[Exercises 6] Samples

6.1 Verify that a function in the sector $[K_1, K_2]$ can be transformed into a function in the sector $[0, \infty]$ by input feedforward followed by output feedback, as shown in Figure 6.7.

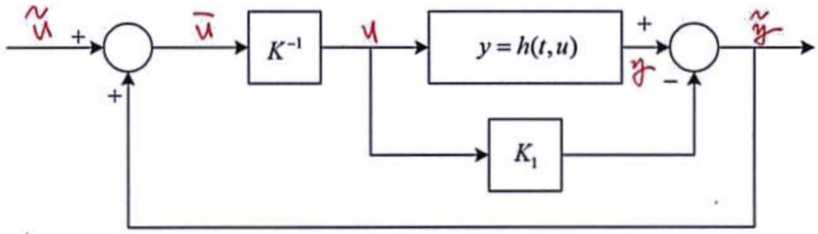


Figure 6.7

$\therefore \tilde{y} = h(t,u) - K_1 u$

Let $\bar{u} = Ku$, $K = K_2 - K_1$

$\therefore \bar{u} = \tilde{u} + \tilde{y}$

By $[h(t,u) - K_1 u]^T [h(t,u) - K_2 u] \le 0$ (6.5),

$[\ \ ]^T[\ \ ] = \tilde{y}^T[h(t,u) - K_2 u] = \tilde{y}^T[h(t,u) - (K+K_1)u] = \tilde{y}^T[\tilde{y} - Ku]$

$= \tilde{y}^T[\tilde{y} - \bar{u}] = \tilde{y}^T[-\tilde{u}] \le 0$

$\therefore \tilde{y}^T \tilde{u} \ge 0$.

6.2. Consider the system

$$a\dot{x} = -x + \frac{1}{k}h(x) + u, \quad y = h(x)$$

where a and k are positive constants and $h \in [0, k]$. Show that the system is passive with $V(x) = a\int_0^x h(\sigma)d\sigma$ as the storage function.

$$- \dot{V} = \frac{\partial V}{\partial x} f(x, u) = \frac{\partial}{\partial x}\left(a \int_0^x h(\sigma)\, d\sigma\right) \cdot \frac{1}{a}\left(-x + \frac{1}{k} h(x) + u\right)$$

$$= \frac{\partial}{\partial x}\int_0^x h(\sigma)\, d\sigma \cdot \left(-x + \frac{1}{k} h(x) + u\right)$$

$$= h(x) \cdot \left(-x + \frac{1}{k} h(x) + u\right)$$

$$= uh(x) - h(x)\left(x - \frac{1}{k}h(x)\right) \leq uh(x)$$

where $x - \frac{1}{k}h(x) \geq x - \frac{1}{k}\cdot kx = 0$, $\forall x \geq 0$

$\therefore uy \geq \dot{V}$

If $x \leq 0$, $x - \frac{1}{k}h(x) \leq 0$ & $-h(x) \geq 0$

$\therefore -h(x)\left(x - \frac{1}{k}h(x)\right) \leq 0$.

6.4. Consider the system

$\dot{x}_1 = x_2,\ \dot{x}_2 = -h(x_1) - ax_2 + u,\ y = kx_2 + u$

Where $a > 0$, $k > 0$, $h \in [\alpha_1, \infty]$, and $\alpha_1 > 0$. Let $V(x) = k\int_0^{x_1} h(s)ds + x^T P x$, where

$p_{11} = ap_{12}$, $p_{22} = k/2$, and $0 < p_{12} < \min\{2\alpha_1, ak/2\}$. Using $V(x)$ as a storage function,

show that the system is strictly passive.

- $x^T P x = [x_1 \quad x_2]\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = [P_{11}x_1 + P_{12}x_2 \quad P_{12}x_1 + P_{22}x_2]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$= P_{11}x_1^2 + P_{12}x_1x_2 + P_{12}x_1x_2 + P_{22}x_2^2$

$\dot{V} = \frac{\partial}{\partial x_1}V\cdot f_1 + \frac{\partial}{\partial x_2}V\cdot f_2 = (K\cdot h(x_1) + 2P_{11}x_1 + 2P_{12}x_2)x_2 + (2P_{12}x_1 + 2P_{22}x_2)\dot{x}_2$

