{R}^n$, contains $x = 0$

$x(t)$: solution of I.V.P. $\sim \subset D$, $\forall t \geq t_0$

f. $\|f(t,x)\|_2 \leq L\|x\|_2$ on $[t_0, \infty) \times D$

(a). show that

$$\left|\frac{d}{dt}[x^T(t)\, x(t)]\right| \leq 2L \cdot \|x(t)\|_2^2$$

$$\Rightarrow \quad \left|\frac{d}{dt}[x^T(t)\, x(t)]\right| = \left|\frac{d}{dt}[x_1^2(t) + \dots + x_n^2(t)]\right|$$

$$= \left|2\left(x_1(t) \cdot \dot{x}_1(t) + \dots + x_n(t) \cdot \dot{x}_n(t)\right)\right|$$

$$= 2\left|x^T(t)\ \dot{x}(t)\right| \leq 2|x^T(t)||\dot{x}(t)| \leq 2\|x^T(t)\| L \|x\|_2$$

$$= 2L \cdot |x^T(t) \cdot x(t)|$$

$$\leq 2L \cdot \|x(t)\|_2^2$$

(b) show that

$$\|x_0\|_2 \exp[-L(t-t_0)] \leq \|x(t)\|_2 \leq \|x_0\|_2 \exp[L(t-t_0)]$$

$\Rightarrow$ Actually, $x^T(t)\ x(t) \equiv \|x(t)\|_2^2$

$\therefore$ let $\|x(t)\|_2^2 = X(t) \geq 0$

then from (a),

$$\left|\frac{d}{dt} X(t)\right| \leq 2L\, X(t)$$

if $X(t) = 0$, trivial. !

if $\Psi(t) \neq 0$,

$$\left| \frac{\dot{\Psi}(t)}{\Psi(t)} \right| \le 2L$$

$$-2L \le \frac{\dot{\Psi}(t)}{\Psi(t)} \le 2L$$

take Integral on $[t_0, t]$

then

$$-2L(t-t_0) \le \ln \frac{\Psi(t)}{\Psi(t_0)} \le 2L(t-t_0)$$

$$\Psi(t_0)\exp(-2L(t-t_0)) \le \Psi(t) \le \Psi(t_0) \cdot \exp(2L(t-t_0))$$

$$\|x_0\|_2^2 \exp(-2L(t-t_0)) \le \|x(t)\|_2^2 \le \|x_0\|_2^2 \exp(2L(t-t_0))$$

finally,

$$\|x_0\|_2 \cdot \exp(-L(t-t_0)) \le \|x(t)\|_2 \le \|x_0\|_2 \cdot \exp(L(t-t_0))$$

4.35. Let α be a class $\mathcal{K}$ function on $[0, a)$

show that. $\alpha(r_1+r_2) \le \alpha(2r_1) + \alpha(2r_2)$, $\forall r_1, r_2 \in [0, \frac{a}{2})$

$\Rightarrow$ If $r_1 \le r_2$

$\alpha(2r_1) \le \alpha(r_1+r_2) \le \alpha(2r_2)$ ($\because$ α : strictly increasing. If $r_1 = r_2 = 0 \sim$ equality)

$\le \alpha(2r_2) + \alpha(2r_1)$

Similarly, ($\because$ $\alpha(0) = 0$ & increasing.)

If $r_2 \le r_1$

$\alpha(2r_2) \le \alpha(r_1+r_2) \le \alpha(2r_1)$

$\le \alpha(2r_1) + \alpha(2r_2)$

$\therefore$

$\alpha(r_1+r_2) \le \alpha(2r_1) + \alpha(2r_2)$ //

Show that the system.

$x_1' = -ax_1 + b. \quad x_2' = -cx_2 + x_1(\alpha - \beta x_1 x_2)$

all coefficients : positive , has a globally exponentially stable equilibrium point.

