Homework Assignment 2

Issued: Oct 1 Due: Oct 15 (Wed)

1. Consider the linear, time-invariant system

$$\dot{x}(t) = Ax(t) + Bu(t)$$

with $x(t) \in \mathbb{R}^n$, and $u(t) \in \mathbb{R}_{n_u}$. Suppose the $Q \in \mathbb{R}^{n \times n}$, with $Q = Q^T \geq 0$. Assume that (A, B) is stabilizable, and $(A, Q^{\frac{1}{2}})$ is detectable. Using a "completion of squares" approach, determine the value of the function $J : \mathbb{R}^n \to \mathbb{R}$

$$J(x_0) := \min_{u \in \mathcal{L}_2} \int_0^\infty [x^T(t)Qx(t) + u^T(t)u(t)]dt$$

subject to

$$\begin{aligned} x(0) &= x_0 \\ \dot{x}(t) &= Ax(t) + Bu(t) \end{aligned}$$

Also determine the input u which achieves the optimum.

2. Two uncertain parameters.

Consider a plant and a compensator shown in Fig. 1, with the following transfer functions.

$$P(s) = \frac{g}{s^2(1 + s\theta)}, \qquad C(s) = \frac{k + T_d s}{1 + T_0 s}.$$

This model includes scaling factors α_1 and β_1 for the uncertainty Δ_g and scaling factors α_2 and β_2 for the uncertainty Δ_θ. $\alpha_1\beta_1 = \varepsilon_1$ is the largest possible uncertainty g, while $\alpha_2\beta_2 = \varepsilon_2$ is the largest possible uncertainty in θ.

(a) Show that the transfer matrix from p_1, p_2 to q_1, q_2 is in the following form:

$$H(s) = \frac{\frac{1}{1+s\theta_0}}{1 + L_0(s)} \begin{bmatrix} \beta_1 C(s) \\ -\beta_2 \end{bmatrix} \begin{bmatrix} -\alpha_1/s^2 & \alpha_2 s \end{bmatrix}.$$

(b) Show that the largest eigenvalue of $H^T(-j\omega)H(j\omega)$ is

$$\begin{aligned} \bar{\sigma}^2(\omega) &= \frac{\frac{1}{1+\omega^2\theta_0^2}}{|1 + L_0(j\omega)|^2}(\beta_1^2|C(j\omega)|^2 + \beta_2^2)(\frac{\alpha_1^2}{\omega^4} + \alpha_2^2\omega^2), \quad \omega \in \mathbb{R}. \\ &= \frac{(\beta_1^2(k^2 + \omega^2 T_d^2) + \beta_2^2(1 + \omega^2 T_0^2))(\alpha_1^2 + \alpha_2^2\omega^6)}{|\chi(j\omega)|^2} \end{aligned}$$

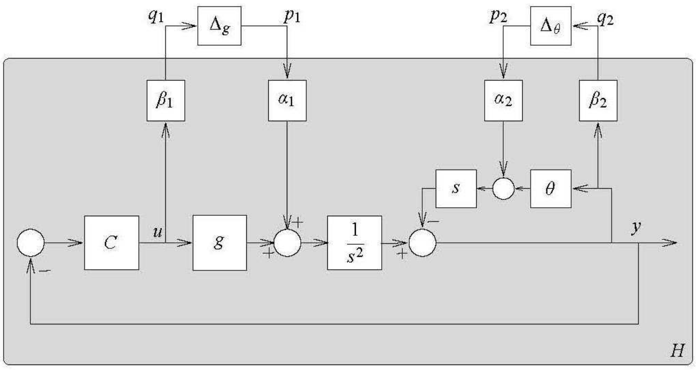

Figure 1: (Prob 2) Two uncertain parameters

with χ the closed-loop characteristic polynomial

$$\chi(s) = \theta_0 T_0 s^4 + (\theta_0 + T_0)s^3 + s^2 + g_0 T_d s + g_0 k.$$

with the nominal values g_0, θ_0.

(c) For $k = 1$, $T_d = \sqrt{2}$, $\theta_0 = 0.1$, $T_0 = 1/10$, $.5 \le g \le 5$, $g_0 = 2.75$, and $\varepsilon_1 = 2.25$, $\varepsilon_2 = 0.1$,

$$\alpha_1 = \beta_1 = \sqrt{\varepsilon_1}, \quad \alpha_2 = \beta_2 = \sqrt{\varepsilon_2}.$$

plot $\bar{\sigma}(\omega)$.

(d) Let $\beta_1 = \varepsilon_1/\alpha_1$, $\beta_2 = \varepsilon_2/\alpha_2$ and $\rho = \alpha_1^2/\alpha_2^2$. Show that, for fixed ω the quantity $\bar{\sigma}^2(\omega)$ is minimized for

$$\rho = \omega^3 \frac{\varepsilon_1}{\varepsilon_2}\sqrt{\frac{k^2 + \omega^2 T_d^2}{1 + \omega^2 T_0^2}},$$

and that for this value of ρ

$$\bar{\sigma}(\omega) = \frac{\varepsilon_1\sqrt{K^2 + \omega^2 T_d^2} + \varepsilon_2 \omega^3 \sqrt{1 + \omega^2 T_0^2}}{|\chi(j\omega)|}, \quad \omega \ge 0.$$

plot $\bar{\sigma}(\omega)$ for the same case as (c). Compare the result with (c).

