

(c) We have

$$x[n] * [h[n] * g[n]] = \left(\frac{1}{2}\right)^n * \delta[n] = \frac{1}{2}^n$$

$$(x[n] * g[n]) * h[n] = 0 * h[n] = 0,$$

and

$$(x[n] * h[n]) * g[n] = \left\{\left(\frac{1}{2}\right)^n \sum_{k=0}^{\infty} 1\right\} * g[n] = \infty.$$

(d) Let $h(t) = u_1(t)$. Then if the input is $x_1(t) = 0$, the output will be $y_1(t) = 0$. Now if $x_2(t) = \text{constant}$, then $y_2(t) = 0$. Therefore, the system is not invertible.

Now note that

$$\int_{-\infty}^t x_2(\tau) d\tau = \begin{cases} 0 & \text{if } x_2(t) = 0 \forall t \\ \infty & \text{if } x_2(t) \neq 0 \end{cases}$$

Therefore, if $\int_{-\infty}^t x_2(\tau) d\tau \neq \infty$, then only $x_2(t) = 0$ will yield $y_2(t) = 0$. Therefore, the system is invertible.

2.72. We have

$$\delta_{\Delta}(t) = \frac{1}{\Delta} u(t) * [\delta(t) - \delta(t-T)].$$

Differentiating both sides we get

$$\begin{aligned} \frac{d}{dt} \delta_{\Delta}(t) &= \frac{1}{\Delta} u'(t) * [\delta(t) - \delta(t-T)] \\ &= \frac{1}{\Delta} \delta(t) * [\delta(t) - \delta(t-T)] \\ &= \frac{1}{\Delta} [\delta(t) - \delta(t-T)] \end{aligned}$$

2.73. For $k = 1$, $u_{-1}(t) = u(t)$. Therefore, the given statement is true for $k = 1$. Now assume that it is true for some $k > 1$. Then,

$$\begin{aligned} u_{-(k+1)}(t) &= u(t) * u_{-k}(t) \\ &= \int_{-\infty}^t u_{-k}(\tau) d\tau = \int_{-\infty}^t u_{-k}(\tau) d\tau \\ &= \int_0^t \frac{\tau^{k-1}}{(k-1)!} d\tau, \quad t \geq 0 \\ &= \frac{t^k}{k!} \Big|_{\tau=0}^t = \frac{t^k}{k!} u(t). \end{aligned}$$

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3.5. Both $x_1(1-t)$ and $x_1(t-1)$ are periodic with fundamental period $T_1 = \frac{2\pi}{\omega_1}$. Since $y(t)$ is a linear combination of $x_1(1-t)$ and $x_1(t-1)$, it is also periodic with fundamental period $T_2 = \frac{2\pi}{\omega_1}$. Therefore, $\omega_2 = \omega_1$.

Since $x_1(t) \xrightarrow{FS} a_k$, using the results in Table 3.1 we have

$$x_1(t+1) \xrightarrow{FS} a_k e^{jk(2\pi/T_1)}$$

$$x_1(t-1) \xrightarrow{FS} a_k e^{-jk(2\pi/T_1)} \Rightarrow x_1(-t+1) \xrightarrow{FS} a_{-k} e^{-jk(2\pi/T_1)}$$

Therefore,

$$x_1(t+1) + x_1(1-t) \xrightarrow{FS} a_k e^{jk(2\pi/T_1)} + a_{-k} e^{-jk(2\pi/T_1)} = e^{-j\omega_1 k} (a_k + a_{-k})$$

3.6. (a) Comparing $x_1(t)$ with the Fourier series synthesis eq. (3.38), we obtain the Fourier series coefficients of $x_1(t)$ to be

$$a_k = \begin{cases} \left(\frac{1}{2}\right)^k, & 0 \leq k \leq 100 \\ 0, & \text{otherwise} \end{cases}$$

From Table 3.1 we know that if $x_1(t)$ is real, then a_k has to be conjugate-symmetric, i.e., $a_k = a_{-k}^*$. Since this is not true for $x_1(t)$, the signal is **not real valued**.

Similarly, the Fourier series coefficients of $x_2(t)$ are

$$a_k = \begin{cases} \cos(k\pi), & 100 \leq k \leq 100 \\ 0, & \text{otherwise} \end{cases}$$

From Table 3.1 we know that if $x_2(t)$ is real, then a_k has to be conjugate-symmetric, i.e., $a_k = a_{-k}^*$. Since this is true for $x_2(t)$, the signal is **real valued**.

Similarly, the Fourier series coefficients of $x_3(t)$ are

$$a_k = \begin{cases} j \sin(k\pi/2), & 100 \leq k \leq 100 \\ 0, & \text{otherwise} \end{cases}$$

From Table 3.1 we know that if $x_3(t)$ is real, then a_k has to be conjugate-symmetric, i.e., $a_k = a_{-k}^*$. Since this is true for $x_3(t)$, the signal is **real valued**.

(b) For a signal to be even, its Fourier series coefficients must be even. This is true only for $x_2(t)$.

3.7. Given that

$$x(t) \xrightarrow{FS} a_k$$

we have

$$g(t) = \frac{dx(t)}{dt} \xrightarrow{FS} b_k = jk \frac{2\pi}{T} a_k.$$

Therefore,

$$a_k = \frac{b_k}{j(2\pi/T)k}, \quad k \neq 0$$

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Chapter 3 Answers

3.1. Using the Fourier series synthesis eq. (3.38),

$$\begin{aligned} x(t) &= a_1 e^{j(2\pi/T)t} + a_{-1} e^{-j(2\pi/T)t} + a_3 e^{j(2\pi/T)t} + a_{-3} e^{-j(2\pi/T)t} \\ &= 2e^{j(2\pi/8)t} + 2e^{-j(2\pi/8)t} + 4je^{j(2\pi/8)t} - 4je^{-j(2\pi/8)t} \\ &= 4\cos\left(\frac{\pi}{4}t\right) - 8\sin\left(\frac{\pi}{4}t\right) \\ &= 4\cos\left(\frac{\pi}{4}t\right) + 8\cos\left(\frac{3\pi}{4}t + \frac{\pi}{2}\right) \end{aligned}$$

3.2. Using the Fourier series synthesis eq. (3.95),

$$\begin{aligned} x[n] &= a_0 + a_2 e^{j2(2\pi/N)n} + a_{-2} e^{-j2(2\pi/N)n} + a_4 e^{j4(2\pi/N)n} + a_{-4} e^{-j4(2\pi/N)n} \\ &= 1 + e^{j(\pi/4)} e^{j2(2\pi/5)n} + e^{-j(\pi/4)} e^{-j2(2\pi/5)n} \\ &\quad + 2e^{j(\pi/3)} e^{j4(2\pi/N)n} + 2e^{-j(\pi/3)} e^{-j4(2\pi/N)n} \\ &= 1 + 2\cos\left(\frac{4\pi}{5}n + \frac{\pi}{4}\right) + 4\cos\left(\frac{8\pi}{5}n + \frac{\pi}{3}\right) \\ &= 1 + 2\sin\left(\frac{4\pi}{5}n + \frac{3\pi}{4}\right) + 4\sin\left(\frac{8\pi}{5}n + \frac{5\pi}{6}\right) \end{aligned}$$

3.3. The given signal is

$$\begin{aligned} x(t) &= 2 + \frac{1}{2} e^{j(2\pi/3)t} + \frac{1}{2} e^{-j(2\pi/3)t} - 2je^{j(5\pi/3)t} + 2je^{-j(5\pi/3)t} \\ &= 2 + \frac{1}{2} e^{j2(2\pi/6)t} + \frac{1}{2} e^{-j2(2\pi/6)t} - 2je^{j5(2\pi/6)t} + 2je^{-j5(2\pi/6)t} \end{aligned}$$

From this, we may conclude that the fundamental frequency of $x(t)$ is $2\pi/6 = \pi/3$. The non-zero Fourier series coefficients of $x(t)$ are:

$$a_0 = 2, \quad a_2 = a_{-2} = \frac{1}{2}, \quad a_5 = a_{-5} = -2j$$

3.4. Since $\omega_0 = \pi$, $T = 2\pi/\omega_0 = 2$. Therefore,

$$a_k = \frac{1}{2} \int_0^2 x(t) e^{-jk\pi t} dt$$

Now,

$$a_0 = \frac{1}{2} \int_0^1 1.5 dt - \frac{1}{2} \int_1^2 1.5 dt = 0$$

and for $k \neq 0$

$$\begin{aligned} a_k &= \frac{1}{2} \int_0^1 1.5 e^{-jk\pi t} dt - \frac{1}{2} \int_1^2 1.5 e^{-jk\pi t} dt \\ &= \frac{3}{2k\pi j} [1 - e^{-jk\pi}] \\ &= \frac{3}{k\pi} e^{-jk(\pi/2)} \sin\left(\frac{k\pi}{2}\right) \end{aligned}$$

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When $k = 0$,

$$a_k = \frac{1}{T} \int_{\langle T \rangle} x(t) dt = \frac{2}{T} \quad \text{using given information}$$

Therefore,

$$a_k = \begin{cases} \frac{2}{T}, & k = 0 \\ \frac{3}{j(2\pi/T)k}, & k \neq 0 \end{cases}$$

