[Exercises 8] Samples

[Problem 8.4] Reconsider Example 8.1 with $a=0$. Show that the origin is stable.

Consider the system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -x_2 + bx_1x_2$$

<Solution>

In Example 8.1, with $a=0$, then we have

$$\dot{y} = -b(yz+z^2), \qquad \dot{z} = -z + b(yz+z^2)$$

Moreover $A_1 = 0$, $g_1(y,0)=0$, $g_2(y,0)=0$. Hence, the origin of the full system is stable. ∎

··· ① the reduced system is $\dot{y}=0$ ⇒ the origin $y=0$ is stable

② $\exists\ V(y)=y^2 \in C^1$ s.t. $\frac{\partial V}{\partial y}\dot{y}=0$

Since ① and ② satisfy the conditions of Corollary 8.1, the origin of the full system is stable //

[Problem 8.15] Consider the system

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 - x_2 - (2x_2 + x_1)(1-x_2^2)$$

(a) Using $V(x) = 5x_1^2 + 2x_1x_2 + 2x_2^2$, show that the origin is AS.

<Solution>

From the given condition

$$\dot{V}(x) = 10x_1x_2 + 2x_2^2 + 2x_1\{-x_1 - x_2 - (2x_2+x_1)(1-x_2^2)\} + 4x_2\{-x_1 - x_2 - (2x_2+x_1)(1-x_2^2)\}$$
$$= -2x_1^2 + 4x_1x_2 - 2x_2^2 - 2(x_1+2x_2)^2(1-x_2^2)$$

For $\forall |x_2|<1$, $\dot{V}(x) \le 0$.

$$\dot{V}(x) = 0 \Rightarrow \dot{V} \le -2(x_1-x_2)^2 = 0 \Rightarrow x_1 = x_2 \Rightarrow \dot{x}_1 = \dot{x}_2$$

Then, we have

$$\dot{x}_2 = -2x_2 - 3x_2(1-x_2^2) \Rightarrow -3x_2(2-x_2^2) = 0 \Rightarrow x_2 = 0 \Rightarrow x_1 = 0$$

By LaSalle's theorem, the origin is AS. ∎

(b) Let

$$S = \{x \in R^2 \mid V(x) \le 5\} \cap \{x \in R^2 \mid |x_2| \le 1\}$$

Show that S is an estimate of the region of attraction.

<Solution>

We know $\dot{V}(x) \le 0$ for $\forall x \in S$. Hence, we have to show that S is positively invariant set. The interior point of S can not leave S through the segment AD and BC since AD and

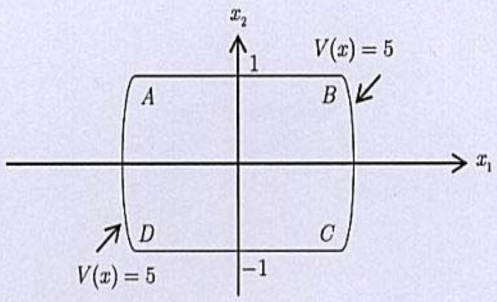
BC are the Lyapunov surface $V(x)=5$.

Let $W=x_2^2$ then

$$\dot{W}=2x_2\{-x_1-x_2-(2x_2+x_1)(1-x_2^2)\}\Rightarrow -2x_2(x_1+x_2)-2x_2(2x_2+x_1)(1-x_2^2)$$

along the segment $|x_2|=1$, $\dot{W}=-2x_2(x_1+x_2)\leq 0$. show that $\begin{cases} x_2=1, & \dot{W}\leq 0 \\ x_2=-1, & \dot{W}\leq 0 \end{cases}$

Thus, the interior point of S can not leave S through the segment AD and BC. Therefore, S is an estimate of the ROA.



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[**Problem 8.20**] Consider the system

$$\dot{x}_1=x_2,\qquad \dot{x}_2=-x_1-g(t)x_2$$

where $g(t)$ is continuously differentiable and $0<k_1\leq g(t)\leq k_2$ for all $t\geq 0$.

(a) Show that the origin is ES.

