1. $\dot{x} = Ax + Bu$

$$J(x_0) := \min_{u \in \mathcal{L}_2} \int_0^\infty [x^TQx + u^Tu]\, dt$$

subject to $\quad x(0) = x_0$

$\qquad\qquad\quad \dot{x} = Ax + Bu$.

sol) Let P be any symmetric matrix.

$$\frac{d}{dt}(x^TPx) = \dot{x}^TPx + x^T\dot{P}x + x^TP\dot{x}$$

$$= (Ax+Bu)^TPx + x^T\dot{P}x + x^TP(Ax+Bu)$$

$$= x^T(A^TP + \dot{P} + PA)x + 2u^TB^TPx$$

$$x^TPx\Big|_0^\infty = \int_0^\infty \{x^T(A^TP + \dot{P} + PA)x + 2u^TB^TPx\}\, dt$$

$$= \overset{\to 0}{x^T(\infty)P(\infty)x(\infty)} - \underline{x^T(0)P(0)x(0)}$$

$$J = \int_0^\infty (x^TQx + u^Tu)\, dt + x^TPx\Big|_0^\infty - x^TPx\Big|_0^\infty$$

$$= x^T(0)P(0)x(0) + \int_0^\infty \{x^T(\dot{P} + A^TP + PA + Q)x + 2x^TPBu + u^Tu\}\, dt$$

$$= x^T(0)P(0)x(0) + J_1$$

where $J_1 = \int_0^\infty \{x^T(\dot{P} + A^TP + PA + Q)x + \underbrace{2x^TPBu}_{\text{or } 2u^TB^TPx} + u^Tu\}$

by "Completion of squares" approach

$$J_1 = \int_0^\infty \{ x^T(\dot{P} + A^TP + PA + Q)x + \underbrace{2x^TPBu}_{= x^TPBu + u^TB^TPx} + u^Tu \} dt$$

$$= \int_0^\infty \{ (B^TPx + u)^T(B^TPx + u)$$

$$- x^TPBB^TPx + x^T(\dot{P} + A^TP + PA + Q)x \} dt$$

$$= \int_0^\infty \{ x^T(\dot{P} + A^TP + PA + Q - PBB^TP)x + (B^TPx + u)^T(B^TPx + u) \, dt$$

if $\dot{P} + A^TP + PA + Q - PBB^TP = 0$ 이고, — ①

$u = -B^TPx$ — ② 이면,

$J_1 = 0$ 이되고, 따라서

$$J(x) = \min_{u \in \mathcal{L}_2} \int_0^\infty [x^TQx + u^Tu] \, dt$$

$$= x^T(0)P(0)x(0) \quad \text{when } J_1 = 0$$

한편 $P(t)$ 는 0 근처에서 일정하고,

$\therefore \dot{P} = 0$, near 0.

$\therefore$ $J(x_0) = Px_0^2$: value of function J

$u = -B^TPx$: input u which achieves the optimum

when P is solution of

$$A^TP + PA + Q - PBB^TP = 0$$

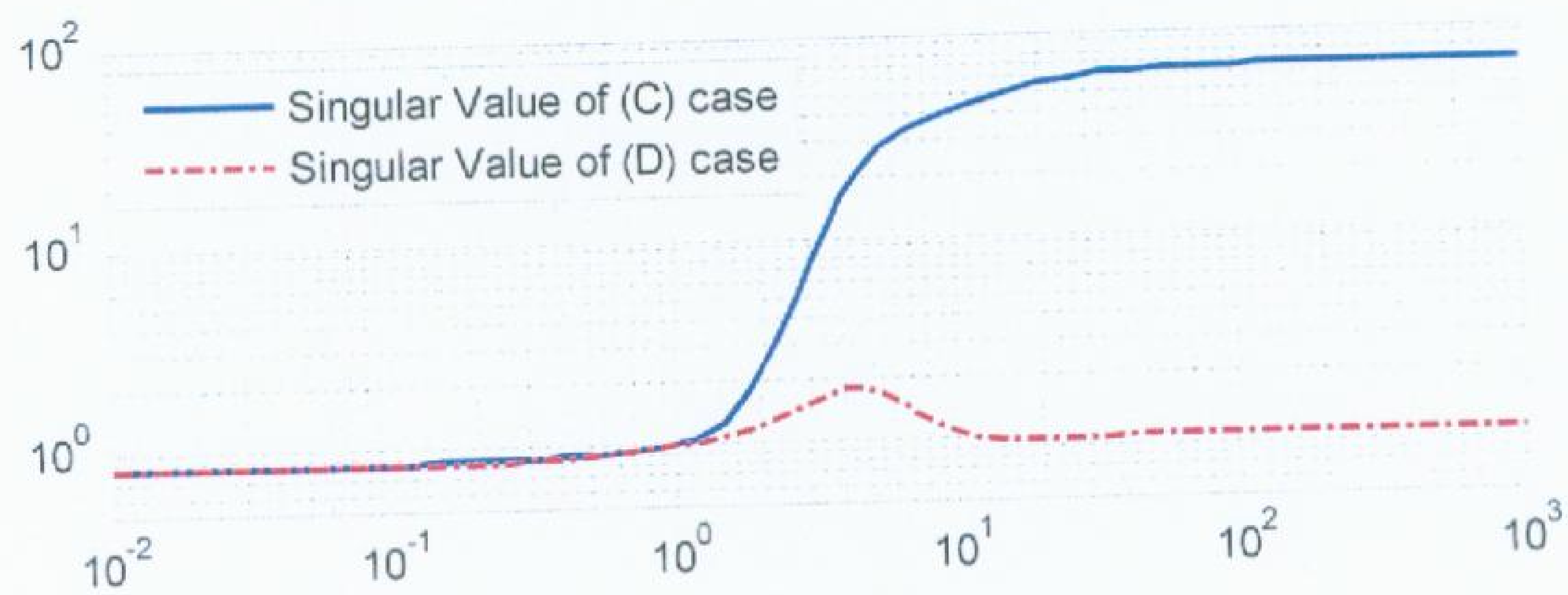
which called ARE (algebraic Riccati equation)