3.8. Since $x(t)$ is real and odd (clue 1), its Fourier series coefficients a_k are purely imaginary and odd (See Table 3.1). Therefore, $a_k = -a_{-k}$ and $a_0 = 0$. Also, since it is given that $a_k = 0$ for $|k| > 1$, the only unknown Fourier series coefficients are a_1 and a_{-1} . Using Parseval's relation,

$$\frac{1}{T} \int_{\langle T \rangle} |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2,$$

for the given signal we have

$$\frac{1}{2} \int_0^2 |x(t)|^2 dt = \sum_{k=-1}^1 |a_k|^2.$$

Using the information given in clue (4) along with the above equation,

$$|a_1|^2 + |a_{-1}|^2 = 1 \Rightarrow 2|a_1|^2 = 1$$

Therefore,

$$a_1 = -a_{-1} = \frac{1}{\sqrt{2}j} \quad \text{or} \quad a_1 = -a_{-1} = -\frac{1}{\sqrt{2}j}$$

The two possible signals which satisfy the given information are

$$x_1(t) = \frac{1}{\sqrt{2}j} e^{j(2\pi/2)t} - \frac{1}{\sqrt{2}j} e^{-j(2\pi/2)t} = -\sqrt{2} \sin(\pi t)$$

and

$$x_2(t) = -\frac{1}{\sqrt{2}j} e^{j(2\pi/2)t} + \frac{1}{\sqrt{2}j} e^{-j(2\pi/2)t} = \sqrt{2} \sin(\pi t)$$

3.9. The period of the given signal is 4. Therefore,

$$\begin{aligned} a_k &= \frac{1}{4} \sum_{n=0}^3 x[n] e^{-j\frac{2\pi}{4}kn} \\ &= \frac{1}{4} [4 + 8e^{-j\frac{3\pi}{2}k}] \end{aligned}$$

This gives

$$a_0 = 3, \quad a_1 = 1 - 2j, \quad a_2 = -1, \quad a_3 = 1 + 2j$$

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3.10. Since the Fourier series coefficients repeat every N , we have

$$a_1 = a_{15}, \quad a_2 = a_{16}, \quad \text{and} \quad a_3 = a_{17}$$

Furthermore, since the signal is real and odd, the Fourier series coefficients a_k will be purely imaginary and odd. Therefore, $a_0 = 0$ and

$$a_1 = -a_{-1}, \quad a_2 = -a_{-2}, \quad a_3 = -a_{-3}$$

Finally,

$$a_{-1} = -j, \quad a_{-2} = -2j, \quad a_{-3} = -3j$$

3.11. Since the Fourier series coefficients repeat every $N = 10$, we have $a_1 = a_{11} = 5$. Furthermore, since $x[n]$ is real and even, a_k is also real and even. Therefore, $a_1 = a_{-1} = 5$. We are also given that

$$\frac{1}{10} \sum_{n=0}^9 |x[n]|^2 = 50.$$

Using Parseval's relation,

$$\begin{aligned} \sum_{k=-\infty}^{\infty} |a_k|^2 &= 50 \\ \sum_{k=-1}^8 |a_k|^2 &= 50 \\ |a_{-1}|^2 + |a_1|^2 + a_0^2 + \sum_{k=2}^8 |a_k|^2 &= 50 \\ a_0^2 + \sum_{k=2}^8 |a_k|^2 &= 0 \end{aligned}$$

Therefore, $a_k = 0$ for $k = 2, \dots, 8$. Now using the synthesis eq.(3.94), we have

$$\begin{aligned} x[n] &= \sum_{k=-\infty}^{\infty} a_k e^{j\frac{2\pi}{N}kn} = \sum_{k=-1}^8 a_k e^{j\frac{2\pi}{10}kn} \\ &= 5e^{j\frac{2\pi}{10}n} + 5e^{-j\frac{2\pi}{10}n} \\ &= 10 \cos\left(\frac{\pi}{5}n\right) \end{aligned}$$

3.12. Using the multiplication property (see Table 3.2), we have

$$\begin{aligned} x_1[n]x_2[n] &\xrightarrow{FS} \sum_{l=-\infty}^{\infty} a_l b_{k-l} = \sum_{k=0}^3 a_k b_{k-l} \\ &\xrightarrow{FS} a_0 b_k + a_1 b_{k-1} + a_2 b_{k-2} + a_3 b_{k-3} \\ &\xrightarrow{FS} b_k + 2b_{k-1} + 2b_{k-2} + 2b_{k-3} \end{aligned}$$

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From the given information, we know that $y[n]$ is

$$\begin{aligned} y[n] &= \cos\left(\frac{5\pi}{2}n + \frac{\pi}{4}\right) \\ &= \cos\left(\frac{\pi}{2}n + \frac{\pi}{4}\right) \\ &= \frac{1}{2}e^{j(\frac{\pi}{2}n + \frac{\pi}{4})} + \frac{1}{2}e^{-j(\frac{\pi}{2}n + \frac{\pi}{4})} \\ &= \frac{1}{2}e^{j(\frac{\pi}{2}n + \frac{\pi}{4})} + \frac{1}{2}e^{j(3\frac{\pi}{2}n - \frac{\pi}{4})} \end{aligned}$$

Comparing this with eq. (S3.14-1), we have

$$H(e^{j0}) = H(e^{j\pi}) = 0$$

and

$$H(e^{j\frac{\pi}{2}}) = 2e^{j\frac{\pi}{4}}, \quad \text{and} \quad H(e^{j\frac{3\pi}{2}}) = 2e^{-j\frac{\pi}{4}}$$

3.15. From the results of Section 3.8,

$$y(t) = \sum_{k=-\infty}^{\infty} a_k H(jk\omega_0) e^{jk\omega_0 t}$$

where $\omega_0 = \frac{2\pi}{T} = 12$. Since $H(j\omega)$ is zero for $|\omega| > 100$, the largest value of $|k|$ for which a_k is nonzero should be such that

$$|k|\omega_0 \leq 100$$

This implies that $|k| \leq 8$. Therefore, for $|k| > 8$, a_k is guaranteed to be zero.

3.16. (a) The given signal $x_1[n]$ is

$$x_1[n] = \begin{cases} 1 & -1 \leq n \leq 1 \\ 0 & \text{elsewhere} \end{cases} = e^{j2\pi n/2} - e^{j2\pi(n-1)/2}$$

Therefore, $x_1[n]$ is periodic with period $N = 2$ and its Fourier series coefficients in the range $0 \leq k \leq 1$ are

$$a_0 = 0, \quad \text{and} \quad a_1 = 1$$

Using the results derived in Section 3.8, the output $y_1[n]$ is given by

$$\begin{aligned} y_1[n] &= \sum_{k=0}^1 a_k H(e^{j2\pi k/2}) e^{jk(2\pi/2)n} \\ &= 0 + a_1 H(e^{j\pi}) e^{j\pi n} \\ &= 0 \end{aligned}$$

(b) The signal $x_2[n]$ is periodic with period $N = 16$. The signal $x_2[n]$ may be written as

$$\begin{aligned} x_2[n] &= e^{j(2\pi/16)(0)n} - (j/2)e^{j(\pi/4)}e^{j(2\pi/16)(3)n} + (j/2)e^{-j(\pi/4)}e^{-j(2\pi/16)(3)n} \\ &= e^{j(2\pi/16)(0)n} - (j/2)e^{j(\pi/4)}e^{j(2\pi/16)(3)n} + (j/2)e^{-j(\pi/4)}e^{j(2\pi/16)(13)n} \end{aligned}$$

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Since b_k is 1 for all values of k , it is clear that $b_k + 2b_{k-1} + 2b_{k-2} + 2b_{k-3}$ will be 6 for all values of k . Therefore,

$$x_1[n]x_2[n] \xrightarrow{FS} 6, \quad \text{for all } k.$$

3.13. Let us first evaluate the Fourier series coefficients of $x(t)$. Clearly, since $x(t)$ is real and odd, a_k is purely imaginary and odd. Therefore, $a_0 = 0$. Now,

$$\begin{aligned} a_k &= \frac{1}{8} \int_0^8 x(t) e^{-j(2\pi/8)kt} dt \\ &= \frac{1}{8} \int_0^4 e^{-j(2\pi/8)kt} dt - \frac{1}{8} \int_4^8 e^{-j(2\pi/8)kt} dt \\ &= \frac{1}{j\pi k} [1 - e^{-j\pi k}] \end{aligned}$$

Clearly, the above expression evaluates to zero for all even values of k . Therefore,

$$a_k = \begin{cases} 0, & k = 0, \pm 2, \pm 4, \dots \\ \frac{2}{j\pi k}, & k = \pm 1, \pm 3, \pm 5, \dots \end{cases}$$

When $x(t)$ is passed through an LTI system with frequency response $H(j\omega)$, the output $y(t)$ is given by (see Section 3.8)

$$y(t) = \sum_{k=-\infty}^{\infty} a_k H(jk\omega_0) e^{jk\omega_0 t}$$

where $\omega_0 = \frac{2\pi}{T} = \frac{\pi}{4}$. Since a_k is non zero only for odd values of k , we need to evaluate the above summation only for odd k . Furthermore, note that

$$H(jk\omega_0) = H(jk(\pi/4)) = \frac{\sin(k\pi)}{k(\pi/4)}$$

is always zero for odd values of k . Therefore,

$$y(t) = 0.$$

3.14. The signal $x[n]$ is periodic with period $N = 4$. Its Fourier series coefficients are

$$\begin{aligned} a_k &= \frac{1}{4} \sum_{n=0}^3 x[n] e^{-j\frac{2\pi}{4}kn} \\ &= \frac{1}{4}, \quad \text{for all } k \end{aligned}$$

