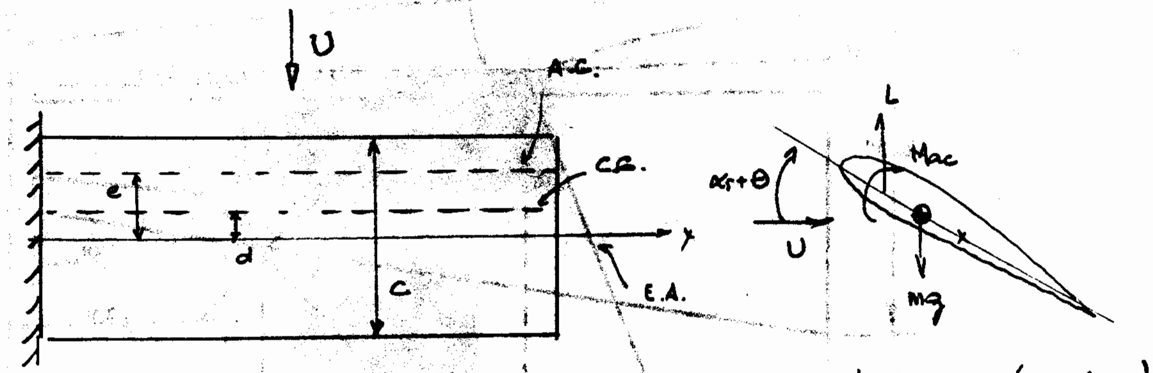
1. Consider the following wind tunnel set up:



α_r = initial angle (rigid α)

(a) Governing equation for torsion

$$GJ\frac{d^2\theta}{dy^2} = -m_{EA} \quad \text{(statics)}$$

From Aerodynamics:

$$l = qc\,c_{l\alpha}(\alpha_r+\theta)$$

$m_{ac} = 0$ ← assuming symmetric airfoil

The moment/length about the elastic axis can then be determi[ned]

$$m_{EA} = e\,qc\,c_{l\alpha}(\alpha_r+\theta) - mgd$$

Therefore:

$$GJ\frac{d^2\theta}{dy^2} = -m_{EA} = -eqc\,c_{l\alpha}(\alpha_r+\theta) + mgd$$

or:
$$\frac{d^2\theta}{dy^2} + \underbrace{\frac{eqc\,c_{\ell\alpha}}{GJ}}_{\lambda^2}\theta = \underbrace{-\frac{eqc}{GJ}c_{\ell\alpha}\alpha_r + \frac{mgd}{GJ}}_{C_1}$$

Note: λ^2, C_1 = constants (not function of y)

The solution of this equation (homogeneous + particular):

$$\theta = A\sin\lambda y + B\cos\lambda y + \frac{C_1}{\lambda^2}$$

The B.C.'s:

$$@\ y=0,\ \theta=0 \longrightarrow 0 = B + \frac{C_1}{\lambda^2} \longrightarrow B = -\frac{C_1}{\lambda^2}$$

semi-span

$$@\ y=\ell,\ GJ\frac{d\theta}{dy} = 0 \longrightarrow 0 = A\lambda\cos\lambda\ell - B\lambda\sin\lambda\ell \longrightarrow A = B\tan\lambda\ell$$

$$\Longrightarrow \boxed{\theta(y) = B\left(\tan\lambda\ell\sin\lambda y + \cos\lambda y - 1\right)}$$

where:
$$B = \left(\alpha_r - \frac{mgd}{eqc\,c_{\ell\alpha}}\right)$$

$$\lambda = \sqrt{\frac{eqc\,c_{\ell\alpha}}{GJ}}$$

The lift distribution becomes:

lift/length

semi-span

$$\boxed{\ell(y) = \underbrace{qc\,c_{\ell\alpha}\alpha_r}_{\text{rigid lift}} + \underbrace{\left(qc\,c_{\ell\alpha}\alpha_r - \frac{mgd}{e}\right)\left(\tan\lambda\ell\sin\lambda y + \cos\lambda y - 1\right)}_{\text{elastic lift}}}$$

Divergence occurs if $\theta \to \infty$. From θ equation, this happens when $\tan\lambda l \to \infty \Rightarrow \cos\lambda l \to 0$

$$\Rightarrow \quad \lambda l = \frac{\pi}{2} = \sqrt{\frac{e\, q_D\, c\, C_{l\alpha}}{GJ}} \cdot l$$

q that satisfies this becomes q_D

Finally:

$$\boxed{q_D = \frac{\pi^2}{4} \frac{GJ}{e c l^2 C_{l\alpha}}} \quad \leftarrow \text{divergence dynamic pressure}$$