<**Solution**>

Let $V(x)=\frac{1}{2}x_1^2+\frac{1}{2}x_2^2$ then,

$$\dot{V}=x_1x_2+x_2(-x_1-g(t)x_2)=-g(t)x_2^2\leq -k_1x_2^2$$

Let

$$A(t)=\begin{bmatrix}0 & 1\\ -1 & -g(t)\end{bmatrix},\bar{A}=\begin{bmatrix}0 & 1\\ -1 & -\end{bmatrix},C(t)=[0\quad \sqrt{g(t)}]$$

Prove this! then the pair $(\bar{A},C)$ is uniformly observable. Also, $A(t)=\bar{A}-C^TC$ and $C(t)$ is uniformly bounded. Then $(A(t),C(t))$ is also uniformly observable.

$$-\dot{P}(t)=P(t)A(t)+A^T(t)P(t)+C^T(t)C(t)$$
$$\Rightarrow P(t)=\begin{bmatrix}\frac{1}{2} & 0\\ 0 & \frac{1}{2}\end{bmatrix}\Rightarrow V(t,x)=x^TP(t)x\Rightarrow \dot{V}(t,x)=-x^TC^TCx\leq 0$$

Therefore, the origin is ES.

■

(b) Would (a) be true if $g(t)$ were not bounded? Consider $g(t) = 2 + \exp(t)$.

<**Solution**>

From the given condition,

$$g(t) = 2 + \exp(t)$$

and

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1 - (2 + \exp^t)x_2$$

Then,

$$x_2 = c\exp^{-t} \Rightarrow x_1 = -(1 + \exp^{-t})c \Rightarrow x_1 \to -c, x_2 \to 0 \text{ as } t \to \infty$$

Therefore, the origin is not AS $\Rightarrow g(t)$ must be bounded.

■

* C.T. Chen 'Linear system theory and design' 2nd Edition.

Consider the n-dim. LTV system

$$\begin{cases} \dot{x} = A(t)x(t) + B(t)u(t) \\ y = C(t)x(t) + D(t)u(t) \end{cases} \quad \text{—①}$$

<Thm.> Assume that $A(t)$ and $C(t) \in C^{n-1}$ (n-1 times continuously differentiable)

Then the system ① is observable at t_0 if there exists a finite $t_1 > t_0$ s.t.

$$\rho \begin{bmatrix} N_0(t_1) \\ N_1(t_1) \\ \vdots \\ N_{n-1}(t_1) \end{bmatrix} = n \quad \text{where} \quad \begin{aligned} & N_{k+1}(t) = N_k(t)A(t) + \frac{d}{dt}N_k(t), \\ & \qquad k = 0, 1, 2, \cdots, n-1 \\ & N_0(t) = C(t). \end{aligned}$$

<Def.> The system ① is said to be uniformly observable in $(-\infty, \infty)$ iff $\exists\ \sigma_0 > 0$ and $\beta_i(\cdot) > 0$ that depends on σ_0 s.t.

$$0 < \beta_1(\sigma_0)I \le V(t, t+\sigma_0) \le \beta_2(\sigma_0)I$$

$$0 < \beta_3(\sigma_0)I \le \Phi^T(t, t+\sigma_0)V(t, t+\sigma_0)\Phi(t, t+\sigma_0) \le \beta_4(\sigma_0)I, \quad \forall t$$

where Φ is the state transition matrix

and $V(t_0, t_1) = \displaystyle\int_{t_0}^{t_1} \Phi^T(t, t_0)C^T(t)C(t)\Phi(t, t_0)\,dt$.

8.4

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_2 + ax_1^2 + bx_1x_2 \end{cases}$$

$a = 0$

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_2 + bx_1x_2 \end{cases}$$

linearization at the origin

$$\frac{\partial f}{\partial x}\Big|_{x=0} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}, \quad \text{eigen-vector } \Phi = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$$

$$T = \Phi^{-1}$$

$$TAT^{-1} = \Phi^{-1}A\Phi = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}^{-1}\begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$$

change of variables

$$\begin{bmatrix} y \\ z \end{bmatrix} = T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ -x_2 \end{bmatrix} \quad ; \quad \underline{x_1 = y + z}, \ \underline{x_2 = -z}$$

reduced system

$$\dot{y} = \dot{x}_1 + \dot{x}_2 = x_2 - x_2 + bx_1x_2 = bx_1x_2 = -b(yz + z^2)$$

$$\dot{z} = -\dot{x}_2 = x_2 - bx_1x_2 = -z + b(yz + z^2)$$

$$N(h(y)) = \frac{\partial h}{\partial y}(y)\left[-b\left(yh(y) + h(y)^2\right)\right] - h(y) + b\left(yh(y) + h(y)^2\right) = 0$$

$$, \ h(0) = h'(0) = 0$$

$h(y) = 0$ is exact solution

$\Rightarrow \ \dot{y} = 0$; reduced system $\dot{y} = 0$ has stable origin with $V(y) = y^2$