$\Rightarrow$ at first find. equil^m point

$-a\bar{x}_1 + b = 0 \rightarrow \underline{\bar{x}_1 = \frac{b}{a}}$

$-c\bar{x}_2 + \bar{x}_1(\alpha - \beta\bar{x}_1\bar{x}_2) = 0 \rightarrow \bar{x}_2 = \frac{\alpha\bar{x}_1}{c + \beta\bar{x}_1^2} = \frac{\alpha\cdot\frac{b}{a}}{c + \beta\frac{b^2}{a^2}} = \underline{\frac{\alpha\cdot ab}{a^2c + \beta b^2}}$

then let. $y_1 = x_1 - \bar{x}_1, \quad y_2 = x_2 - \bar{x}_2$

$\dot{y}_1 = \dot{x}_1 = -ax_1 + b = -a\cdot(y_1 + \bar{x}_1) + b = -ay_1 \quad \cdots (1)$

$\dot{y}_2 = \dot{x}_2 = -cx_2 + x_1(\alpha - \beta x_1 x_2)$

$= -c\cdot(y_2 + \bar{x}_2) + (y_1 + \bar{x}_1)\{\alpha - \beta(y_1 + \bar{x}_1)(y_2 + \bar{x}_2)\}$

$= -c\cdot(y_2 + \bar{x}_2) + \alpha(y_1 + \bar{x}_1) - \beta\cdot(y_1 + \bar{x}_1)^2\cdot(y_2 + \bar{x}_2)$

$= -(c + \beta(y_1 + \bar{x}_1)^2)y_2 - c\bar{x}_2 + \alpha\cdot(y_1 + \bar{x}_1) - \beta(y_1 + \bar{x}_1)^2\cdot\bar{x}_2$

$= -\left(c + \beta(y_1 + \bar{x}_1)^2\right)y_2 + \left(-c\bar{x}_2 + \alpha\cdot(y_1 + \bar{x}_1) - \beta(y_1 + \bar{x}_1)^2\bar{x}_2\right) \quad \cdots (2)$

from. $y_1 = e^{-a\cdot(t - t_0)}\cdot y_1(0)$

then let. $c + \beta(y_1 + \bar{x}_1)^2 = A(t), \quad -c\bar{x}_2 + \alpha(y_1 + \bar{x}_1) - \beta(y_1 + \bar{x}_1)^2\bar{x}_2 = B(t)$

$\therefore \dot{y}_2 = -A(t)\cdot y_2 + B(t)$

$\therefore$ time variant Linear system.

$\Phi_{s_2} = e^{\int -A(t)\,dt}, \quad \phi(t,\tau) = e^{\int_\tau^t -A(t)\,dt}$

time varying

then. $y_2 = e^{\int_{t_0}^{t} -A(t)\,dt} \cdot y_2(0) + \int_{t_0}^{t} e^{\int_{\tau}^{t} -A(t)\,dt} \cdot B(\tau)\, d\tau.$

$A(t) \to$ const.

$A(t) > 0$ $\quad\therefore$ first term : exponential term & goes to zero as $t \to \infty$

second term : " & "

$\therefore$ Although we don't know exact form of y_2,
y_2 is composed of exponential terms and
goes to zero as t goes to infinity.

You need more analysis.

$$y_1 = e^{-a(t-t_0)} \cdot y_1(0)$$

$\therefore$ Globally Exponentially Stable !!

How about using Lyapunov Stability?

investigate input-to-state stability,

(1) $\dot{x}_1 = -x_1 + x_1^2 x_2$

$\dot{x}_2 = -x_1^3 - x_2 + u$

$\Rightarrow$ if $u=0$

let $V = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 \geq 0$: radially unbounded.

then $\dot{V} = x_1\dot{x}_1 + x_2\dot{x}_2$

$= -x_1^2 + \cancel{x_1^3 x_2} - \cancel{x_1^3 x_2} - x_2^2 < 0$, $\forall x$ except origin

$\therefore$ G.A.S.