(b) Using the giving numerical data, the lift expression become

$$l_{rigid} = q \cdot 12 \cdot 2\pi \cdot 10 \frac{\pi}{180}$$

$$l_{elastic} = \left(q \cdot 12 \cdot 2\pi \frac{10\pi}{180} - 12.5 \frac{5\cancel{c}}{10\cancel{c}} \right) \left(\tan 60\lambda \sin\lambda y + \cos\lambda y - 1 \right)$$

with: $\lambda = \sqrt{\dfrac{1.2\, q \cdot 12 \cdot 2\pi}{6.0\, 10^4}} = \lambda(q)$

Note: $q_D = \dfrac{\pi^2}{4} \dfrac{6.0\, 10^4}{1.2\; 12\; 60^2\; 2\pi} = 0.4545$ psi

In the next page you can find the plots of $l(y)$ as well as $\theta(y)$ for $\dfrac{q}{q_D}$ (< 1).

Problem Set 2 - Solution

Question 1 (b)

Given Constants

```
GJ=6 10^4;
c=12;
alphar=10 Pi/180.;
mg=12.5;
e=0.1 c;
cla=2 Pi;
length=60;
d=0.05 c;
```

Divergence Dynamic Pressure

```
qd = Pi^2/4 GJ/(e c length^2 cla)

0.454513
```

Lift Distribution

Rigid Lift

```
lrigid[y_,q_]:= q c cla alphar
```

Elastic Lift

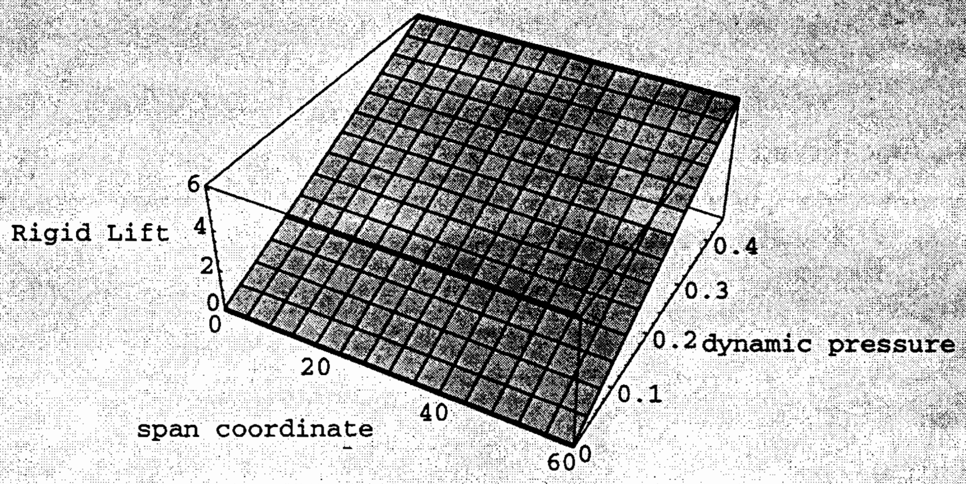
```
λ[q_]:=Sqrt[e q c cla/GJ]
```

```
lelastic[y_,q_]:= (q c cla alphar - mg d/e)(Tan[λ[q] length] Sin[λ[q] y] + Cos[λ[q] y] - 1)
```

■ Plot

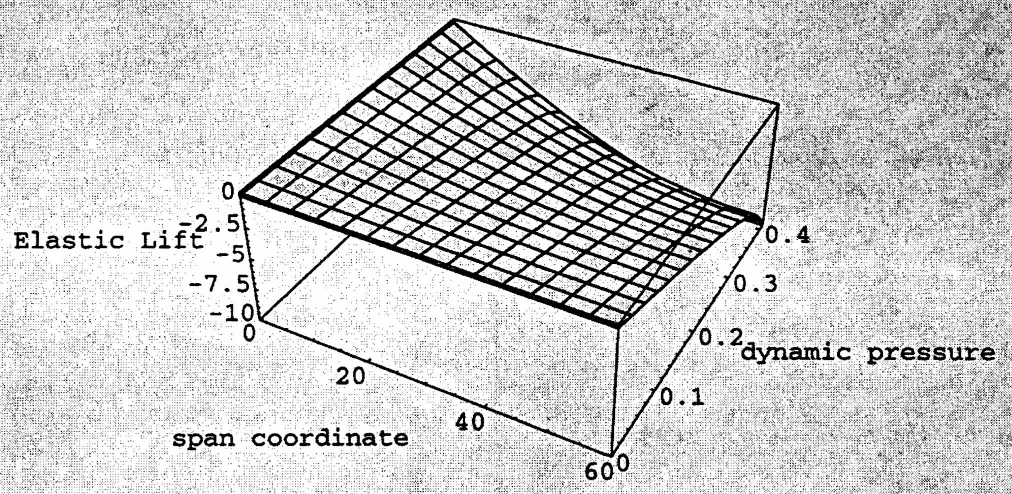
■ **Rigid Lift distribution over the span as function of dynamic pressure**

```
Plot3D[lrigid[y,q],{y,0,60},{q,0,qd},AxesLabel->{"span coordinate",
"dynamic pressure", "Rigid Lift"}]
```