$\dot{V}(y) = 2y\dot{y} = 0$ by Cor. 8.1.,

$\therefore$ The origin of the full-system is stable

8.15

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 - x_2 - (2x_2 + x_1)(1 - x_2^2) \end{cases}$$

(a) $V(x) = 5x_1^2 + 2x_1x_2 + 2x_2^2$

$$\dot{V}(x) = 10x_1\dot{x}_1 + 2\dot{x}_1x_2 + 2x_1\dot{x}_2 + 4x_2\dot{x}_2$$

$$= 10x_1x_2 + 2x_2^2 + 2(x_1+2x_2)\left[-x_1 - x_2 - (2x_2+x_1)(1-x_2^2)\right]$$

$$= 10x_1x_2 + 2x_2^2 + 2(x_1+2x_2)(-x_1-x_2) - 2(x_1+2x_2)^2(1-x_2^2)$$

$$= -2x_1^2 - 2x_2^2 + 4x_1x_2 - 2(x_1+2x_2)^2(1-x_2^2)$$

$$= -2(x_1-x_2)^2 - 2(x_1+2x_2)^2(1-x_2^2)$$

$$\le -2(x_1-x_2)^2, \quad \forall x_2^2 \le 1 \ ; \ |x_2| \le 1$$

원점 외에서 $\dot{V}(x) \le 0$ negative semi-definite

∴ The origin of system is asymptotically stable

Use LaSalle's Thm.

(b) $|x_2| \le 1$: $|b^T x| \le 1 \Rightarrow b = [0 \ 1]^T, \ r = 1$

$$c < \frac{r^2}{b^T P^{-1} b} = \frac{1}{[0 \ 1]\frac{1}{9}\begin{bmatrix} 2 & -1 \\ -1 & 5 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}} = \frac{9}{5} = 1.8$$

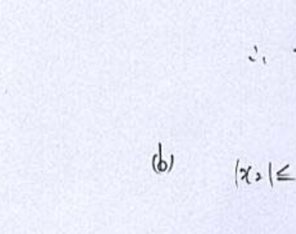
If S is not positively invariant, a state may go through ① (or ②,③,④) line

but, we can obtain a better estimate of R_A

To show that the trajectories cannot leave S through ③ or ④,

$\sigma = x_2$

$$\frac{d}{dt}\sigma^2 = 2\sigma\dot{\sigma} = 2\sigma\dot{x}_2 = 2\sigma(-x_1 - x_2 - (2x_2+x_1)(1-x_2^2))$$



$$\le -2\sigma(x_1+x_2) \qquad |x_2| \le 1$$

on the boundary $\sigma = 1$

$$\frac{d}{dt}\sigma^2 \le -2\sigma(x_1+x_2) \le 0 \ ; \quad -2(x_1+1) \le 0$$

$$x_1 + 1 \ge 0 \ ; \quad x_1 \ge -1$$

on the boundary $\sigma = -1$

$$\frac{d}{dt}\sigma^2 \le -2(-1)(x_1-1) \le 0 \ ; \quad (x_1-1) \le 0 \ ; \quad x_1 \le 1$$

From (a), we can know that, on ① and ②, $\dot{V}(x) \le 0$. Therefore, the state trajectories cannot leave S through ① or ②

$c_1 = V(x)|_{x_1=-1, x_2=1} = 5$, $c_2 = V(x)|_{x_1=1, x_2=-1} = 5$

$$\therefore \ S = \{x \in R^2 \mid V(x) \le 5\} \cap \{x \in R^2 \mid |x_2| \le 1\}$$