if $u \neq 0$

$\dot{V} = -x_1^2 - x_2^2 + u \cdot x_2$

$= -\|x\|_2^2 + u \cdot x_2 \leq -\|x\|_2^2 + \|x\|_2 \cdot u.$

$= -(1-\theta)\|x\|_2^2 - \theta\|x\|_2^2 + \|x\|_2|u|$

$\leq -(1-\theta)\|x\|_2^2$ if $-\theta\|x\|_2^2 + \|x\|_2|u| < 0$

(where $0<\theta<1$) i.e. $\|x\| > \frac{|u|}{\theta}$

$\therefore$ Input-to-state stable, w/ $\gamma(r) = \frac{r}{\theta}$ //

(4)

$$\dot{x}_1 = (x_1 - x_2 + u)(x_1^2 - 1)$$

$$\dot{x}_2 = (x_1 + x_2 + u)(x_1^2 - 1)$$

$\Rightarrow$ let. $V = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2$.

then $V(0) = 0$ & $V(x) > 0$. $\forall x$, except 0

$$\dot{V} = x_1\dot{x}_1 + x_2\dot{x}_2$$

$$= x_1(x_1 - x_2 + u)(x_1^2 - 1) + x_2(x_1 + x_2 + u)(x_1^2 - 1)$$

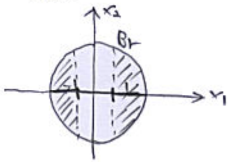
$$= x_1(x_1 + u)(x_1^2 - 1) - \cancel{x_1 x_2 (x_1^2 - 1)}$$

$$+ x_2(x_2 + u)(x_1^2 - 1) + \cancel{x_1 x_2 (x_1^2 - 1)}$$

$$= (x_1^2 + x_2^2)(x_1^2 - 1) + (x_1 + x_2)u(x_1^2 - 1)$$

if $u = 0$, equilibrium point $= 0$

then.



for $-1 \leq x_1 \leq 1$

$\dot{V}(x) \leq 0$

for $x_1 \leq -1$, $x_1 \geq 1$

$\dot{V}(x) \geq 0$

$\therefore$ locally stable about origin.

So consider $\|x\| < 1$, (i.e. locally)

then $\dot{V} = (x_1^2 + x_2^2)(x_1^2 - 1) + (x_1 + x_2)u \cdot (x_1^2 - 1)$

$$\leq -(x_1^2 + x_2^2)(1 - x_1^2) - (x_1 + x_2)u(1 - x_1^2)$$

$$\dot{V} = -(x_1^2+x_2^2)(1-x_1^2) - (x_1+x_2)\,u\,(1-x_1^2)$$

$$\le -(x_1^2+x_2^2) - (x_1+x_2)\cdot u$$

$$= -(x_1^2+x_2^2)(1-\theta) - \theta(x_1^2+x_2^2) + |x_1+x_2|\,u \quad (0 \le \theta \le 1)$$

$$\le -(x_1^2+x_2^2)(1-\theta) - \theta(x_1^2+x_2^2) + 2\|x\|_2 u.$$

$$= -\|x\|_2^2(1-\theta) - \theta\|x\|_2^2 + 2\|x\|_2 \cdot u$$

$\therefore$ locally ISS. w/ $-\theta\|x\|_2^2 + 2\|x\|_2 u \le 0$

$\|x\|_2 \ge \frac{2|u|}{\theta}$

ISS X

(i.e. $\gamma(r) = \frac{2r}{\theta}$)

Are all conditions of Exercise 4.60 satisfied?

3.17 Consider the initial-value problem (3.1) and let $D \subset R^n$ be a domain that contains $x=0$. Suppose $x(t)$, the solution (3.1), belongs to D for all $t \geq t_0$. and $\| f(t,x)\|_2 \leq L\|x\|_2$ on $[t_0, \infty) \times D$. Show that