Prob 2

Why do (c) & (d) differ by a very wide margin?

- The test in (c) is based on full uncertainties rather than the structured uncertainties given in the problem.
- (The test also allows dynamical uncertainties rather than the real uncertainties implied by the problem setting.)
- (d) is less conservative by allowing $\alpha_1, \alpha_2, \beta_1, \beta_2$ to be freq-dependent and choosing them to minimize $\bar{\sigma}(\omega)$ for each freq ω.

(d) Compare the (b) case result with (d) case result



[Fig3. Maximum singular value]

$\boxed{\text{Prob. 3}}$ structured sing. val. of a dyadic matrix

$$M = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ \vdots \end{bmatrix} [b_1 b_2 \cdots] \qquad a_i, b_j \in \mathbb{C}$$

with

$$\Delta = \operatorname{diag}[\Delta_1, \Delta_2, \cdots] \qquad \Delta_i \in \mathbb{C}.$$

is

$$\mu_\Delta(M) = \sum_i |a_i b_i|.$$

Pf

$$\underbrace{\det(I - M\Delta)}_{①} = \det(I - \overset{\text{col.}}{a}\overset{\text{row vec.}}{b}\Delta) = \det(I - b\Delta a)$$

$$= 1 - \sum_i a_i b_i \Delta_i$$

$$\geq 1 - \sum_i |a_i b_i| \max_i |\Delta_i| = \underbrace{1 - \sum_i |a_i b_i| \bar{\sigma}(\Delta)}_{②}$$

To make $① = 0$, we need at least $② \leq 0$.

thus, $\bar{\sigma}(\Delta) \geq \dfrac{1}{\sum_i |a_i b_i|}$.

Choose

$$\Delta = \frac{1}{\sum_i |a_i b_i|} \begin{bmatrix} \operatorname{sgn}(a_1 b_1) & & O \\ & \operatorname{sgn}(a_2 b_2) & \\ O & & \ddots \end{bmatrix},$$

then

$$\bar{\sigma}(\Delta) = \frac{1}{\sum |a_i b_i|} \quad \text{and}$$

$$① = 1 - \sum_j a_j b_j \Delta_j = 1 - \sum_j a_j b_j \frac{\operatorname{sgn}(a_j b_j)}{\sum_i |a_i b_i|} = 0.$$

$\therefore$ The smallest Δ has $\bar{\sigma}(\Delta) = \dfrac{1}{\sum_i |a_i b_i|}$,

thus, $\mu_\Delta(M) = \sum_i |a_i b_i|$.

In prob 2,

$$H(j\omega) = \frac{\frac{1}{1+j\omega\theta_0}}{1+L_0(j\omega)} \begin{array}{c} \\ \left[\begin{array}{c} \beta_1 C(j\omega) \\ -\beta_2 \end{array}\right] \end{array} \begin{array}{l} a_1 \\ a_2 \end{array} \begin{array}{c} \begin{array}{cc} b_1 & b_2 \end{array} \\ \left[\begin{array}{cc} +\frac{\alpha_1}{\omega^2} & \alpha_2 j\omega \end{array}\right] \end{array}$$

← dyadic structure.

$$\Delta = \mathrm{diag}[\Delta_\rho \ \Delta_\theta]$$

$$\therefore \mu_\Delta(H) = \left| \frac{\frac{1}{1+j\omega\theta_0}}{1+L_0(j\omega)} \right| \left(\frac{\alpha_1\beta_1}{\omega^2} |C(j\omega)| + \alpha_2\beta_2\omega \right).$$

$$= \text{same as } 2(d).$$