3. (a) Show that the structured singular value of the 2×2 dyadic matrix

$$M = \begin{bmatrix} a_1 b_1 & a_1 b_2 \\ a_2 b_1 & a_2 b_2 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \begin{bmatrix} b_1 & b_2 \end{bmatrix}$$

with a_1, a_2, b_1, and b_2 complex numbers, with respect to the perturbation structure

$$\Delta = \begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix}, \quad \Delta_1 \in \mathbb{C},\ \Delta_2 \in \mathbb{C}$$

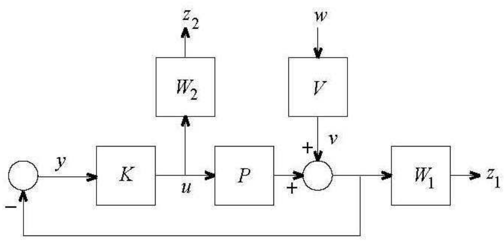

Figure 2: (Prob 4) mixed sensitivity design

is

$$\mu(M) = |a_1 b_1| + |a_2 b_2|.$$

(b) Apply this fact to the prob 3, and compute $\mu(H(j\omega))$. Compare the result with 2(d).

4. We want to design a mixed-sensitivity controller for $P(s) = 1/s^2$ using

$$\left\| \begin{bmatrix} W_1 SV \\ W_2 UV \end{bmatrix} \right\|_\infty$$

where $U = K(I + PK)^{-1}$ $S = (I + PK)^{-1}$ and the weighting functions are:

$$V(s) = \frac{s^2 + s\sqrt{2} + 1}{s^2}$$

$W_1 = 1$, and $W_2(s) = c(1 + rs)$.

(a) Show that, when $r = 0$, the plant is

$$\begin{aligned} \dot{x} &= \underbrace{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix}}_{B_1} w + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{B_2} u, \\ z &= \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{C_1} x + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{D_{11}} w + \underbrace{\begin{bmatrix} 0 \\ c \end{bmatrix}}_{D_{12}} u, \\ y &= \underbrace{\begin{bmatrix} 0 & -1 \end{bmatrix}}_{C_2} x + \underbrace{\begin{bmatrix} -1 \end{bmatrix}}_{D_{21}} w. \end{aligned}$$

(b) What do you think we can expect from the given choices of weighting functions (when $r = 0$ and $r \neq 0$)?

(c) Design H_∞ controller using Matlab for $r = 0$, $c = 0.1$. What is the value of the resulting H_∞ norm?

5. `[EXTRA]` State-space calculation of $||\cdot||_\infty$ norm :Consider a linear system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned}$$

In this analysis, it is not necessary to assume that A is stable, but we must assume that A has no imaginary-axis eigenvalues. Introduce the notation $G^{\sim}(s) = [G(-\bar{s})]^*$.

First, check that

$$M(s) := [I - G(s)G^{\sim}(s)]^{-1} = \left[\begin{array}{cc|c} A & BB^* & 0 \\ -CC^* & -A^* & -C^* \\ \hline C & 0 & I \end{array}\right].$$

(a) Consider a given frequency $\bar{\omega}$. Show that $G(j\bar{\omega})$ has a singular value equal to 1 (some singular value - not necessarily the maximum) if and only if $I - G(s)G^{\sim}(s)$ is singular at $s = j\bar{\omega}$.

(b) Consider a given frequecy $\bar{\omega}$. Show that $G(j\bar{\omega})$ has a singular value equal to 1 if and only if $M(s)$ has a pole at $s = j\bar{\omega}$.

(c) Hence, the imaginary axis poles of $M(s)$ are the same as the points on the imaginary axis where G has a singular value equal to 1. In this part, we will show that the imaginary-axis poles of $M(s)$ are exactly equal to the imaginary axis eigenvalues of the "A" matrix for M (it could be possible that some eigenvalues of the "A" matrix are uncontrollable and/or unobservable, so they would not show up in the transfer function - this calculation will rule that possibility out). To do this, show the any imaginary-axis eigenvalues of

$$\begin{bmatrix} A & BB^* \\ -CC^* & -A^* \end{bmatrix}$$

are controllable through

$$\begin{bmatrix} 0 \\ -C^* \end{bmatrix},$$

and unobservable through

$$\begin{bmatrix} C & 0 \end{bmatrix}.$$

`[Hint]` Use the Popov-Bellman-Hautus (Kailath) test for controllability and observability.

(d) Hence, we have proven the statement: $G(j\omega)$ has a singular value equal to 1 if and only if

$$\begin{bmatrix} A & BB^* \\ -CC^* & -A^* \end{bmatrix}$$

has eigenvalue equal to $j\omega$. In other words: for all $\omega \in \mathbb{R}, G(j\omega)$ has no singular values equal to 1 if and only if

$$\begin{bmatrix} A & BB^* \\ -CC^* & -A^* \end{bmatrix}$$

has no imaginary axis eigenvalues. Generalize these two statements to the case where $G(j\omega)$ has a singular value equal to some positive number $\gamma \neq 1$.

(e) Prove the following: For $\gamma > 0$,

$$\sup_{\omega \in \mathbb{R}} \bar{\sigma}[G(j\omega)] < \gamma$$

if and only if

$$\begin{bmatrix} A & \frac{1}{\gamma^2} BB^* \\ -CC^* & -A^* \end{bmatrix}$$

has no imaginary axis eigenvalues.

Two good reference for this problem are

- Boyd, Balakkrishnan and Kabamba, "A bisection method for computing the H_∞ norm of a transfer matrix and related problems," *Math Control Signals and Systems*, 2(3):207-219, 1989.

- Bruinsma and Steinbuch, "A fast algorithm to compute the H_∞ norm of a transfer function matrix, *Systems and Control Letters*, 14, pp. 287-293, 1990.