$= K\cdot h(x_1)x_2 + 2P_{11}x_1x_2 + 2P_{12}x_2^2 - 2P_{12}x_1h(x_1) - 2P_{22}x_2 h(x_1) - 2aP_{12}x_1x_2$
$- 2aP_{22}x_2^2 + 2uP_{12}x_1 + 2uP_{22}x_2$

$= K\cdot h(x_1)x_2 + 2aP_{12}x_1x_2 + 2P_{12}x_2^2 - 2P_{12}x_1h(x_1) - Kx_2h(x_1)$
$- 2aP_{12}x_1x_2 - aKx_2^2 + 2uP_{12}x_1 + uKx_2 \quad (\because P_{11} = aP_{12}, \quad P_{22} = \frac{K}{2})$

$= 2P_{12}(u - h(x_1))x_1 + (2P_{12} - aK)x_2^2 + uKx_2$

since. $uy = uKx_2 + u^2$, $\quad uKx_2 = uy - u^2$

$\dot{V} = uy - u^2 + 2P_{12}(u - h(x_1))x_1 + (2P_{12} - aK)x_2^2$

where, $(2P_{12} - aK)x_2^2 \le 0$ $\quad (\because P_{12} < \min(2\alpha_1, \frac{aK}{2}))$ $\quad \left((2P_{12} - aK)x_2^2 = 0 \text{ when. } x_2 = 0.\right)$

$\dot{V} \le uy - u^2 + 2P_{12}(u - h(x_1))x_1 = uy - u^2 + 2P_{12}ux_1 - 2P_{12}h(x_1)x_1$

$= uy - (u - P_{12}x_1)^2 + (P_{12}x_1)^2 - 2P_{12}h(x_1)x_1$

$\le uy - P_{12}x_1(2h(x_1) - P_{12}x_1) \overset{uy}{\le} + (p_{12} - 2\alpha_1)p_{12}x_1^2$

If $x_1 < 0$? (where, $2h(x_1) - P_{12}x_1 > 2h_1(x_1) - 2\alpha_1 x_1 \ge 2\alpha_1 x_1 - 2\alpha_1 x_1 = 0$)

$\therefore \dot{V} \le uy - P_{12}x_1(2h(x_1) - P_{12}x_1) \le uy$ $\quad \therefore u^T y \ge \dot{V} + \psi(x)$. for some positive definite function ψ. $x_1 x_2 \ne 0$.

이때, $x_1 = 0$ 이면, $\dot{V} = uy$.

so this system is strictly passive.

6.10. Consider the equations of motion of an m-link robot, described in Exercise 1.4. Assume that $P(q)$ is a positive definite function of q and $g(q)=0$ has an isolated root at $q=0$.

(a) Using the total energy $V=\frac{1}{2}\dot{q}^T M(q)\dot{q}+P(q)$ as a storage function, show that the map from u to $\dot{q}$ is passive.

- 운동 방정식은

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + D\dot{q} + g(q) = u, \quad y = \dot{q}$$

이고, V 의 미분은.

$$\dot{V} = \dot{q}^T M(q)\ddot{q} + \frac{1}{2}\dot{q}^T \dot{M} q + \frac{\partial P}{\partial q}\dot{q}$$

$$= \dot{q}^T[u - C(q,\dot{q})\dot{q} - D\dot{q} - g(q)] + \frac{1}{2}\dot{q}^T\dot{M}\dot{q} + g^T(q)\dot{q}$$

$$= y^T u - y^T D y \le y^T u, \quad (\because \dot{M}-2C \text{ is skew-symmetric matrix})$$

$\therefore$ This system is passive.

(b) With $u = -K_d\dot{q} + v$, where K_d is positive diagonal constant matrix, show that the map from v to $\dot{q}$ is output strictly passive.

이 경우, $\dot{V} \le y^T u = y^T(-K_d y + v)$

$= y^T v - y^T K_d y \Rightarrow v^T y \ge \dot{V} + y^T(\underline{K_d y})_{>0}$

$y^T v = y^T u + y^T K_d y > y^T u$.

$\therefore \dot{V} \le y^T v$. 이므로, output strictly passive.

$+ y^T \rho(y)$

(c) Show that $u = -K_d \dot{q}$, where K_d is a positive diagonal constant matrix, makes the origin $(q = 0, \dot{q} = 0)$ asymptotically stable. Under what additional conditions will it globally asymptotically stable?