From the results presented in Section 3.8, we know that the output $y[n]$ is given by

$$\begin{aligned} y[n] &= \sum_{k=0}^3 a_k H(e^{j2\pi k/4}) e^{jk(2\pi/4)n} \\ &= \frac{1}{4} H(e^{j0}) e^{j0} + \frac{1}{4} H(e^{j\pi/2}) e^{j(\pi/2)n} \\ &\quad + \frac{1}{4} H(e^{j3\pi/2}) e^{j(3\pi/2)n} + \frac{1}{4} H(e^{j\pi}) e^{j\pi n} \end{aligned} \quad (\text{S3.14-1})$$

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Therefore, the non-zero Fourier series coefficients of $x_2[n]$ in the range $0 \leq k \leq 15$ are

$$a_0 = 1, \quad a_3 = -(j/2)e^{j(\pi/4)}, \quad a_{13} = (j/2)e^{-j(\pi/4)}$$

Using the results derived in Section 3.8, the output $y_2[n]$ is given by

$$\begin{aligned} y_2[n] &= \sum_{k=0}^{15} a_k H(e^{j2\pi k/16}) e^{jk(2\pi/16)n} \\ &= 0 - (j/2)e^{j(\pi/4)} e^{j(2\pi/16)(3)n} + (j/2)e^{-j(\pi/4)} e^{j(2\pi/16)(13)n} \\ &= \sin\left(\frac{3\pi}{8}n + \frac{\pi}{4}\right) \end{aligned}$$

(c) The signal $x_3[n]$ may be written as

$$x_3[n] = \left[\left(\frac{1}{2}\right)^n u[n]\right] * \sum_{k=-\infty}^{\infty} \delta[n-4k] = g[n] * r[n]$$

where $g[n] = \left(\frac{1}{2}\right)^n u[n]$ and $r[n] = \sum_{k=-\infty}^{\infty} \delta[n-4k]$. Therefore, $y_3[n]$ may be obtained

by passing the signal $r[n]$ through the filter with frequency response $H(e^{j\omega})$, and then convolving the result with $g[n]$.

The signal $r[n]$ is periodic with period 4 and its Fourier series coefficients are

$$a_k = \frac{1}{4}, \quad \text{for all } k \quad (\text{See Problem 3.14})$$

The output $q[n]$ obtained by passing $r[n]$ through the filter with frequency response $H(e^{j\omega})$ is

$$\begin{aligned} q[n] &= \sum_{k=0}^3 a_k H(e^{j2\pi k/4}) e^{jk(2\pi/4)n} \\ &= (1/4) \{ H(e^{j0}) e^{j0} + H(e^{j\pi/2}) e^{j(\pi/2)n} + H(e^{j\pi}) e^{j\pi n} + H(e^{j3\pi/2}) e^{j(3\pi/2)n} \} \\ &= 0 \end{aligned}$$

Therefore, the final output $y_3[n] = q[n] * g[n] = 0$.

3.17. (a) Since complex exponentials are Eigen functions of LTI systems, the input $x_1(t) = e^{j5t}$ has to produce an output of the form Ae^{j5t} , where A is a complex constant. But clearly, in this case the output is not of this form. Therefore, system S_1 is definitely not LTI.

(b) This system may be LTI because it satisfies the Eigen function property of LTI systems.

(c) In this case, the output is of the form $y_3(t) = (1/2)e^{j5t} + (1/2)e^{-j5t}$. Clearly, the output contains a complex exponential with frequency -5 which was not present in the input $x_3(t)$. We know that an LTI system can never produce a complex exponential of frequency -5 unless there was complex exponential of the same frequency at its input. Since this is not the case in this problem, S_3 is definitely not LTI.

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3.18. (a) By using an argument similar to the one used in part (a) of the previous problem, we conclude that S_1 is definitely not LTI.

(b) The output in this case is $y_2[n] = e^{j(3\pi/2)n} = e^{-j(\pi/2)n}$. Clearly this violates the eigenfunction property of LTI systems. Therefore, S_2 is definitely not LTI.

(c) The output in this case is $y_3[n] = 2e^{j(5\pi/2)n} = 2e^{j(\pi/2)n}$. This does not violate the eigenfunction property of LTI systems. Therefore, S_3 could possibly be an LTI system.

3.19. (a) Voltage across inductor $= L \frac{dy(t)}{dt}$.
Current through resistor $= \frac{1}{R} \frac{dy(t)}{dt}$.
Input current $x(t) =$ current through resistor + current through inductor
Therefore,

$$x(t) = \frac{L}{R} \frac{dy(t)}{dt} + y(t).$$

Substituting for R and L we obtain

$$\frac{dy(t)}{dt} + y(t) = x(t).$$

(b) Using the approach outlined in Section 3.10.1, we know that the output of this system will be $H(j\omega)e^{j\omega t}$ when the input is $e^{j\omega t}$. Substituting in the differential equation of part (a),

$$j\omega H(j\omega)e^{j\omega t} + H(j\omega)e^{j\omega t} = e^{j\omega t}$$

Therefore,

$$H(j\omega) = \frac{1}{1 + j\omega}$$

(c) The signal $x(t)$ is periodic with period 2π . Since $x(t)$ can be expressed in the form

$$x(t) = \frac{1}{2}e^{j(2\pi/2\pi)t} + \frac{1}{2}e^{-j(2\pi/2\pi)t},$$

the non-zero Fourier series coefficients of $x(t)$ are

$$a_1 = a_{-1} = \frac{1}{2}.$$

Using the results derived in Section 3.8 (see eq.(3.124)), we have

$$\begin{aligned} y(t) &= a_1 H(j) e^{jt} + a_{-1} H(-j) e^{-jt} \\ &= (1/2) \left(\frac{1}{1+j} e^{jt} + \frac{1}{1-j} e^{-jt} \right) \\ &= (1/2\sqrt{2}) (e^{-j\pi/4} e^{jt} + e^{j\pi/4} e^{-jt}) \\ &= (1/\sqrt{2}) \cos(t - \pi/4) \end{aligned}$$

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3.22. (a) (i) $T = 1$, $a_0 = 0$, $a_k = \frac{j(-1)^k}{k\pi}$, $k \neq 0$.

(ii) Here,

$$x(t) = \begin{cases} t+2, & -2 < t < -1 \\ 1, & -1 < t < 1 \\ 2-t, & 1 < t < 2 \end{cases}$$

$T = 6$, $a_0 = 1/2$, and

$$a_k = \begin{cases} 0, & k \text{ even} \\ \frac{6}{\pi^2 k^2} \sin(\frac{\pi k}{2}) \sin(\frac{\pi k}{6}), & k \text{ odd} \end{cases}$$

(iii) $T = 3$, $a_0 = 1$, and

$$a_k = \frac{3j}{2\pi^2 k^2} [e^{j2\pi/3} \sin(k2\pi/3) + 2e^{j\pi/3} \sin(k\pi/3)], \quad k \neq 0.$$

(iv) $T = 2$, $a_0 = -1/2$, $a_k = \frac{1}{2} - (-1)^k$, $k \neq 0$.

(v) $T = 6$, $\omega_0 = \pi/3$, and

$$a_k = \frac{\cos(2k\pi/3) - \cos(k\pi/3)}{jk\pi/3}.$$

Note that $a_0 = 0$ and a_k even = 0.

(vi) $T = 4$, $\omega_0 = \pi/2$, $a_0 = 3/4$ and

$$a_k = \frac{e^{-j\pi/2} \sin(k\pi/2) + e^{-j\pi/4} \sin(k\pi/4)}{k\pi}, \quad \forall k.$$

(b) $T = 2$, $a_k = \frac{-1^k}{j(1+jk\pi)} [e - e^{-1}]$ for all k .

(c) $T = 3$, $\omega_0 = 2\pi/3$, $a_0 = 1$ and

$$a_k = \frac{2e^{-j\pi/3}}{\pi k} \sin(2\pi k/3) + \frac{e^{-j\pi k}}{\pi k} \sin(\pi k).$$

3.23. (a) First let us consider a signal $y(t)$ with FS coefficients

$$b_k = \frac{\sin(k\pi/4)}{k\pi}.$$

From Example 3.5, we know that $y(t)$ must be a periodic square wave which over one period is

$$y(t) = \begin{cases} 1, & |t| < 1/2 \\ 0, & 1/2 < |t| < 2 \end{cases}$$

Now, note that $b_0 = 1/4$. Let us define another signal $z(t) = -1/4$ whose only nonzero FS coefficient is $c_0 = -1/4$. The signal $p(t) = y(t) + z(t)$ will have FS coefficients

$$d_k = a_k + c_k = \begin{cases} 0, & k = 0 \\ \frac{\sin(k\pi/4)}{k\pi}, & \text{otherwise.} \end{cases}$$

Now note that $a_k = d_k e^{j(\pi/2)k}$. Therefore, the signal $x(t) = p(t+1)$ which is as shown in Figure S2.23(a).

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3.20. (a) Current through the capacitor $= C \frac{dy(t)}{dt}$.