8.20 $\dot{x}_1 = x_2$, $\dot{x}_2 = -x_1 - q(t)x_2$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -q(t) \end{bmatrix} \quad \& \quad 0 < k_1 \le q(t) \le k_2 \quad \text{for all } t \ge 0$$

(a) $V = \frac{1}{2} x^T p x = \frac{1}{2}[x_1 \ x_2]^T \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$\dot{V} = x_1\dot{x}_1 + x_2\dot{x}_2 = x_1x_2 - x_1x_2 - q(t)x_2^2 = -q(t)x_2^2$; $\dot{V} \le 0$

$0 < \frac{1}{2}I \le p \le 2I$ $\qquad$ $p = \frac{1}{2}I$

$-\dot{p}(t) = p(t)A(t) + A^T(t)p(t) + C^T(t)C(t)$

$\dot{V}(t,x) = \frac{1}{2}[\dot{x}^T p x + x^T p \dot{x} + x^T \dot{p} x]$

$= \frac{1}{2}[x^T A^T p x + x^T p A x + x^T \dot{p} x]$

$= \frac{1}{2} x^T [A^T p + pA + \dot{p}] x$

$= -\frac{1}{2} x^T C^T C x$

$\Rightarrow \dot{V}(t,x) = -\frac{1}{2} x^T C^T C x = -q(t) x_2^2 \le 0$

$= -[x_1 \ x_2] \begin{bmatrix} 0 & 0 \\ 0 & q(t) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$C = \frac{1}{2} \begin{bmatrix} 0 \\ \sqrt{q(t)} \end{bmatrix}^T$

Cannot this approach for LTV systems

observability check

$\mathcal{O} = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2}\sqrt{q(t)} \\ \frac{-\sqrt{q(t)}}{2} & \frac{-q(t)\sqrt{q(t)}}{2} \end{bmatrix}$, $\text{rank}(\mathcal{O}) = 2$

observability

All $t \ge 0$ and $\sqrt{q(t)} \ne 0$ $\Rightarrow$ uniformly observable

let $x(t) = \Phi(t,t_0) \cdot x(t_0)$

$V(t+\delta, \phi(t+\delta; t, x)) - V(t,x) = \int_t^{t+\delta} \dot{V}(\tau, \phi(\tau; t, x))\, d\tau$

$= -x^T W(t, t+\delta) x$

From uniform observability of $(A(t), C(t))$,

$W(t+\delta) \ge kI > 0$, let $k < 1$

$$V(t+\delta, \phi(t+\delta; t, x)) - V(t,x) \le -k\|x\|^2 \le -\frac{k}{2}V(t,x)$$

$$\frac{k}{2} = \lambda \ ; \quad 0<k<2$$

$$\underline{0< \lambda <1}$$

by Theorem 8.5 The origin of system is asymptotically exp. stable.

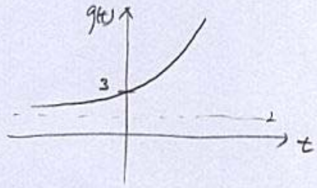
(b) 문제 (a)에서 $\dot{V}(t,x) = -g(t)x_2^2$

i) 만약 $g(t)$가 unbound라면, $g(t)<0$ 일때 $\dot{V}(t,x) \ge 0$가 된다.

만약 $g(t) = 2 + \exp(t)$ 라면

$$V(t,x) = -g(t)x_2^2$$
$$= -(2+e^t)x_2^2 \le 0$$

$g(t) \le 0$인 구간 존재 X



ii) $\mathcal{O} = \begin{bmatrix} C \\ CA \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 & \sqrt{g(t)} \\ -\sqrt{g(t)} & -g(t)\sqrt{g(t)} \end{bmatrix}$

If $g(t)=0$, then $\sqrt{g(t)}=0 \Rightarrow \text{rank}(\mathcal{O}) \neq 2$

finally, observable 하지 않다.

$g(t) = 2+e^t$ 라면 $g(t) = 2+e^t \neq 0$.

$\sqrt{g(t)}$인 구간 존재하지 않는다.

i) & ii) ; $\dot{V}(t,x) < 0$ & $\text{rank}(\mathcal{O}) = 2$

finally, If $g(t) = 2+\exp(t)$; unbounded $g(t)$

then, the origin of system is exponentially stable