If $u = 0$,

$\dot{V} \le -\dot{q}^T K_d \dot{q} = -\dot{q}^T K_d \dot{q} \le 0$.

$\equiv$: identically (ex) $g(t) \equiv 0$ means $g(t)$ is zero for all t.

$\dot{V} = 0$ 이면, $\dot{q}(t) \equiv 0 \Rightarrow \ddot{q}(t) \equiv 0 \Rightarrow g(q(t)) \equiv 0$.

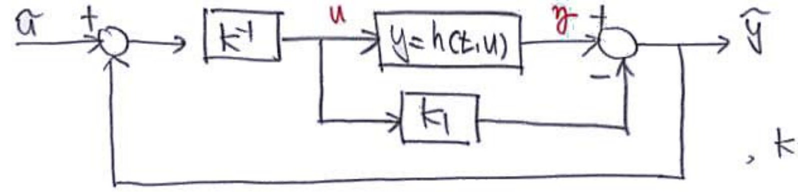
$\therefore q(t) \equiv 0$ 이다

따라서, the origin is asymptotically stable.

그리고, $q = 0$ 이 unique root of $g(q) = 0$ 이고, $P(q)$ is radially unbounded 이라면, GAS 이다.

#6.1. sector $[k_1, k_2]$ $\xrightarrow{\text{into}}$ sector $[0, \infty]$

by input feedward followed by output feedback.



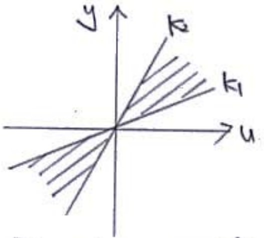
, $k = k_2 - k_1 = k^T > 0$

$\Rightarrow$ $u, \tilde{u}, y, \tilde{y} \in \mathbb{R}^p$ (i.e. same dimension)

from the shown structure,

$u = k^{-1}(\tilde{u} + \tilde{y})$, $\tilde{y} = y - k_1 u$.

for sector $[k_1, k_2]$



$[y - k_1 u]^T [y - k_2 u] \le 0$

for this modified system,

$\tilde{u}^T \tilde{y} = (ku - \tilde{y})^T (y - k_1 u) = (ku + k_1 u - y)^T (y - k_1 u)$

$= (y - k_1 u)^T (\underbrace{k}_{= k_2 - k_1} u + k_1 u - y)$ (∵ $\tilde{u}^T \tilde{y} \in$ scalar)

$= (y - k_1 u)^T (\boxed{2k_1 u} + k_2 u - y) = -(y - k_1 u)^T (y - k_2 u - \cancel{2k_1} u)$

$= \underbrace{-(y - k_1 u)^T (y - k_2 u)} + \underbrace{2(y - \cancel{k_1 u})^T (k_1 u)}$

i) 1st term: $-(y - k_1 u)^T (y - k_2 u) \ge 0$ (∵ sector $[k_1, k_2]$)

ii) 2nd term: $2(y - k_1 u)^T (k_1 u) \ge 0$

since for 1st quadrant : $y - k_1 u \ge 0$, $k_1 u \ge 0$

for 3rd quadrant : $y - k_1 u \le 0$, $k_1 u \le 0$

So, $\tilde{u}^T \tilde{y} \ge 0$. $\Longrightarrow$ this means this modified system is passive

i.e. in the sector $[0, \infty]$

#6.2. Consider the system.

$$a\dot{x} = -x + \frac{1}{k}h(x) + u. \qquad y = h(x)$$

$a, k > 0$. & constant. $\qquad h \in [0, k]$

show the system is passive with $V(x) = a\int_0^x h(\sigma)\,d\sigma$

as the storage function

$\Rightarrow$ $V(x)$: storage fcn. $\Rightarrow$ $V(x) \geq 0$

$$\dot{V} = \frac{\partial V}{\partial x}\cdot\dot{x} = a\cdot h(x)\dot{x}$$

$$= h(x)\cdot\left(-x + \frac{1}{k}h(x) + u\right)$$

$$= h(x)u + \frac{1}{k}h(x)^2 - h(x)x.$$

from $h \in [0, k]$,

for 1st quadrant, $\frac{1}{k}h(x)^2 - h(x)x \leq \frac{h(x)}{k}(kx) - h(x)x = 0$