Voltage across resistor $= RC \frac{dy(t)}{dt}$.

Voltage across inductor $= LC \frac{d^2 y(t)}{dt^2}$.

Input voltage = Voltage across resistor + Voltage across inductor + Voltage across capacitor.

Therefore,

$$x(t) = LC \frac{d^2 y(t)}{dt^2} + RC \frac{dy(t)}{dt} + y(t)$$

Substituting for R , L and C , we have

$$\frac{d^2 y(t)}{dt^2} + \frac{dy(t)}{dt} + y(t) = x(t)$$

(b) We will now use an approach similar to the one used in part (b) of the previous problem. If we assume that the input is of the form $e^{j\omega t}$, then the output will be of the form $H(j\omega)e^{j\omega t}$. Substituting in the above differential equation and simplifying, we obtain

$$H(j\omega) = \frac{1}{-\omega^2 + j\omega + 1}$$

(c) The signal $x(t)$ is periodic with period 2π . Since $x(t)$ can be expressed in the form

$$x(t) = \frac{1}{2j} e^{j(2\pi/2\pi)t} - \frac{1}{2j} e^{-j(2\pi/2\pi)t},$$

the non-zero Fourier series coefficients of $x(t)$ are

$$a_1 = a_{-1} = \frac{1}{2j}.$$

Using the results derived in Section 3.8 (see eq.(3.124)), we have

$$\begin{aligned} y(t) &= a_1 H(j) e^{jt} + a_{-1} H(-j) e^{-jt} \\ &= (1/2j) \left(\frac{1}{j} e^{jt} - \frac{1}{-j} e^{-jt} \right) \\ &= (-1/2) (e^{jt} + e^{-jt}) \\ &= -\cos(t) \end{aligned}$$

3.21. Using the Fourier series synthesis eq. (3.38),

$$\begin{aligned} x(t) &= a_1 e^{j(2\pi/7)t} + a_{-1} e^{-j(2\pi/7)t} + a_5 e^{j5(2\pi/7)t} + a_{-5} e^{-j5(2\pi/7)t} \\ &= j e^{j(2\pi/7)t} - j e^{-j(2\pi/7)t} + 2e^{j5(2\pi/7)t} + 2e^{-j5(2\pi/7)t} \\ &= -2 \sin(\frac{\pi}{4}t) + 4 \cos(\frac{5\pi}{4}t) \\ &= -2 \cos(\frac{\pi}{4}t - \pi/2) + 4 \cos(\frac{5\pi}{4}t). \end{aligned}$$

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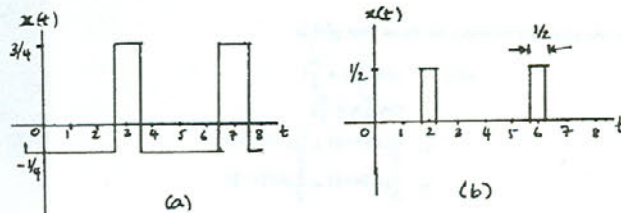


Figure S3.23

(b) First let us consider a signal $y(t)$ with FS coefficients

$$b_k = \frac{\sin(k\pi/8)}{2k\pi}.$$

From Example 3.5, we know that $y(t)$ must be a periodic square wave which over one period is

$$y(t) = \begin{cases} 1/2, & |t| < 1/4 \\ 0, & 1/4 < |t| < 2 \end{cases}$$

Now note that $a_k = b_k e^{j\pi k}$. Therefore, the signal $x(t) = y(t+2)$ which is as shown in Figure S2.23(b).

(c) The only nonzero FS coefficients are $a_1 = a_{-1} = j$ and $a_2 = a_{-2} = 2j$. Using the FS synthesis equation, we get

$$\begin{aligned} x(t) &= a_1 e^{j(2\pi/7)t} + a_{-1} e^{-j(2\pi/7)t} + a_2 e^{j2(2\pi/7)t} + a_{-2} e^{-j2(2\pi/7)t} \\ &= j e^{j(2\pi/7)t} - j e^{-j(2\pi/7)t} + 2j e^{j2(2\pi/7)t} - 2j e^{-j2(2\pi/7)t} \\ &= -2 \sin(\frac{\pi}{2}t) - 4 \sin(\pi t) \end{aligned}$$

(d) The FS coefficients a_k may be written as the sum of two sets of FS coefficients b_k and c_k , where

$$b_k = 1, \quad \text{for all } k$$

and

$$c_k = \begin{cases} 1, & k \text{ odd} \\ 0, & k \text{ even} \end{cases}$$

The FS coefficients b_k correspond to the signal

$$y(t) = \sum_{k=-\infty}^{\infty} \delta(t - 4k)$$

and the FS coefficients c_k correspond to the signal

$$z(t) = \sum_{k=-\infty}^{\infty} e^{j(\pi/2)k} \delta(t - 2k).$$

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Therefore,

$$x(t) = y(t) + p(t) = \sum_{k=-\infty}^{\infty} \delta(t-4k) + \sum_{k=-\infty}^{\infty} e^{j(\pi/2)t} \delta(t-2k).$$

3.24. (a) We have

$$a_0 = \frac{1}{2} \int_0^1 t dt + \frac{1}{2} \int_1^2 (2-t) dt = 1/2.$$

(b) The signal $g(t) = dx(t)/dt$ is as shown in Figure S3.24.

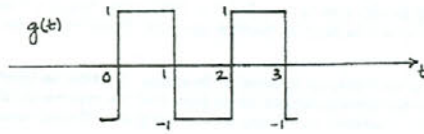


Figure S3.24

The FS coefficients b_k of $g(t)$ may be found as follows:

$$b_0 = \frac{1}{2} \int_0^1 dt - \frac{1}{2} \int_1^2 dt = 0$$

and

$$b_k = \frac{1}{2} \int_0^1 e^{-j\pi k t} dt - \frac{1}{2} \int_1^2 e^{-j\pi k t} dt = \frac{1}{j\pi k} [1 - e^{-j\pi k}].$$

(c) Note that

$$g(t) = \frac{dx(t)}{dt} \xrightarrow{FS} b_k = jk\pi a_k.$$

Therefore,

$$a_k = \frac{1}{j\pi k} b_k = -\frac{1}{\pi^2 k^2} (1 - e^{-j\pi k}).$$

3.25. (a) The nonzero FS coefficients of $x(t)$ are $a_1 = a_{-1} = 1/2$.

(b) The nonzero FS coefficients of $x(t)$ are $b_1 = b_{-1} = 1/2j$.

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(c) Using the multiplication property, we know that

$$z(t) = x(t)y(t) \xrightarrow{FS} c_k = \sum_{l=-\infty}^{\infty} a_l b_{k-l}.$$

Therefore,

$$c_k = a_k * b_k = \frac{1}{4j} \delta[k-2] - \frac{1}{4j} \delta[k+2].$$

This implies that the nonzero Fourier series coefficients of $z(t)$ are $c_2 = c_{-2} = (1/4j)$.

(d) We have

$$z(t) = \sin(4t) \cos(4t) = \frac{1}{2} \sin(8t).$$

Therefore, the nonzero Fourier series coefficients of $z(t)$ are $c_2 = c_{-2} = (1/4j)$.

3.26. (a) If $x(t)$ is real, then $x(t) = x^*(t)$. This implies that for $x(t)$ real $a_k = a_{-k}^*$. Since this is not true in this case problem, $x(t)$ is not real.

(b) If $x(t)$ is even, then $x(t) = x(-t)$ and $a_k = a_{-k}$. Since this is true for this case, $x(t)$ is even.

(c) We have

$$g(t) = \frac{dx(t)}{dt} \xrightarrow{FS} b_k = jk \frac{2\pi}{T_0} a_k.$$

Therefore,

$$b_k = \begin{cases} 0, & k = 0 \\ -k(1/2)^{|k|}(2\pi/T_0), & \text{otherwise} \end{cases}$$

Since b_k is not even, $g(t)$ is not even.

3.27. Using the Fourier series synthesis eq. (3.38),

$$\begin{aligned} x[n] &= a_0 + a_2 e^{j2(2\pi/N)n} + a_{-2} e^{-j2(2\pi/N)n} + a_4 e^{j4(2\pi/N)n} + a_{-4} e^{-j4(2\pi/N)n} \\ &= 2 + 2e^{j\pi/6} e^{j(4\pi/5)n} + 2e^{-j\pi/6} e^{-j(4\pi/5)n} + e^{j\pi/3} e^{j(8\pi/5)n} + e^{-j\pi/3} e^{-j(8\pi/5)n} \\ &= 2 + 4 \cos[(4\pi n/5) + \pi/6] + 2 \cos[(8\pi n/5) + \pi/3] \\ &= 2 + 4 \sin[(4\pi n/5) + 2\pi/3] + 2 \sin[(8\pi n/5) + 5\pi/6] \end{aligned}$$

3.28. (a) $N = 7$,

$$a_k = \frac{1}{7} \frac{e^{-j4\pi k/7} \sin(5\pi k/7)}{\sin(\pi k/7)}.$$

(b) $N = 6$, a_k over one period ($0 \leq k \leq 5$) may be specified as: $a_0 = 4/6$,

$$a_k = \frac{1}{6} e^{-j\pi k/2} \frac{\sin(\frac{2\pi k}{3})}{\sin(\frac{\pi k}{6})}, \quad 1 \leq k \leq 5.$$

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(c) $N = 6$,

$$a_k = 1 + 4 \cos(\pi k/3) - 2 \cos(2\pi k/3).$$

(d) $N = 12$, a_k over one period ($0 \leq k \leq 11$) may be specified as: $a_1 = \frac{1}{4j} = a_{11}^*$, $a_5 = -\frac{1}{4j} = a_7^*$, $a_k = 0$ otherwise.