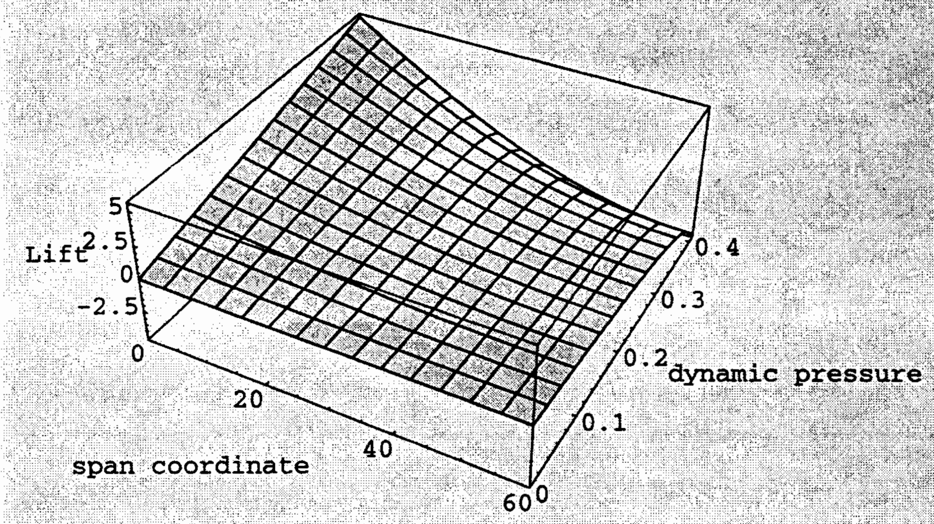
- **Elastic Lift distribution over the span as function of dynamic pressure**

```
Plot3D[lelastic[y,q],{y,0,60},{q,0,0.9 qd},AxesLabel->{"span
coordinate", "dynamic pressure", "Elastic Lift"}]
```



- **Total Lift distribution**

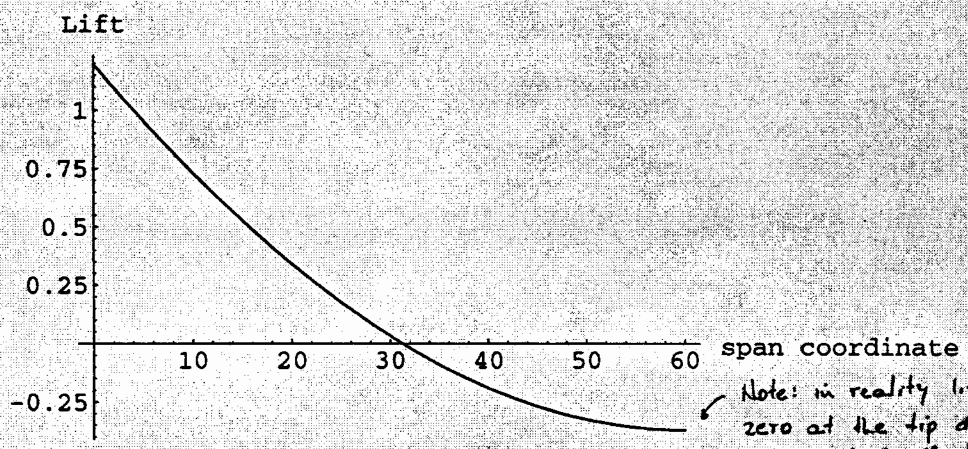
```
Plot3D[lrigid[y,q]+lelastic[y,q],{y,0,60},{q,0,0.9 qd},AxesLabel->
{"span coordinate", "dynamic pressure", "Lift"}]
```



combined lift

- **Lift distribution at q=0.2 qd**

```
Plot[lrigid[y,0.2 qd]+lelastic[y, 0.2 qd],{y,0,60},AxesLabel->{"span
coordinate", "Lift"}]
```