for 3rd quadrant, $\frac{1}{k}h(x)^2 - h(x)x \leq \frac{1}{k}(kx)^2 - (kx)x = 0$

$\therefore$ $\dot{V} \leq h(x)u$ $\Rightarrow$ "passive" //

#6.7. $\dot{x}_1 = x_2$, $\dot{x}_2 = -h(x_1) - ax_2 + u$. $y = kx_2 + u$

$\left(a > 0,\ k > 0 \quad h \in [\alpha_1, \infty]. \quad \alpha_1 > 0. \right.$

$\left(V(x) = k \int_0^{x_1} h(s)ds + x^T P x. \right.$

$\left(p_{11} = a p_{12} \ , \ p_{22} = k/2 \ , \quad 0 < p_{12} < \min\{2\alpha_1 , \frac{ak}{2}\} \right.$

show the system is strictly passive

$$\Rightarrow \quad \dot{V} = \frac{\partial V}{\partial x} \cdot \dot{x} = k \cdot h(x_1) \cdot \dot{x}_1 + 2x^T P \dot{x}$$

$$= k \cdot h(x_1) x_2 + 2[x_1 \ x_2] \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}$$

$$= k \cdot h(x_1) x_2 + 2(a p_{12} x_1 + p_{12} \cdot x_2)\dot{x}_1 + 2\left(p_{12} x_1 + \frac{k}{2} x_2\right)\dot{x}_2$$

$$= k h(x_1) x_2 + 2(a \cdot p_{12} x_1 + p_{12} x_2) x_2 + 2\left(p_{12} x_1 + \frac{k}{2} x_2\right)\left(-h(x_1) - a x_2 + u\right)$$

$$= \cancel{k h(x_1) x_2} + 2(\cancel{a p_{12} x_1 x_2} + p_{12} x_2^2) + 2 p_{12} x_1 (-h(x_1) - \cancel{a x_2} + u)$$

$$+ k x_2 (-\cancel{h(x_1)} - a x_2 + u)$$

$$= (2p_{12} - ak) x_2^2 - 2p_{12} h(x_1) x_1 + 2p_{12} x_1 u + k x_2 u.$$

$$uy - \dot{V} = u.(\cancel{kx_2} + u) - (2p_{12} - ak) x_2^2 + 2p_{12} h(x_1) x_1 - 2p_{12} x_1 u - \cancel{k x_2 u}$$

$$= u^2 - (2p_{12} - ak) x_2^2 + 2p_{12} . x_1 (h(x_1) - u)$$

for $h \in [\alpha_1, \infty]$, $x_1 (h(x_1) - \alpha_1 x_1) \geq 0$

$$uy - \dot{V} \geq u^2 - (2p_{12} - ak) x_2^2 + 2p_{12} \alpha_1 \cdot x_1^2 - 2p_{12} x_1 u.$$

since $0 < p_{12} < \min\{2\alpha_1 , \frac{ak}{2}\}$.

$$uy - \dot{V} > u^2 + 2p_{12} \alpha_1 x_1^2 - 2p_{12} x_1 u.$$

$$= (u - p_{12} x_1)^2 + p_{12} (2\alpha_1 - p_{12}) x_1^2$$

#6.7. (continued)

$$u y - \dot{V} > (u - p_{12} x_1)^2 + p_{12}(2\alpha_1 - p_{12}) x_1^2$$

if $2\alpha_1 < \frac{ak}{2}$, $0 < p_{12} < 2\alpha_1$

$\therefore$ $u y - \dot{V} > (u - p_{12} x_1)^2 > 0$

if $2\alpha_1 > \frac{ak}{2}$. $0 < p_{12} < \frac{ak}{2} < 2\alpha_1$.

$\therefore$ $u y - \dot{V} > (u - p_{12} x_1)^2 > 0$ $(uy > \dot{V} + (\quad)^2)$

$\Rightarrow$ "strictly passive"

#6.10 m-link robot.