(e) $N = 4$,

$$a_k = 1 + 2(-1)^k (1 - \frac{1}{\sqrt{2}}) \cos(\frac{\pi k}{2}).$$

(f) $N = 12$,

$$a_k = 1 + (1 - \frac{1}{\sqrt{2}}) 2 \cos(\frac{\pi k}{6}) + 2(1 - \frac{1}{\sqrt{2}}) \cos(\frac{\pi k}{2}) + 2(1 + \frac{1}{\sqrt{2}}) \cos(\frac{5\pi k}{6}) + 2(-1)^k + 2 \cos(\frac{2\pi k}{3}).$$

3.29. (a) $N = 8$. Over one period ($0 \leq n \leq 7$),

$$x[n] = 4\delta[n-1] + 4\delta[n-7] + 4j\delta[n-3] - 4j\delta[n-5].$$

(b) $N = 8$. Over one period ($0 \leq n \leq 7$),

$$x[n] = \frac{1}{2j} \left[\frac{-e^{j\frac{3\pi n}{4}} \sin(\frac{7}{2}(\frac{\pi n}{4} + \frac{\pi}{8}))}{\sin(\frac{1}{2}(\frac{\pi n}{4} + \frac{\pi}{8}))} + \frac{e^{j\frac{3\pi n}{4}} \sin(\frac{7}{2}(\frac{\pi n}{4} - \frac{\pi}{8}))}{\sin(\frac{1}{2}(\frac{\pi n}{4} - \frac{\pi}{8}))} \right].$$

(c) $N = 8$. Over one period ($0 \leq n \leq 7$),

$$x[n] = 1 + (-1)^n + 2 \cos(\frac{\pi n}{4}) + 2 \cos(\frac{3\pi n}{4}).$$

(d) $N = 8$. Over one period ($0 \leq n \leq 7$),

$$x[n] = 2 + 2 \cos(\frac{\pi n}{4}) + \cos(\frac{\pi n}{2}) + \frac{1}{2} \cos(\frac{3\pi n}{4}).$$

3.30. (a) The nonzero FS coefficients of $x(t)$ are $a_0 = 1$, $a_1 = a_{-1} = 1/2$.

(b) The nonzero FS coefficients of $x(t)$ are $b_1 = b_{-1} = e^{-j\pi/4}/2$.

(c) Using the multiplication property, we know that

$$z[n] = x[n]y[n] \xrightarrow{FS} c_k = \sum_{l=-\infty}^{\infty} a_l b_{k-l}.$$

This implies that the nonzero Fourier series coefficients of $z[n]$ are $c_0 = \cos(\pi/4)/2$, $c_1 = c_{-1}^* = e^{-j\pi/4}/2$, $c_2 = c_{-2}^* = e^{-j\pi/4}/4$.

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(d) We have

$$\begin{aligned} z[n] &= \sin\left(\frac{2\pi n}{6} + \frac{\pi}{4}\right) + \sin\left(\frac{2\pi n}{6} + \frac{\pi}{4}\right) \cos\left(\frac{2\pi n}{6}\right) \\ &= \sin\left(\frac{2\pi n}{6} + \frac{\pi}{4}\right) + \frac{1}{2} \left[\sin\left(\frac{4\pi n}{6} + \frac{\pi}{4}\right) + \sin\left(\frac{\pi}{4}\right) \right] \end{aligned}$$

This implies that the nonzero Fourier series coefficients of $z[n]$ are $c_0 = \cos(\pi/4)/2$, $c_1 = c_{-1}^* = e^{-j\pi/4}/2$, $c_2 = c_{-2}^* = e^{-j\pi/4}/4$.

3.31. (a) $g[n]$ is as shown in Figure S3.31. Clearly, $g[n]$ has a fundamental period of 10.

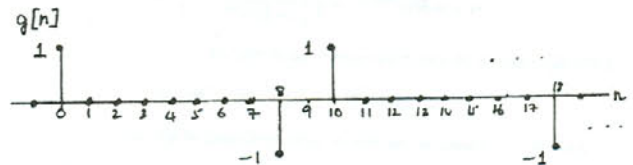


Figure S3.31

(b) The Fourier series coefficients of $g[n]$ are $b_k = (1/10)[1 - e^{-j(2\pi/10)k}]$.

(c) Since $g[n] = x[n] - x[n-1]$, the FS coefficients a_k and b_k must be related as

$$b_k = a_k - e^{-j(2\pi/10)k} a_k.$$

Therefore,

$$a_k = \frac{b_k}{1 - e^{-j(2\pi/10)k}} = \frac{(1/10)[1 - e^{-j(2\pi/10)k}]}{1 - e^{-j(2\pi/10)k}}.$$

3.32. (a) The four equations are

$$a_0 + a_1 + a_2 + a_3 = 1, \quad a_0 + ja_1 - a_2 - ja_3 = 0$$

$$a_0 - a_1 + a_2 - a_3 = 2, \quad a_0 - ja_1 - a_2 + ja_3 = -1.$$

Solving, we get $a_0 = 1/2$, $a_1 = -\frac{1+j}{4}$, $a_2 = -1$, $a_3 = -\frac{1-j}{4}$.

(b) By direct calculation,

$$a_k = \frac{1}{4} [1 + 2e^{-jk\pi} - e^{-jk3\pi/2}].$$

This is the same as the answer we obtained in part (a) for $0 \leq k \leq 3$.

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3.33. We will first evaluate the frequency response of the system. Consider an input $x(t)$ of the form $e^{j\omega t}$. From the discussion in Section 3.9.2 we know that the response to this input will be $y(t) = H(j\omega)e^{j\omega t}$. Therefore, substituting these in the given differential equation, we get

$$H(j\omega)j\omega e^{j\omega t} + 4e^{j\omega t} = e^{j\omega t}.$$

Therefore,

$$H(j\omega) = \frac{1}{j\omega + 4}.$$

From eq. (3.124), we know that

$$y(t) = \sum_{k=-\infty}^{\infty} a_k H(jk\omega_0) e^{jk\omega_0 t}$$

when the input is $x(t)$. $x(t)$ has the Fourier series coefficients a_k and fundamental frequency ω_0 . Therefore, the Fourier series coefficients of $y(t)$ are $a_k H(jk\omega_0)$.

(a) Here, $\omega_0 = 2\pi$ and the nonzero FS coefficients of $x(t)$ are $a_1 = a_{-1} = 1/2$. Therefore, the nonzero FS coefficients of $y(t)$ are

$$b_1 = a_1 H(j2\pi) = \frac{1}{2(4 + j2\pi)}, \quad b_{-1} = a_{-1} H(-j2\pi) = \frac{1}{2(4 - j2\pi)}.$$

(b) Here, $\omega_0 = 2\pi$ and the nonzero FS coefficients of $x(t)$ are $a_2 = a_{-2} = 1/2j$ and $a_3 = a_{-3} = e^{j\pi/4}/2$. Therefore, the nonzero FS coefficients of $y(t)$ are

$$b_2 = a_2 H(j4\pi) = \frac{1}{2j(4 + j4\pi)}, \quad b_{-2} = a_{-2} H(-j4\pi) = -\frac{1}{2j(4 - j4\pi)},$$

$$b_3 = a_3 H(j6\pi) = \frac{e^{j\pi/4}}{2(4 + j6\pi)}, \quad b_{-3} = a_{-3} H(-j6\pi) = -\frac{e^{-j\pi/4}}{2(4 - j6\pi)}.$$

3.34. The frequency response of the system is given by

$$H(j\omega) = \int_{-\infty}^{\infty} e^{-4|t|} e^{-j\omega t} dt = \frac{1}{4 + j\omega} + \frac{1}{4 - j\omega}.$$

(a) Here, $T = 1$ and $\omega_0 = 2\pi$ and $a_k = 1$ for all k . The FS coefficients of the output are

$$b_k = a_k H(jk\omega_0) = \frac{1}{4 + j2\pi k} + \frac{1}{4 - j2\pi k}.$$

(b) Here, $T = 2$ and $\omega_0 = \pi$ and

$$a_k = \begin{cases} 0, & k \text{ even} \\ 1, & k \text{ odd} \end{cases}.$$

Therefore, the FS coefficients of the output are

$$b_k = a_k H(jk\omega_0) = \begin{cases} 0, & k \text{ even} \\ \frac{1}{4 + j\pi k} + \frac{1}{4 - j\pi k}, & k \text{ odd} \end{cases}.$$

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3.37. The frequency response of the system may be easily shown to be

$$H(e^{j\omega}) = \frac{1}{1 - \frac{1}{2}e^{-j\omega}} - \frac{1}{1 - 2e^{-j\omega}}.$$