(a) $\left| \frac{d}{dt}[x^T(t)x(t)] \right| \leq 2L\|x(t)\|_2^2$

$\dot{x} = f(t,x) \qquad x(t_0) = x_0 \qquad -(3.1)$

$\frac{d}{dt}(x^T(t)x(t)) = 2x^T(t)\dot{x}(t)$

$= 2x^T(t)f(t,x) \qquad — ①$

문제에서 $\| f(t,x)\|_2 \leq L\|x\|_2$ 이므로 Lipschitz 조건을 이용하면

$\| f(t,x) - f(t,0)\|_2 \leq L\|(x-0)\|_2$

$\rightarrow f(t,0) = 0$ 임을 알 수 있다

① 식에서. $\frac{d}{dt}(x^T(t)x(t)) = 2x^T(t)f(t,x)$

$\rightarrow \left| \frac{d}{dt}[x^T(t)x(t)] \right| \leq 2\|x\|_2 \|f(t,x)\|_2$

$\leq 2L\|x\|_2^2$

$\therefore \left| \frac{d}{dt}[x^T(t)x(t)] \right| \leq 2L\|x\|_2^2 \qquad — ②$

(b) $\|x_0\|_2 \exp[-L(t-t_0)] \leq \|x(t)\|_2 \leq \|x_0\|_2 \exp[L(t-t_0)]$

let $\begin{cases} V(t) = x^T(t)x(t) \\ V_0 = x_0^T x_0 = \|x_0\|_2^2 \end{cases}$ if $x(t)=0$, trivial

② 식에 의해, $-2L\|x\|_2^2 \leq \frac{d}{dt}[x^T(t)x(t)] \leq 2L\|x\|_2^2$

$-2LV(t) \leq \dot{V}(t) \leq 2LV(t)$

if $x(t) \neq 0$

$-2L \leq \frac{\dot{V}(t)}{V(t)} \leq 2L$

$-\int_{t_0}^{t} 2L\,dt \leq \int_{V_0}^{V} \frac{dV}{V} \leq \int_{t_0}^{t} 2L\,dt$

$-2L(t-t_0) \leq \ln\left(\frac{V(t)}{V_0}\right) \leq 2L(t-t_0)$

$\exp[-2L(t-t_0)] \leq \frac{V(t)}{V_0} \leq \exp[2L(t-t_0)]$

$\Rightarrow V_0 \exp[-2L(t-t_0)] \leq V(t) \leq V_0 \exp[2L(t-t_0)]$

$\|x_0\|_2^2 \exp[-2L(t-t_0)] \leq \|x\|_2^2 \leq \|x_0\|_2^2 \exp[2L(t-t_0)]$

$\therefore \|x_0\|_2 \exp[-L(t-t_0)] \leq \|x(t)\|_2 \leq \|x_0\|_2 \exp[L(t-t_0)]$

4.35 Let α be a class K function on $[0, a)$.

Show that $\alpha(r_1+r_2) \le \alpha(2r_1) + \alpha(2r_2)$ $\quad \forall r_1, r_2 \in [0, \frac{a}{2})$

① $r_1 \ge r_2$, $\quad r_1+r_2 \le 2r_1$ $\quad \forall r_1+r_2 \in [0, a)$,

$\alpha(r_1+r_2) \le \alpha(2r_1) \le \alpha(2r_1) + \alpha(2r_2)$

② $r_1 \le r_2$, $\quad r_1+r_2 \le 2r_2$ $\quad \forall r_1+r_2 \in [0, a)$

$\alpha(r_1+r_2) \le \alpha(2r_2) \le \alpha(2r_2) + \alpha(2r_1)$

$\therefore$ $\alpha(r_1+r_2) \le \alpha(2r_1) + \alpha(2r_2)$ $\quad \forall r_1, r_2 \in [0, \frac{a}{2})$

위의 부등식은 항상 성립한다.

4.49 Show that the system $\begin{cases} \dot{x}_1 = -a x_1 + b \\ \dot{x}_2 = -c x_2 + x_1(\alpha - \beta x_1 x_2) \end{cases}$

where all coefficients are positive, has a globally exponentially stable equilibrium point.

① $- \dot{x}_1 = -a x_1 + b = 0 \quad \rightarrow \quad \bar{x}_1 = \frac{b}{a}$

② $- \dot{x}_2 = -c x_2 + x_1(\alpha - \beta x_1 x_2) = 0$

$$-c x_2 + \alpha\left(\frac{b}{a}\right) - \beta\left(\frac{b}{a}\right)^2 x_2 = 0 \quad \rightarrow \quad \bar{x}_2 = \frac{\alpha\left(\frac{b}{a}\right)}{c + \beta\left(\frac{b}{a}\right)^2}$$

equilibrium point $(\bar{x}_1, \bar{x}_2) = \left(\frac{b}{a}, \frac{\alpha(b/a)}{c+\beta(b/a)^2}\right)$

equilibrium point 가 원점이 되도록 하자.