Note: in reality lift drops to zero at the tip due to end effect (3-D effect) not capture in this formulation

- **Twist distribution**

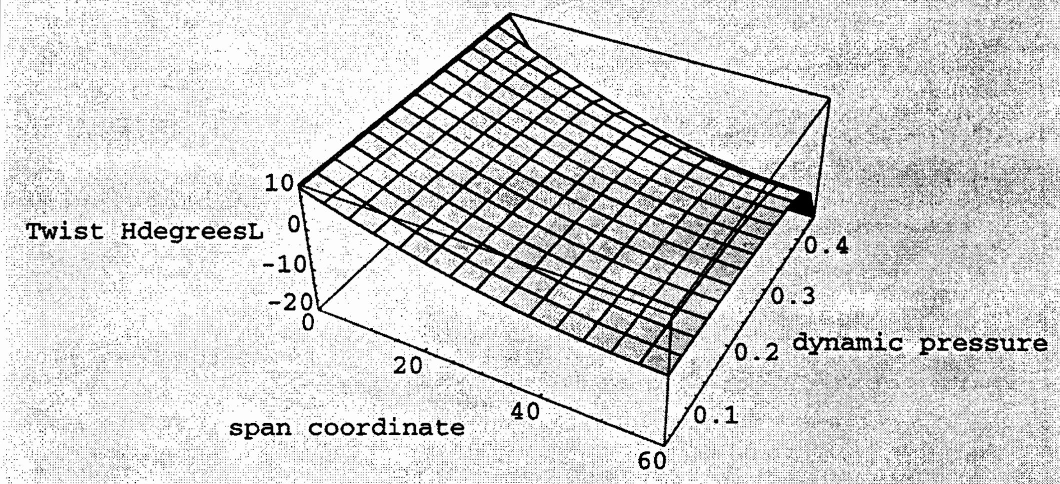
```
θ[y_,q_]:=(alphar-mg d/(e q c cla))(Tan[λ[q] length] Sin[λ[q] y] + Cos[
λ[q] y] - 1)
```

I decided to look at the total twist of the wing to see why lift was negative at the outboard. As one can see on the next page, when the moment caused by the offset weight induces an elastic twist higher than the preset one ($\alpha_r + \underbrace{\alpha_t(y)}_{0 \text{ in this case}}$), the initial angle of attack will be negative, causing negative lift.

This could be fixed by:

- increasing α_r
- adding $\alpha_t(y)$
- increasing GJ
- reducing $(mg).d$

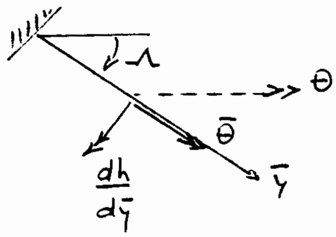
```
Plot3D[180./Pi (alphar+ θ[y,q]),{y,0,60},{q,0.1 qd,0.99 qd},AxesLabel->
{"span coordinate", "dynamic pressure", "Twist (degrees)"}]
```



2. From the problem statement, $EI, GJ = const.$, $y_i < y_j$, find

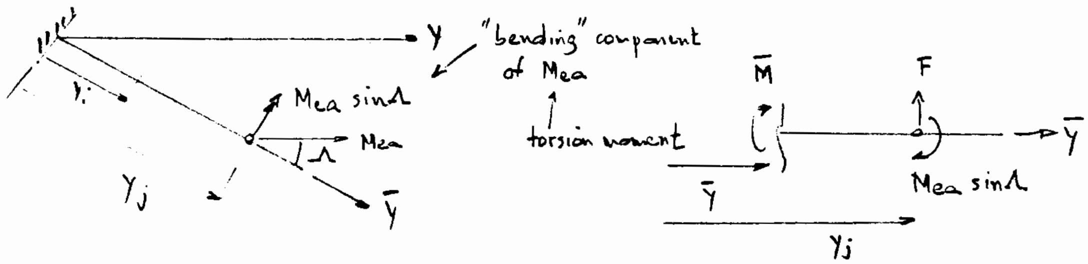
$$\theta = C^{\theta z} F_{ea} + C^{\theta\theta} M_{ea}$$