(a) The Fourier series coefficients of $x[n]$ are

$$a_k = \frac{1}{4}, \quad \text{for all } k.$$

Also, $N = 4$. Therefore, the Fourier series coefficients of $y[n]$ are

$$b_k = a_k H(e^{j2\pi k/N}) = \frac{1}{4} \left[\frac{1}{1 - \frac{1}{2}e^{-j\pi k/2}} - \frac{1}{1 - 2e^{-j\pi k/2}} \right].$$

(b) In this case, the Fourier series coefficients of $x[n]$ are

$$a_k = \frac{1}{6} [1 + 2 \cos(k\pi/3)], \quad \text{for all } k.$$

Also, $N = 6$. Therefore, the Fourier series coefficients of $y[n]$ are

$$b_k = a_k H(e^{j2\pi k/N}) = \frac{1}{6} [1 + 2 \cos(k\pi/3)] \left[\frac{1}{1 - \frac{1}{2}e^{-j\pi k/3}} - \frac{1}{1 - 2e^{-j\pi k/3}} \right].$$

3.38. The frequency response of the system may be evaluated as

$$H(e^{j\omega}) = -e^{2j\omega} - e^{j\omega} + 1 + e^{-j\omega} + e^{-2j\omega}.$$

For $x[n]$, $N = 4$ and $\omega_0 = \pi/2$. The FS coefficients of the input $x[n]$ are

$$a_k = \frac{1}{4}, \quad \text{for all } n.$$

Therefore, the FS coefficients of the output are

$$b_k = a_k H(e^{jk\omega_0}) = \frac{1}{4} [1 - e^{jk\pi/2} + e^{-jk\pi/2}].$$

3.39. Let the FS coefficients of the input be a_k . The FS coefficients of the output are of the form

$$b_k = a_k H(e^{jk\omega_0}),$$

where $\omega_0 = 2\pi/3$. Note that in the range $0 \leq k \leq 2$, $H(e^{jk\omega_0}) = 0$ for $k = 1, 2$. Therefore, only b_0 has a nonzero value among b_k in the range $0 \leq k \leq 2$.

3.40. Let the Fourier series coefficients of $x(t)$ be a_k .

(c) Here, $T = 1$, $\omega_0 = 2\pi$ and

$$a_k = \begin{cases} 1/2, & k = 0 \\ 0, & k \text{ even}, k \neq 0 \\ \frac{\sin(\pi k/2)}{\pi k}, & k \text{ odd} \end{cases}.$$

Therefore, the FS coefficients of the output are

$$b_k = a_k H(jk\omega_0) = \begin{cases} 1/4, & k = 0 \\ 0, & k \text{ even}, k \neq 0 \\ \frac{\sin(\pi k/2)}{\pi k} \left[\frac{1}{4 + j2\pi k} + \frac{1}{4 - j2\pi k} \right], & k \text{ odd} \end{cases}.$$

3.35. We know that the Fourier series coefficient of $y(t)$ are $b_k = H(jk\omega_0)a_k$, where ω_0 is the fundamental frequency of $x(t)$ and a_k are the FS coefficients of $x(t)$.

If $y(t)$ is identical to $x(t)$, then $b_k = a_k$ for all k . Noting that $H(j\omega) = 0$ for $|\omega| \geq 250$, we know that $H(jk\omega_0) = 0$ for $|k| \geq 18$ (because $\omega_0 = 14$). Therefore, a_k must be zero for $|k| \geq 18$.

3.36. We will first evaluate the frequency response of the system. Consider an input $x[n]$ of the form $e^{j\omega n}$. From the discussion in Section 3.9 we know that the response to this input will be $y[n] = H(e^{j\omega})e^{j\omega n}$. Therefore, substituting these in the given difference equation, we get

$$H(e^{j\omega})e^{j\omega n} - \frac{1}{4}e^{-j\omega}e^{j\omega n}H(e^{j\omega}) = e^{j\omega n}.$$

Therefore,

$$H(j\omega) = \frac{1}{1 - \frac{1}{4}e^{-j\omega}}.$$

From eq. (3.131), we know that

$$y[n] = \sum_{k=-\infty}^{\infty} a_k H(e^{j2\pi k/N}) e^{j2\pi k n/N}$$

when the input is $x[n]$. $x[n]$ has the Fourier series coefficients a_k and fundamental frequency $2\pi/N$. Therefore, the Fourier series coefficients of $y[n]$ are $a_k H(e^{j2\pi k/N})$.

(a) Here, $N = 4$ and the nonzero FS coefficients of $x[n]$ are $a_3 = a_{-3} = 1/2j$. Therefore, the nonzero FS coefficients of $y[n]$ are

$$b_3 = a_3 H(e^{j3\pi/4}) = \frac{1}{2j(1 - (1/4)e^{-j3\pi/4})}, \quad b_{-3} = a_{-3} H(e^{-j3\pi/4}) = \frac{-1}{2j(1 - (1/4)e^{j3\pi/4})}.$$

(b) Here, $N = 8$ and the nonzero FS coefficients of $x[n]$ are $a_1 = a_{-1} = 1/2$ and $a_2 = a_{-2} = 1$. Therefore, the nonzero FS coefficients of $y[n]$ are

$$b_1 = a_1 H(e^{j\pi/4}) = \frac{1}{2(1 - (1/4)e^{-j\pi/4})}, \quad b_{-1} = a_{-1} H(e^{-j\pi/4}) = \frac{1}{2(1 - (1/4)e^{j\pi/4})},$$

$$b_2 = a_2 H(e^{j\pi/2}) = \frac{1}{(1 - (1/4)e^{-j\pi/2})}, \quad b_{-2} = a_{-2} H(e^{-j\pi/2}) = \frac{1}{(1 - (1/4)e^{j\pi/2})}.$$

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(a) $x(t - t_0)$ is also periodic with period T . The Fourier series coefficients b_k of $x(t - t_0)$ are

$$b_k = \frac{1}{T} \int_T x(t - t_0) e^{-jk(2\pi/T)t} dt$$

$$= \frac{e^{-jk(2\pi/T)t_0}}{T} \int_T x(\tau) e^{-jk(2\pi/T)\tau} d\tau$$

$$= e^{-jk(2\pi/T)t_0} a_k$$

Similarly, the Fourier series coefficients of $x(t + t_0)$ are

$$c_k = e^{jk(2\pi/T)t_0} a_k.$$

Finally, the Fourier series coefficients of $x(t - t_0) + x(t + t_0)$ are

$$d_k = b_k + c_k = e^{-jk(2\pi/T)t_0} a_k + e^{jk(2\pi/T)t_0} a_k = 2 \cos(k2\pi t_0/T) a_k.$$

(b) Note that $\mathcal{E}v\{x(t)\} = [x(t) + x(-t)]/2$. The FS coefficients of $x(-t)$ are

$$b_k = \frac{1}{T} \int_T x(-t) e^{-jk(2\pi/T)t} dt$$

$$= \frac{1}{T} \int_T x(\tau) e^{jk(2\pi/T)\tau} d\tau$$

$$= a_{-k}$$

Therefore, the FS coefficients of $\mathcal{E}v\{x(t)\}$ are

$$c_k = \frac{a_k + b_k}{2} = \frac{a_k + a_{-k}}{2}.$$

(c) Note that $\mathcal{R}e\{x(t)\} = [x(t) + x^*(t)]/2$. The FS coefficients of $x^*(t)$ are

$$b_k = \frac{1}{T} \int_T x^*(t) e^{-jk(2\pi/T)t} dt.$$

Conjugating both sides, we get

$$b_k^* = \frac{1}{T} \int_T x(t) e^{jk(2\pi/T)t} dt = a_{-k}.$$

Therefore, the FS coefficients of $\mathcal{R}e\{x(t)\}$ are

$$c_k = \frac{a_k + b_k}{2} = \frac{a_k + a_{-k}^*}{2}.$$

(d) The Fourier series synthesis equation gives

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j(2\pi/T)kt}.$$

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Differentiating both sides wrt t twice, we get

$$\frac{d^2 x(t)}{dt^2} = \sum_{k=-\infty}^{\infty} -k^2 \frac{4\pi^2}{T^2} a_k e^{j(2\pi/T)kt}$$

By inspection, we know that the Fourier series coefficients of $d^2 x(t)/dt^2$ are $-k^2 \frac{4\pi^2}{T^2} a_k$.

(e) The period of $x(3t)$ is a third of the period of $x(t)$. Therefore, the signal $x(3t-1)$ is periodic with period $T/3$. The Fourier series coefficients of $x(3t)$ are still a_k . Using the analysis of part (a), we know that the Fourier series coefficients of $x(3t-1)$ is $e^{-jk(6\pi/T)} a_k$.

3.41. Since $a_k = a_{-k}$, we require that $x(t) = x(-t)$. Also, note that since $a_k = a_{k+2}$, we require that

$$x(t) = x(t)e^{-j(4\pi/3)t}$$

This in turn implies that $x(t)$ may have nonzero values only for $t = 0, \pm 1.5, \pm 3, \pm 4.5, \dots$

Since $\int_{-0.5}^{0.5} x(t) dt = 1$, we may conclude that $x(t) = \delta(t)$ for $-0.5 \leq t \leq 0.5$. Also, since

$\int_{0.5}^{1.5} x(t) dt = 2$, we may conclude that $x(t) = 2\delta(t-3/2)$ in the range $0.5 \leq t \leq 3/2$. Therefore, $x(t)$ may be written as

$$x(t) = \sum_{k=-\infty}^{\infty} \delta(t-k/3) + 2 \sum_{k=-\infty}^{\infty} \delta(t-3k-3/2)$$

3.42. (a) From Problem 3.40 (and Table 3.1), we know that FS coefficients of $x^*(t)$ are a_k^* . Now, we know that $x(t)$ is real, then $x(t) = x^*(t)$. Therefore, $a_k = a_{-k}^*$. Note that this implies $a_0 = a_0^*$. Therefore, a_0 must be real.