$$\begin{cases} y_1 = x_1 - \bar{x}_1 \\ y_2 = x_2 - \bar{x}_2 \end{cases}$$

② 식에 의해, $-c\bar{x}_2 + \alpha \bar{x}_1 - \beta \bar{x}_1^2 \bar{x}_2 = 0$

$$\rightarrow \begin{cases} \dot{y}_1 = -a y_1 \\ \dot{y}_2 = -c(y_2 + \bar{x}_2) + (y_1 + \bar{x}_1)(\alpha - \beta(y_1 + \bar{x}_1)(y_2 + \bar{x}_2)) \end{cases}$$

$$= -c y_2 - c\bar{x}_2 + (y_1 + \bar{x}_1)(\alpha - \beta y_1 y_2 - \beta y_1 \bar{x}_2 - \beta \bar{x}_1 y_2 - \beta \bar{x}_1 \bar{x}_2)$$

$$= -c y_2 - c\bar{x}_2 + \alpha y_1 - \beta y_1^2 y_2 - \beta y_1^2 \bar{x}_2 - \beta \bar{x}_1 y_1 y_2 - \beta \bar{x}_1 \bar{x}_2 y_1$$
$$+ \alpha \bar{x}_1 - \beta \bar{x}_1 y_1 y_2 - \beta y_1 \bar{x}_1 \bar{x}_2 - \beta \bar{x}_1^2 y_2 - \beta \bar{x}_1^2 \bar{x}_2$$

$$= \alpha y_1 - c y_2 - 2\beta \bar{x}_1 \bar{x}_2 y_1 - 2\beta \bar{x}_1 y_1 y_2 - \beta \bar{x}_1^2 y_2 - \beta \bar{x}_2 y_1^2 - \beta y_1^2 y_2$$

$$= \alpha y_1 - [c + \beta(y_1^2 + 2\bar{x}_1 y_1 + \bar{x}_1^2)] y_2 - \beta \bar{x}_2 y_1^2 - 2\beta \bar{x}_1 \bar{x}_2 y_1$$

$$= \alpha y_1 - [c + \beta(y_1 + \bar{x}_1)^2] y_2 - \beta \bar{x}_2 y_1^2 - 2\beta \bar{x}_1 \bar{x}_2 y_1$$

$$\rightarrow \begin{cases} \dot{y}_1 = -a y_1 \\ \dot{y}_2 = \alpha y_1 - [c + \beta(y_1 + \bar{x}_1)^2] y_2 - \beta \bar{x}_2 y_1^2 - 2\beta \bar{x}_1 \bar{x}_2 y_1 \end{cases}$$

Lyapunov function $V = k_1 y_1^2 + k_2 y_2^2 + k_3 y_1^4$

$$\dot{V} = 2k_1 y_1 \dot{y}_1 + 2k_2 y_2 \dot{y}_2 + 4k_3 y_1^3 \dot{y}_1$$

$$= 2k_1 y_1(-a y_1) + 2k_2 y_2(\alpha y_1 - [c + \beta(y_1 + \bar{x}_1)^2] y_2 - \beta \bar{x}_2 y_1^2 - 2\beta \bar{x}_1 \bar{x}_2 y_1) + 4k_3 y_1^3(-a y_1)$$

$$= -2a k_1 y_1^2 - 4a k_3 y_1^4 + 2\alpha k_2 y_1 y_2 - 2k_2[c + \beta(y_1 + \bar{x}_1)^2] y_2^2 - 2\beta k_2 \bar{x}_2 y_1^2 y_2 - 4k_2 \beta \bar{x}_1 \bar{x}_2 y_1 y_2$$

$$= -2a k_1 y_1^2 - 4a k_3 y_1^4 - 2k_2 c y_2^2 - 2k_2 \beta(y_1 + \bar{x}_1)^2 y_2^2 + 2\alpha k_2 y_1 y_2$$
$$- 2\beta k_2 \bar{x}_2 y_1^2 y_2 - 4k_2 \beta \bar{x}_1 \bar{x}_2 y_1 y_2$$