find



$$\theta = \bar{\theta}\cos\Lambda - \frac{dh}{d\bar{y}}\sin\Lambda$$

<u>Bending h along $\bar{y}$</u> :



$$\overset{+}{\curvearrowleft} \sum M = 0 \Rightarrow \bar{M} + M_{ea}\sin\Lambda - F(\bar{y}_j - \bar{y})$$

$$\text{or:} \quad \bar{M} = F(\bar{y}_j - \bar{y}) - M_{ea}\sin\Lambda$$

From beam constitutive relation:

$$EI\frac{d^2h}{d\bar{y}^2} = \bar{M} = F(\bar{y}_j - \bar{y}) - M_{ea}\sin\Lambda$$

and integrating from 0 to $\bar{y}$:

$$EI \frac{dh}{d\bar{y}} = F\left(\bar{y}_j \bar{y} - \frac{\bar{y}^2}{2}\right) - M_{ea} \bar{y} \sin\Lambda + C_1$$

but $@\ \bar{y} = 0$, $\frac{dh}{d\bar{y}} = 0$ (zero slope) $\Longrightarrow$ $C_1 = 0$

Once more:

$$EI\, h = F\left(\bar{y}_j \frac{\bar{y}^2}{2} - \frac{\bar{y}^3}{6}\right) - M_{ea} \frac{\bar{y}^2}{2} \sin\Lambda + C_2$$

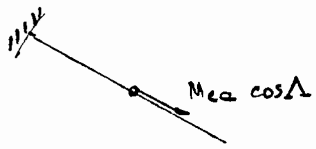
and also $@\ \bar{y} = 0$, $h = 0$ $\Longrightarrow$ $C_2 = 0$

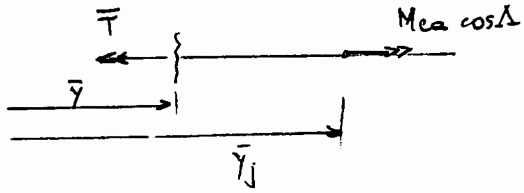
So, at $\bar{y} = \bar{y}_i$:

$$h = \left(\bar{y}_j \frac{\bar{y}_i^2}{2} - \frac{\bar{y}_i^3}{6}\right) \frac{F}{EI} - \frac{\bar{y}_i^2}{2} \sin\Lambda \frac{M_{ea}}{EI}$$

$$\theta = \left(\bar{y}_j \bar{y}_i - \frac{\bar{y}_i^2}{2}\right) \frac{F}{EI} - \bar{y}_i \sin\Lambda \frac{M_{ea}}{EI}$$

<u>Torsion $\bar{\theta}$ about $\bar{y}$:</u>





$$\Sigma T = 0 \Longrightarrow \bar{T} - M_{ea} \cos\Lambda = 0$$

$$\bar{T} = M_{ea} \cos\Lambda$$

From beam torsion constitutive relation:

$$GJ\,\frac{d\bar\theta}{d\bar y} = \bar T = M_{ea}\cos\Lambda$$

$$\Rightarrow \quad GJ\,\bar\theta = M_{ea}\,\bar y\cos\Lambda + C_3$$

but @ $\bar y = 0$, $\bar\theta = 0$ $\longrightarrow$ $C_3 = 0$

So, at $\bar y = \bar y_i$:

$$\bar\theta = \bar y_i \cos\Lambda\,\frac{M_{ea}}{GJ}$$

<u>Putting all together</u>

Now, putting all together into :

$$\theta = \bar\theta\cos\Lambda - \frac{dh}{d\bar y}\sin\Lambda$$

gives:

$$\theta = \bar y_i\cos^2\Lambda\,\frac{M_{ea}}{GJ} - \left(\bar y_j\,\bar y_i - \frac{\bar y_i^2}{2}\right)\frac{F}{EI}\sin\Lambda + \bar y_i\sin^2\Lambda\,\frac{M_{ea}}{EI}$$