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + D\dot{q} + g(q) = u \quad , \quad q, u \in \mathbb{R}^m$$

$$M(q) = M^T(q) > 0 \quad , \quad D \geq 0$$

$$(\dot{M}-2C)^T = -(\dot{M}-2C) \sim \text{skew-symmetric mtx.}$$

$$g(q) = \left[\frac{\partial P(q)}{\partial q}\right]^T \quad , \quad P(q) \sim \text{P.D.} \quad g(q)=0 : \text{soln}, \ q=0$$

(a) $V = \frac{1}{2}\dot{q}^T M(q)\dot{q} + P(q)$. show the map from u to $\dot{q}$ is passive.

$\Rightarrow$ let $q = x_1$, $\dot{q} = x_2$

then $\dot{x}_1 = x_2$

$$\dot{x}_2 = M^{-1}(x_1)\{u - C(x_1,x_2)x_2 - D x_2 - g(x_1)\} \quad , \quad V = \frac{1}{2}x_2^T M(x_1) x_2 + P(x_1)$$

$$y = x_2$$

$$\dot{V} = \frac{\partial V}{\partial x}\cdot\dot{x} = \frac{\partial V}{\partial x_1}\dot{x}_1 + \frac{\partial V}{\partial x_2}\dot{x}_2 = \frac{1}{2}x_2^T\dot{M}x_2 + \frac{\partial P(x_1)}{\partial x_1}\cdot\dot{x}_1 + x_2^T M\cdot\dot{x}_2$$

$$= \frac{1}{2}x_2^T\dot{M}x_2 + g^T(x_1)x_2 + x_2^T \cancel{M}\left(\cancel{M^{-1}}\left(u - C(x_1,x_2)x_2 + Dx_2 + g(x_1)\right)\right)$$

$$= \frac{1}{2}x_2^T\dot{M}x_2 + \cancel{g^T(x_1)x_2} + x_2^T u - x_2^T C(x_1,x_2)x_2 - x_2^T D x_2 - \cancel{x_2^T g(x_1)}$$

$$= \frac{1}{2}x_2^T(\dot{M}-2C)x_2 + x_2^T u - x_2^T D x_2$$

if $\dot{M}-2C = A$. $\quad A^T = -A$

$$\therefore \ (x_2^T A x_2)^T = -x_2^T A x_2 = x_2^T A x_2 \quad (\text{since } x_2^T A x_2 \in \mathbb{R})$$

$$\therefore \ x_2^T A x_2 = x_2^T(\dot{M}-2C)x_2 = 0$$

$$\therefore \quad \dot{V} = x_2^T u - x_2^T D x_2$$

$$\Rightarrow x_2^T u = \dot{V} + x_2^T D x_2 \geq \dot{V} \qquad \therefore \ \underline{u \text{ to } x_2 \text{ passive!}}$$

#6.10. (Continued)

(b) $u = -k_d \cdot \dot{q} + v$. k_d: positive diagonal constant mtx

show that the map from v to $\dot{q}$ is output strictly passive

$$\Rightarrow \dot{V} = x_2^T u - x_2^T D x_2$$

$$= x_2^T (-k_d \cdot x_2 + v) - x_2^T D x_2$$

$$= x_2^T v - x_2^T (k_d + D) x_2.$$

$$x_2^T v - \dot{V} = x_2^T (k_d + D) x_2 > 0 \quad \left(\begin{array}{l} \forall x_2 \neq 0 \\ \text{since } k_d + D : \text{positive definite} \end{array} \right)$$

$\therefore$ map from v to $x_2 (= \dot{q})$

~ output strictly passive!

(c) $u = -k_d \cdot \dot{q}$ makes the origin $(q=0, \dot{q}=0)$ asymptotically stable.

What conditions for G.A.S

$$\Rightarrow \dot{V} = x_2^T u - x_2^T D x_2 = x_2^T (-k_d \cdot x_2) - x_2^T D x_2$$

$$= -x_2^T (k_d + D) x_2 \leq 0.$$

$$\dot{V} = 0 \rightarrow x_2 = 0 \rightarrow g(x_1) \rightarrow x_1 = 0 \quad \text{since } \dot{x}_2 = M^{-1}(x_1) \{ -\cancel{k_d x_2} - \cancel{C x_2} - \cancel{D x_2} - g(x_1) \} = 0$$

$\therefore$ Origin $(\equiv (q=0, \dot{q}=0))$ ~ Asymptotically stable

For G.A.S.

V: radially unbounded, so $M(q), P(q)$ have to be unbounded!!