$$\le -2a k_1 y_1^2 - 2c k_2 y_2^2 - 4a k_3 y_1^4 + 2k_2(\alpha - 2\beta \bar{x}_1 \bar{x}_2) y_1 y_2 - 2\beta k_2 \bar{x}_2 y_1^2 y_2$$

$$= -2a k_1 y_1^2 - 2c k_2 y_2^2 - 4a k_3 y_1^4 + 2k_2 A y_1 y_2 - 2k_2 B y_1^2 y_2$$

where $\begin{cases} A = \alpha - 2\beta \bar{x}_1 \bar{x}_2 & > 0 \\ B = \beta \bar{x}_2 & > 0 \end{cases}$ show

$$\le -ak_1y_1^2 - ck_2y_2^2 - ak_3y_1^4$$
$$\quad -ak_1y_1^2 - ck_2y_2^2 - 3ak_3y_1^4 + 2k_2|A||y_1||y_2| + 2k_2|B|y_1^2|y_2|$$
$$= -ak_1y_1^2 - ck_2y_2^2 - ak_3y_1^4$$
$$\quad -\left(ak_1y_1^2 - 2k_2A|y_1||y_2| + \frac{c}{2}k_2y_2^2\right)$$
$$\quad -\left(3ak_3y_1^4 - 2k_2B|y_2|y_1^2 + \frac{c}{2}k_2y_2^2\right)$$
$$= -ak_1y_1^2 - ck_2y_2^2 - ak_3y_1^4 - \begin{bmatrix}|y_1| & |y_2|\end{bmatrix}\begin{bmatrix}ak_1 & -k_2A \\ -k_2A & \frac{ck_2}{2}\end{bmatrix}\begin{bmatrix}|y_1| \\ |y_2|\end{bmatrix}$$
$$\quad - \begin{bmatrix}|y_1|^2 & |y_2|\end{bmatrix}\begin{bmatrix}3ak_3 & -k_2B \\ -k_2B & \frac{ck_2}{2}\end{bmatrix}\begin{bmatrix}|y_1|^2 \\ |y_2|\end{bmatrix}$$

let. $Q_1 = \begin{bmatrix}ak_1 & -k_2A \\ -k_2A & \frac{ck_2}{2}\end{bmatrix}$ $\quad Q_2 = \begin{bmatrix}3ak_3 & -k_2B \\ -k_2B & ck_2/2\end{bmatrix}$

$$\begin{cases} ak_1 > 0 \rightarrow k_1 > 0 \\ \frac{ac}{2}k_1k_2 - k_2^2A^2 > 0 \rightarrow k_2\left(\frac{ac}{2}k_1 - A^2k_2\right) > 0 \end{cases}$$
$$k_2 > 0 \text{ and } k_2 < \frac{ac}{2A^2}k_1 \Rightarrow 0 < k_2 < \frac{ac}{2A^2}k_1$$

위의 조건을 만족할때 Q_1 : positive definite

$$\begin{cases} 3ak_3 > 0 \rightarrow k_3 > 0 \\ \frac{3}{2}ack_2k_3 - k_2^2B^2 > 0 \rightarrow k_2\left(\frac{3}{2}ack_3 - k_2B^2\right) > 0 \end{cases}$$
$$k_2 > 0 \text{ and } k_2 < \frac{3ac}{2B^2}k_3 \Rightarrow 0 < k_2 < \frac{3ac}{2B^2}k_3$$

위의 조건 만족할때 Q_2 : positive definite

$$\therefore \dot{V} \le -ak_1y_1^2 - ck_2y_2^2 - ak_3y_1^4$$
$$\le -M(k_1y_1^2 + k_2y_2^2 + k_3y_1^4) \quad \leftarrow M = \min\{a, c\}$$
$$= -MV$$
$$\Rightarrow \dot{V} \le -MV$$

$$\begin{cases} k_1 > 0,\ k_3 > 0. \\ 0 < k_2 < N, \text{ where } N = \min\left\{\frac{ac}{2A^2}k_1, \frac{3ac}{3B^2}k_3\right\} \end{cases}$$

$\Rightarrow$ Q_1 과 Q_2 가 positive definite

만족하면, $\dot{V} \le -MV$ 이므로, origin 은 globally exponentially stable 하다.