But since $\bar y = \dfrac{y}{\cos\Lambda}$:

$$\boxed{\theta = \overbrace{\frac{-\sin\Lambda}{EI\cos^2\Lambda}\left(y_i y_j - \frac{y_i^2}{2}\right)}^{C^{\theta z}} F + \overbrace{\left(\frac{\sin^2\Lambda}{EI\cos\Lambda} + \frac{\cos\Lambda}{GJ}\right) y_i}^{C^{\theta\theta}} M_{ea}}$$

fo

$y_i <$

Also, for C^{zz} and $C^{z\theta}$ influence coefficients :

$$h = \left(\bar{y}_j \frac{\bar{y}_i^2}{2} - \frac{\bar{y}_i^3}{6} \right) \frac{F}{EI} - \frac{\bar{y}_i^2}{2} \sin\Lambda \frac{M_{ea}}{EI}$$

and then:

$$h = \underbrace{\frac{1}{EI \cos^3\Lambda} \left(y_i^2 y_j - \frac{y_i^3}{6} \right)}_{C^{zz}} F \underbrace{- \frac{\sin\Lambda}{EI \cos^2\Lambda} \frac{y_i^2}{2}}_{C^{z\theta}} M_{ea} \quad ; \quad y_i < y_j$$

<u>Note:</u> For $y_i > y_j$ values, obtain from symmetry :

$$C_{ij}^{\theta\theta} = C_{ji}^{\theta\theta} \quad ; \quad C_{ij}^{zz} = C_{ji}^{zz} \quad ; \quad C_{ij}^{\theta z} = C_{ji}^{z\theta}$$

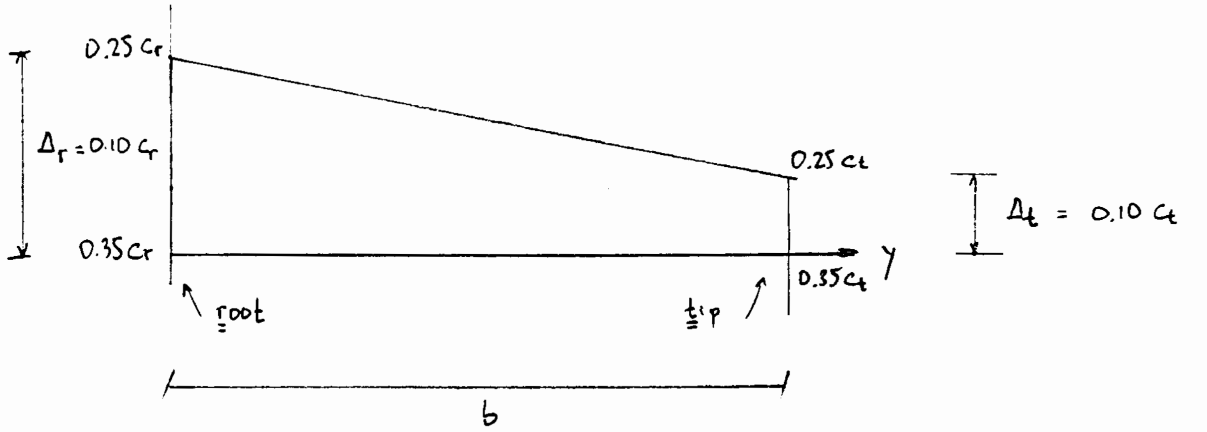
note change in superscripts !

3. Torsional divergence:

$$[A]\,[E]\,\{cc_e\} = \frac{1}{q}\,\{cc_e\}$$

where: $[E] = [C^{\theta\theta}]\,[\lceil W \rfloor]\,[\lceil e \rfloor]$

For the aerodynamic center located at $0.25\,c$ and considering the wing geometry given, one gets:



$$e = 0.10\,\frac{(c_t - c_r)}{b}\,y + 0.10\,c_r$$ ← eq. of straight line

and the y-coordinate related to each station (node):

$$y_i = b\,\cos\theta_i$$

$$\therefore \quad e_i = 0.10\,(c_t - c_r)\cos\theta_i + 0.10\,c_r$$

with: $c_r = 225''$

$c_t = 100''$

$$\Rightarrow \quad e_i = -12.5\cos\theta_i + 22.5$$

$$\theta_1 = \frac{\pi}{8} \longrightarrow e_1 = 10.95$$

$$\theta_2 = \frac{\pi}{4} \longrightarrow e_2 = 13.66$$

$$\theta_3 = \frac{3\pi}{8} \longrightarrow e_3 = 17.72$$

$$\theta_4 = \frac{\pi}{2} \longrightarrow e_4 = 22.50$$

and then:

$$[{}^\backslash e_\backslash] = \begin{bmatrix} 10.95 & & & \\ & 13.66 & & \\ & & 17.72 & \\ & & & 22.50 \end{bmatrix}$$