(b) From Problem 3.40 (and Table 3.1), we know that FS coefficients of $x(-t)$ are a_{-k} . If $x(t)$ is even, then $x(t) = x(-t)$. This implies that

$$a_k = a_{-k} \quad (\text{S3.42-1})$$

This implies that the FS coefficients are even. From the previous part, we know that if $x(t)$ is real, then

$$a_k = a_{-k}^* \quad (\text{S3.42-2})$$

Using eqs. (S3.42-1) and (S3.42-2), we know that $a_k = a_k^*$. Therefore, a_k is real for all k . Hence, we may conclude that a_k is real and even.

(c) From Problem 3.40 (and Table 3.1), we know that FS coefficients of $x(-t)$ are a_{-k} . If $x(t)$ is odd, then $x(t) = -x(-t)$. This implies that

$$a_k = -a_{-k} \quad (\text{S3.42-3})$$

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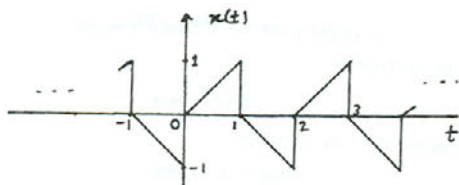


Figure S3.43

(d) (1) If a_1 or a_{-1} is nonzero, then

$$x(t) = a_{\pm 1} e^{\pm j2\pi t/T} + \dots$$

and

$$x(t+t_0) = a_{\pm 1} e^{\pm j2\pi(t+t_0)/T} + \dots$$

The smallest value of $|t_0|$ (other than $|t_0| = 0$ for which $e^{\pm j2\pi t_0/T} = 1$) is the fundamental period. Only then is

$$x(t+t_0) = a_{\pm 1} e^{\pm j2\pi t/T} + \dots = x(t)$$

Therefore, t_0 has to be the fundamental period.

(2) The period of $x(t)$ is the least common multiple of the periods of $e^{jk(2\pi/T)t}$ and $e^{jl(2\pi/T)t}$. The period of $e^{jk(2\pi/T)t}$ is T/k and the period of $e^{jl(2\pi/T)t}$ is T/l . Since k and l have no common factors, the least common multiple of T/k and T/l is T .

4. The only unknown FS coefficients are a_1, a_{-1}, a_2 , and a_{-2} . Since $x(t)$ is real, $a_1 = a_{-1}^*$, and $a_2 = a_{-2}^*$. Since a_1 is real, $a_1 = a_{-1}$. Now, $x(t)$ is of the form

$$x(t) = A_1 \cos(\omega_0 t) + A_2 \cos(2\omega_0 t + \theta)$$

where $\omega_0 = 2\pi/6$. From this we get

$$x(t-3) = A_1 \cos(\omega_0 t - 3\omega_0) + A_2 \cos(2\omega_0 t + \theta - 6\omega_0)$$

Now if we need $x(t) = -x(t-3)$, then $3\omega_0$ and $6\omega_0$ should both be odd multiples of π . Clearly, this is impossible. Therefore, $a_2 = a_{-2} = 0$ and

$$x(t) = A_1 \cos(\omega_0 t)$$

Now, using Parseval's relation on Clue 5, we get

$$\sum_{k=-\infty}^{\infty} |a_k|^2 = |a_1|^2 + |a_{-1}|^2 = \frac{1}{2}$$

Therefore, $|a_1| = 1/2$. Since a_1 is positive, we have $a_1 = a_{-1} = 1/2$. Therefore, $x(t) = \cos(\pi t/3)$.

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This implies that the FS coefficients are odd. From the previous part, we know that if $x(t)$ is real, then

$$a_k = a_{-k}^* \quad (\text{S3.42-4})$$

Using eqs. (S3.42-3) and (S3.42-4), we know that $a_k = -a_k^*$. Therefore, a_k is imaginary for all k . Hence, we may conclude that a_k is real and even. Noting that eq. (S3.42-3) requires that $a_0 = -a_0$, we may also conclude that $a_0 = 0$.

(d) Note that $\mathcal{E}v\{x(t)\} = [x(t) + x(-t)]/2$. From the previous parts, we know that the FS coefficients of $\mathcal{E}v\{x(t)\}$ will be $[a_k + a_{-k}]/2$. Using eq. (S3.43-2), we may write the FS coefficients of $\mathcal{E}v\{x(t)\}$ as $[a_k + a_k^*]/2 = \mathcal{R}\{a_k\}$.

(e) Note that $\mathcal{O}d\{x(t)\} = [x(t) - x(-t)]/2$. From the previous parts, we know that the FS coefficients of $\mathcal{O}d\{x(t)\}$ will be $[a_k - a_{-k}]/2$. Using eq. (S3.43-2), we may write the FS coefficients of $\mathcal{O}d\{x(t)\}$ as $[a_k - a_k^*]/2 = j\mathcal{I}m\{a_k\}$.

3.43. (a) (i) We have

$$x(t) = \sum_{\text{odd } k} a_k e^{jk\frac{\pi}{2}t}$$

Therefore,

$$x(t+T/2) = \sum_{\text{odd } k} a_k e^{jk\frac{\pi}{2}(t+T/2)} e^{jk\pi}$$

Since $e^{jk\pi} = -1$ for k odd,

$$x(t+T/2) = -x(t)$$

(ii) The Fourier series coefficients of $x(t)$ are

$$\begin{aligned} a_k &= \frac{1}{T} \int_0^{T/2} x(t) e^{-jk\omega_0 t} dt + \frac{1}{T} \int_{T/2}^T x(t) e^{-jk\omega_0 t} dt \\ &= \frac{1}{T} \int_0^{T/2} [x(t) + x(t+T/2)] e^{-jk\omega_0 t} dt \end{aligned}$$

Note that the right-hand side of the above equation evaluates to zero for even values of k if $x(t) = -x(t+T/2)$.

(b) The function is as shown in Figure S3.43.

Note that $T = 2$ and $\omega_0 = \pi$. Therefore,

$$a_k = \begin{cases} 0 & k \text{ even} \\ \frac{1}{j\pi k} + \frac{1}{k^2\pi^2} & k \text{ odd} \end{cases}$$

(c) No. For an even harmonic signal we may follow the reasoning of part (a-i) to show that $x(t) = x(t+T/2)$. In this case, the fundamental period is $T/2$.

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3.45. By inspection, we may conclude that the FS coefficients of $x(t)$ are

$$\gamma_k = \begin{cases} a_0, & k = 0 \\ B_k + jC_k, & k > 0 \\ B_k - jC_k, & k < 0 \end{cases}$$

(a) We know from Problem 3.42 that if $x(t)$ is real, the FS coefficients of $\mathcal{E}v\{x(t)\}$ are $\mathcal{R}\{a_k\}$. Therefore,

$$a_0 = a_0, \quad a_k = B_k$$

We know from Problem 3.42 that if $x(t)$ is real, the FS coefficients of $\mathcal{O}d\{x(t)\}$ are $j\mathcal{I}m\{\gamma_k\}$. Therefore,

$$\beta_0 = 0, \quad \beta_k = \begin{cases} jC_k, & k > 0 \\ -jC_k, & k < 0 \end{cases}$$

(b) $\alpha_k = \alpha_{-k}$ and $\beta_k = -\beta_{-k}$

(c) The signal is

$$y(t) = 1 + \mathcal{E}v\{x(t)\} + \frac{1}{2} \mathcal{E}v\{x(t)\} - \mathcal{O}d\{x(t)\}$$

This is as shown in Figure S3.45.

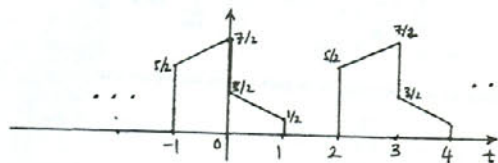


Figure S3.45

3.46. (a) The Fourier series coefficients of $x(t)$ are

$$\begin{aligned} c_k &= \frac{1}{T} \int_T \sum_n \sum_l a_n b_l e^{j(n+l)\omega_0 t} e^{-jk\omega_0 t} dt \\ &= \frac{1}{T} \sum_n \sum_l a_n b_l \delta(k - (n+l)) \\ &= \sum_n a_n b_{k-n} \end{aligned}$$

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(b) (i) Here, $T_0 = 3$ and $\omega_0 = 2\pi/3$. Therefore,

$$c_k = \left[\frac{1}{2} \delta(k-30) + \frac{1}{2} \delta(k+30) \right] * \frac{2 \sin(k2\pi/3)}{3k2\pi/3}.$$

Simplifying,

$$c_k = \frac{\sin\{(k-30)2\pi/3\}}{3(k-30)2\pi/3} + \frac{\sin\{(k+30)2\pi/3\}}{3(k+30)2\pi/3}$$

and $c_{\pm 30} = 1/3$.