4.55 For each of the following systems, investigate input-to-state stability.

(1) $\begin{cases} \dot{x}_1 = -x_1 + x_1^2 x_2 \\ \dot{x}_2 = -x_1^3 - x_2 + u \end{cases}$

① $u=0$ 일때, $\begin{cases} \dot{x}_1 = -x_1 + x_1^2 x_2 \\ \dot{x}_2 = -x_1^3 - x_2 \end{cases}$

→ equilibrium point $(x_1, x_2) = (0,0)$

Lyapunov function $V(x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 > 0$

$$\dot{V}(x) = x_1\dot{x}_1 + x_2\dot{x}_2$$
$$= x_1(-x_1 + x_1^2 x_2) + x_2(-x_1^3 - x_2)$$
$$= -x_1^2 + \cancel{x_1^3 x_2} - \cancel{x_1^3 x_2} - x_2^2 = -(x_1^2 + x_2^2) < 0$$

∴ the origin 은 globally asymptotically stable 이다

② $u \neq 0$ 일때

$$\dot{V}(x) = x_1\dot{x}_1 + x_2\dot{x}_2$$
$$= x_1(-x_1 + x_1^2 x_2) + x_2(-x_1^3 - x_2 + u)$$
$$= -x_1^2 - x_2^2 + u x_2$$
$$\le -x_1^2 - x_2^2 + |x_2||u|$$
$$= -(1-\theta)(x_1^2 + x_2^2) - \theta(x_1^2 + x_2^2) + |x_2||u| \quad \text{where } 0<\theta<1$$

$-\theta(x_1^2 + x_2^2) + |x_2||u| \le 0$ 이 되도록 하는 조건을 구하자.

$$-\theta x_1^2 - \theta x_2^2 + |x_2||u| \le 0$$
$$|x_2|(-\theta|x_2| + |u|) \le 0$$
$$|x_2| \ge \frac{|u|}{\theta}$$

$|x_2| \le \frac{|u|}{\theta}$ 일때, $-\theta x_1^2 + \frac{|u|^2}{\theta} \le 0$

$$x_1^2 \ge \frac{|u|^2}{\theta^2}$$
$$|x_1| \ge \frac{|u|}{\theta}$$

∴ $|x_2| \ge \frac{|u|}{\theta}$ or $|x_2| \le \frac{|u|}{\theta}$ and $|x_1| \ge \frac{|u|}{\theta}$

→ $\max\{|x_1|, |x_2|\} \ge \max\{\frac{|u|}{\theta}, \frac{|u|}{\theta}\} = \frac{|u|}{\theta}$

the norm $\|x\|_\infty = \max\{|x_1|, |x_2|\}$

the class K function $\rho(r) = \max\{\frac{r}{\theta}, \frac{r}{\theta}\} = \frac{r}{\theta}$

Simply,

$\dot{V} \le -(1-\theta)\|x\|_2^2 - \theta\|x\|_2^2 + \|x\|_2|u|$

$\le -(1-\theta)\|x\|_2^2$, $\forall \|x\|_2 \ge \frac{|u|}{\theta}$

⇒ $\dot{V}(x) \le -(1-\theta)(x_1^2 + x_2^2) \quad \forall \|x\|_\infty \ge \rho(|u|) = \frac{|u|}{\theta}$

$\forall \|x\|_2 \ge \frac{|u|}{\theta}$

$V(x)$가 positive definite 하고 radially unbounded 하므로 ($\because$ GAS 이므로)
Lemma 4.3 에 의해서, 다음 부등식을 얻을 수 있다.

$$\alpha_1(\|x\|) \le V(x) \le \alpha_2(\|x\|)$$

Theorem 4.19 에 의해,

$$\alpha_1(\|x\|) \le V(x) \le \alpha_2(\|x\|)$$

$$\dot{V} \le -(1-\theta)(x_1^2+x_2^2) = -W_3(x) \qquad \forall \|x\|_\infty \ge \rho(|u|)$$