From the attached Mathematica file:

$$[E] = \begin{bmatrix} 3.4917 & 8.0477 & 13.6357 & 0 \\ 3.4917 & 3.5392 & 5.9967 & 0 \\ 3.4917 & 3.5392 & 2.5211 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \cdot 10^{-4}$$

The eigenvalues of $[A][E]$ was found to be:

$$\lambda = \{ 0.7333 \; , \; -0.1366 \; , \; -0.02676 \; , \; 0. \}$$

and since we are looking for the highest positive eigenvalue of thi[s] system (minimum q, since $\lambda = \frac{1}{q}$),

$$\frac{1}{q_D} = 0.7333$$

$$\Longrightarrow \quad q_D = 1.3637 \ lb/in^2$$

or

$$\boxed{q_D = 196.373 \ lb/ft^2}$$

Now, the associated divergence speed:

$$U_D = \sqrt{\frac{2\,q_D}{\rho}} \qquad ; \qquad \rho = \rho(h)$$

Using the given ISA relation for ρ vs. h (altitude),

$$\rho = 0.002377 \left(1 - 6.8348\, 10^{-6}\, h\right)^{4.2586} \quad slug/ft^3$$

Note: h in ft

the divergence velocity as function of altitude:

$$\boxed{U_D = 406.48 \left(1 - 6.8348\, 10^{-6}\, h\right)^{-2.1293} \quad ft/s}$$

where $0 \leq h \leq 30{,}000\, ft$

Note: For sea level: $U_{D_{SL}} = 406.48\ ft/s$

Problem Set 2 - Solution

Question 3 (a)

■ Given

```
Cθθ = {{424.3,424.3,424.3,0},
       {424.3,186.6,186.6,0},
       {424.3,186.6,78.45,0},
       {    0,    0,    0,    0}} 10^(-9);
MatrixForm[Cθθ]
```

$$\begin{pmatrix} 4.243\times10^{-7} & 4.243\times10^{-7} & 4.243\times10^{-7} & 0 \\ 4.243\times10^{-7} & 1.866\times10^{-7} & 1.866\times10^{-7} & 0 \\ 4.243\times10^{-7} & 1.866\times10^{-7} & 7.845\times10^{-8} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

```
A = {{245.25,90.89,32.31,17.56},
     {49.19,402.38,125.82,31.69},
     {13.38,96.29,550.29,130.43},
     {13.34,44.82,241.00,610.50}};
MatrixForm[A]
```

$$\begin{pmatrix} 245.25 & 90.89 & 32.31 & 17.56 \\ 49.19 & 402.38 & 125.82 & 31.69 \\ 13.38 & 96.29 & 550.29 & 130.43 \\ 13.34 & 44.82 & 241. & 610.5 \end{pmatrix}$$

```
W = DiagonalMatrix[{75.14,138.84,181.40,98.17}];
MatrixForm[W]
```

$$\begin{pmatrix} 75.14 & 0 & 0 & 0 \\ 0 & 138.84 & 0 & 0 \\ 0 & 0 & 181.4 & 0 \\ 0 & 0 & 0 & 98.17 \end{pmatrix}$$

■ Calculated from the given geometry

```
e = DiagonalMatrix[{10.95,13.66,17.72,22.50}];
MatrixForm[e]
```

$$\begin{pmatrix} 10.95 & 0 & 0 & 0 \\ 0 & 13.66 & 0 & 0 \\ 0 & 0 & 17.72 & 0 \\ 0 & 0 & 0 & 22.5 \end{pmatrix}$$

■ Calculation of [E] matrix

```
Ematrix = Cθθ . W . e;
MatrixForm[Ematrix]
```

$$\begin{pmatrix} 0.000349107 & 0.000804708 & 0.00136387 & 0. \\ 0.000349107 & 0.000353897 & 0.000599809 & 0. \\ 0.000349107 & 0.000353897 & 0.00025217 & 0. \\ 0. & 0. & 0. & 0. \end{pmatrix}$$

■ Eigenvalue Problem in [A][E]

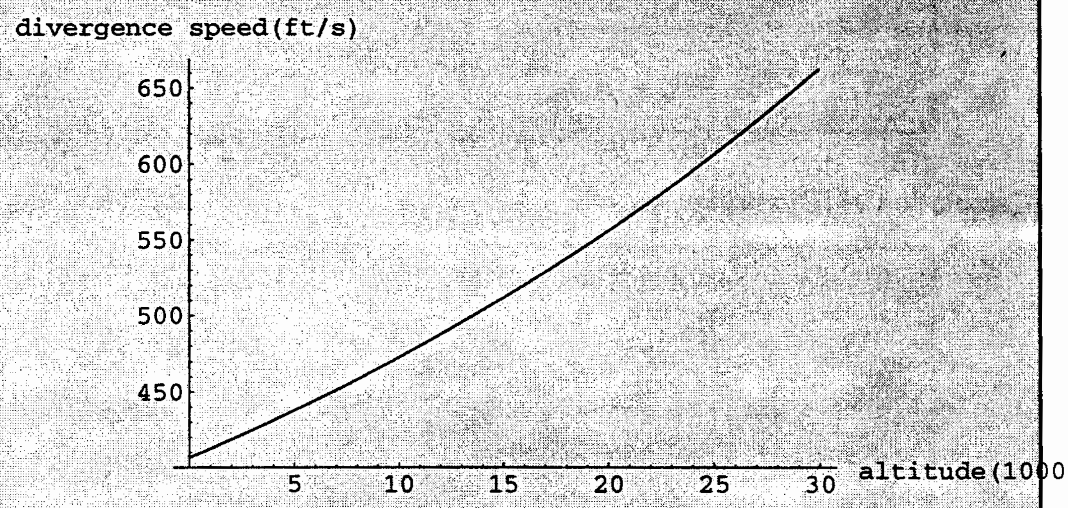
```
Eigenvalues[A.Ematrix]
```

{0.733308, -0.136637, -0.0267588, 0.}

Divergence Velocity as function of Altitude

```
Ud[h_]:=406.48 (1 - h 10^3 6.8348 10^(-6))^(-2.1293);
```

```
Plot[Ud[h],{h,0,30},AxesLabel->{"altitude(1000 ft)","divergence speed(
ft/s)"}]
```



b) Let us consider the two extremum points:

a) $h = 0 \rightarrow U_D \approx 406.5$ ft/s — $a = \sqrt{\gamma R T} = 1123$ ft/s

b) $h = 30{,}000$ ft $\rightarrow U_D \approx 662.6$ ft/s — $a = 1001$ ft/s

where a = speed of sound $\left(\gamma = 1.44 \, , \, R = 287 \, \frac{m^2}{s^2 K} \, , \, T = \text{temperature (ISA} \right.$

Then, the corresponding (torsional divergence) Mach numbers:

a) $h = 0 \longrightarrow \boxed{M_{D_0} = 0.36}$

b) $h = 30{,}000 \longrightarrow \boxed{M_D = 0.66}$

$\longrightarrow \Delta M = \dfrac{M_D - M_{D_0}}{M_{D_0}} = 84\%$

which means that the Mach # range changes considerably. Since we know that the A.I.C. is function of Mach # ($C_{l\alpha} = C_{l\alpha}(M$ we cannot use the same $[A]$ to evaluate U_D at $h=0$ and $h = 30{,}000$ ft, as done in this problem. Compressibility effects need to be taken into account.

One way of dealing with compressibility effects is by the Prandtl-Glauert compressibility correction. The "compressible" A.I.C. is written as:

$$[A]_c = \underbrace{\frac{1}{\sqrt{1-M^2}}}_{\text{Prandtl-Glauert correction term}} \; \underbrace{[A]}_{\text{incompressible } [A]}$$

From the eigenproblem, this yields:

$$\lambda = \sqrt{1-M^2} \; q_c^{-1}$$

... or for this problem:

$$\frac{q_c}{\sqrt{1-M^2}} = 196.37 \qquad (ft/s)$$

(For $M \neq 0 \Rightarrow q_c \sim q_i$)
compressible incom

but : $q_c = \frac{1}{2}\rho U_c^2$; $M^2 = \frac{U^2}{a^2}$

$( )^2 \Rightarrow$

$$q_c^2 = (196.37)^2 \left(1 - \frac{U_c^2}{a^2}\right) = \left[\frac{1}{2}\rho U_c^2\right]^2$$

$$\left(\frac{\frac{1}{2}\rho}{196.37}\right)^2 U^4 + \left(\frac{1}{a^2}\right) U^2 - 1 = 0$$

and the roots are:

$$U_{D_c}^2 = \left[\frac{196.37}{a(h)\,\rho(h)/2}\right]\left[-\frac{196.37}{a(h)\,\rho(h)} + \sqrt{\left(\frac{196.37}{a(h)\,\rho(h)}\right)^2 + a(h)^2}\right]$$

where: $a(h) = \sqrt{\gamma R T(h)}$

from ISA as well