이므로 input-to-state stable 이다. (with $\gamma(r) = \frac{r}{\theta}$)

$$V(x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 \le \frac{1}{2}\|x\|_\infty^2 + \frac{1}{2}\|x\|_\infty^2$$

$$V(x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 \ge \begin{cases} \frac{1}{2}|x_1|^2 = \frac{1}{2}\|x\|_\infty^2 & \text{if } |x_2| \le |x_1| \\ \frac{1}{2}|x_2|^2 = \frac{1}{2}\|x\|_\infty^2 & \text{if } |x_2| \ge |x_1| \end{cases}$$

the class K_∞ function, $\begin{cases} \alpha_1(r) = \frac{1}{2}r^2 \\ \alpha_2(r) = \frac{1}{2}r^2 + \frac{1}{2}r^2 = r^2 \end{cases}$

(4) $\begin{cases} \dot{x}_1 = (x_1 - x_2 + u)(x_1^2 - 1) \\ \dot{x}_2 = (x_1 + x_2 + u)(x_1^2 - 1) \end{cases}$

① $u=0$ 일때, $\begin{cases} \dot{x}_1 = (x_1 - x_2)(x_1^2 - 1) = x_1^3 - x_1 - x_1^2 x_2 + x_2 \\ \dot{x}_2 = (x_1 + x_2)(x_1^2 - 1) = x_1^3 - x_1 + x_1^2 x_2 - x_2 \end{cases}$

equilibrium point 가 $(0,0)$ 이외에도 $(1,1)$, $(-1,-1)$ 등이 존재하므로,

Not globally asymptotically stable 이다.

그러므로 이 system은 global input-to-state stable 하지않다.

② Local input-to-state stable 을 check해보자.

origin 에 대해 위의 system 의 Jacobian matrix A 를 구하면 다음과 같다

$$A = \left[\begin{matrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{matrix}\right]\Bigg|_{x=0} = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}$$

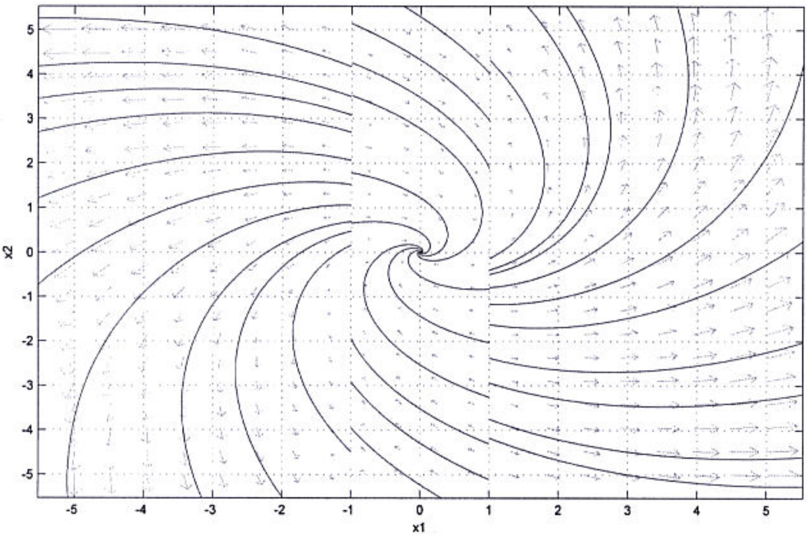
A가 Hurwitz 하므로, 이 system은 exponentially stable 하다.

Lemma 4.6 에 의해, Local input-to-state stable 하다.

all conditions of locally ISS are satisfied?

u=0 일때 phase portrait.

x1 ' = (x1 - x2) (x1^2 - 1)
x2 ' = (x1 + x2) (x1^2 - 1)



origin 근처에서만 수렴하고 그 외의 부분은 발산함을 알 수 있다.

input 의 boundary가 있을 때 즉, input이 bound 되어있을 때 출력도 bound 되어 있어야 input-to-state stable 이라 할수있다. 하지만 이 system은 input이 bound 되어도 state 가 발산하므로 input-to-state stable 이라 할 수 없다.

작은 영역, 원점 주변 영역에 대해서만 locally input-to-state stable 이라 할 수 있다