(ii) We may express $x_2(t)$ as

$$x_2(t) = \text{sum of two shifted square waves} \times \cos(20\pi t).$$

Here, $T_0 = 3$, $\omega_0 = 2\pi/3$. Therefore,

$$c_k = \frac{1}{3} e^{-j(k-30)(2\pi/3)} \frac{\sin\{(k-30)2\pi/3\}}{(k-30)2\pi/3} + \frac{1}{3} e^{-j(k+30)(2\pi/3)} \frac{\sin\{(k+30)2\pi/3\}}{(k+30)2\pi/3} \\ + \frac{1}{3} e^{-j(k-30)(\pi/3)} \frac{\sin\{(k-30)\pi/3\}}{(k-30)2\pi/3} + \frac{1}{3} e^{-j(k+30)(\pi/3)} \frac{\sin\{(k+30)\pi/3\}}{(k+30)2\pi/3}$$

(iii) Here, $T_0 = 4$, $\omega_0 = \pi/2$. Therefore,

$$c_k = \left[\frac{1}{2} \delta(k-40) + \frac{1}{2} \delta(k+40) \right] * \frac{j[k\omega_0 + e^{-1} \{\sin k\omega_0 - \cos k\omega_0\}]}{2[1 + (k\omega_0)^2]}.$$

Simplifying,

$$c_k = \frac{j[(k-40)\omega_0 + e^{-1} \{\sin(k-40)\omega_0 - \cos(k-40)\omega_0\}]}{4[1 + \{(k-40)\omega_0\}^2]} \\ + \frac{j[(k+40)\omega_0 + e^{-1} \{\sin(k+40)\omega_0 - \cos(k+40)\omega_0\}]}{4[1 + \{(k+40)\omega_0\}^2]}.$$

(c) From Problem 3.42, we know that $b_k = a_{-k}^*$. From part (a), we know that the FS coefficients of $x(t) = x(t)y(t) = x(t)x^*(t) = |x(t)|^2$ will be

$$c_k = \sum_{n=-\infty}^{\infty} a_n b_{n-k} = \sum_{n=-\infty}^{\infty} a_n a_{n+k}^*.$$

From the Fourier series analysis equation, we have

$$c_k = \frac{1}{T_0} \int_0^{T_0} |x(t)|^2 e^{-j(2\pi/T_0)kt} dt = \sum_{n=-\infty}^{\infty} a_n a_{n+k}^*.$$

Putting $k = 0$ in this equation, we get

$$\frac{1}{T_0} \int_0^{T_0} |x(t)|^2 dt = \sum_{n=-\infty}^{\infty} |a_n|^2.$$

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(b) Here,

$$y[n] = \frac{1}{2} [x[n] + (-1)^n x[n]].$$

For N even,

$$\hat{a}_k = \frac{1}{2} [a_k + a_{k-N/2}].$$

For N odd,

$$\hat{a}_k = \begin{cases} \frac{1}{2} [a_k + a_{k-N/2}], & k \text{ even} \\ \frac{1}{2} a_k, & k \text{ odd} \end{cases}$$

3.49. (a) The FS coefficients are given by

$$a_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/N} \\ = \frac{1}{N} \sum_{n=0}^{(N/2)-1} x[n] e^{-j2\pi kn/N} + \frac{1}{N} \sum_{n=N/2}^{N-1} x[n] e^{-j2\pi kn/N} \\ = \frac{1}{N} \sum_{n=0}^{(N/2)-1} x[n] e^{-j2\pi kn/N} + \frac{e^{-j\pi k}}{N} \sum_{n=0}^{(N/2)-1} x[n+N/2] e^{-j2\pi kn/N} = 0 \\ = \frac{1}{N} \sum_{n=0}^{(N/2)-1} x[n] e^{-j2\pi kn/N} - \frac{e^{-j\pi k}}{N} \sum_{n=0}^{(N/2)-1} x[n] e^{-j2\pi kn/N} \\ = 0, \quad \text{for } k \text{ even.}$$

(b) By adopting an approach similar to part (a), we may show that

$$a_k = \frac{1}{N} \left[\sum_{n=0}^{N/2-1} \{1 - e^{-j2\pi k/2} + e^{-j\pi k} - e^{-j3\pi k/2}\} x[n] e^{-j2\pi kn/N} \right] \\ = 0, \quad \text{for } k = 4r, r \in \mathbb{Z}$$

(c) If N/M is an integer, we may generalize the approach of part (a) to show that

$$a_k = \frac{1}{N} \left[\sum_{r=0}^{B-1} \{1 - e^{-j2\pi r} + e^{-j4\pi r} - \dots + e^{-j2\pi(M-1)r}\} x[n] e^{-j2\pi kn/N} \right],$$

where $B = N/M$ and $r = k/m$. From the above equation, it is clear that

$$a_k = 0, \quad \text{if } k = rM, r \in \mathbb{Z}.$$

3.50. From Table 3.2, we know that if

$$x[n] \xrightarrow{FS} a_k,$$

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3.47. Considering $x(t)$ to be periodic with period 1, the nonzero FS coefficients of $x(t)$ are $a_1 = a_{-1} = 1/2$. If we now consider $x(t)$ to be periodic with period 3, then the nonzero FS coefficients of $x(t)$ are $b_3 = b_{-3} = 1/2$.

3.48. (a) The FS coefficients of $x[n - n_0]$ are

$$\hat{a}_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n - n_0] e^{-j2\pi kn/N} \\ = \frac{1}{N} e^{-j2\pi kn_0/N} \sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/N} \\ = e^{-j2\pi kn_0/N} a_k$$

(b) Using the results of part (a), the FS coefficients of $x[n] - x[n-1]$ are given by

$$\hat{a}_k = a_k - e^{-j2\pi k/N} a_k = [1 - e^{-j2\pi k/N}] a_k.$$

(c) Using the results of part (a), the FS coefficients of $x[n] - x[n-N/2]$ are given by

$$\hat{a}_k = a_k [1 - e^{-j\pi k}] = \begin{cases} 0, & k \text{ even} \\ 2a_k, & k \text{ odd} \end{cases}$$

(d) Note that $x[n] + x[n+N/2]$ has a period of $N/2$. The FS coefficients of $x[n] + x[n+N/2]$ are given by

$$\hat{a}_k = \frac{2}{N} \sum_{n=0}^{N/2-1} [x[n] + x[n+N/2]] e^{-j4\pi kn/N} = 2a_{2k}$$

for $0 \leq k \leq (N/2 - 1)$.

(e) The FS coefficients of $x^*[-n]$ are

$$\hat{a}_k = \frac{1}{N} \sum_{n=0}^{N-1} x^*[-n] e^{-j2\pi kn/N} = a_k^*.$$

(f) With N even the FS coefficients of $(-1)^n x[n]$ are

$$\hat{a}_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/N} (-1)^n = a_{k-N/2}$$

(g) With N odd, the period of $(-1)^n x[n]$ is $2N$. Therefore, the FS coefficients are

$$\hat{a}_k = \frac{1}{2N} \left[\sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/(2N)} + \sum_{n=N}^{2N-1} x[n] e^{-j2\pi kn/(2N)} (-1)^{n-N} \right].$$

Note that for k odd $\frac{k-N}{2}$ is an integer and $k-N$ is an even integer. Also, for k even, $k-N$ is an odd integer and $e^{-j\pi(k-N)} = -1$. Therefore,

$$\hat{a}_k = \begin{cases} a_{\frac{k-N}{2}}, & k \text{ odd} \\ 0, & k \text{ even} \end{cases}$$

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then

$$(-1)^n x[n] \xrightarrow{FS} a_{k-N/2}.$$

In this case, $N = 8$. Therefore,

$$(-1)^n x[n] \xrightarrow{FS} a_{k-4}.$$

Since it is given that $a_k = -a_{k-4}$, we have

$$x[n] = -(-1)^n x[n].$$

This implies that $x[0] = x[\pm 2] = x[\pm 4] = \dots = 0$.

We are also given that $x[1] = x[5] = \dots = 1$ and $x[3] = x[7] = -1$. Therefore, one period of $x[n]$ is as shown in Figure S3.50.

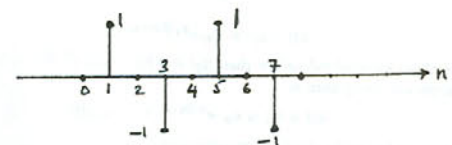


Figure S3.50

3.51. We have

$$e^{j4(2\pi/8)n} x[n] = e^{j\pi n} x[n] = (-1)^n x[n] \xrightarrow{FS} a_{k-4}$$

and therefore,

$$(-1)^{n+1} x[n] \xrightarrow{FS} -a_{k-4}.$$

If $a_k = -a_{k-4}$, then $x[0] = x[\pm 2] = x[\pm 4] = \dots = 0$. Now, note that in the signal $p[n] = x[n-1]$, $p[\pm 1] = p[\pm 3] = \dots = 0$. Now let us plot the signal $z[n] = (1 + (-1)^n)/2$. This is as shown in Figure S3.51.

Clearly, the signal $y[n] = z[n]p[n] = p[n]$ because $p[n]$ is zero whenever $z[n]$ is zero. Therefore, $y[n] = x[n-1]$. The FS coefficients of $y[n]$ are $a_k e^{-j(2\pi/8)k}$.

3.52. (a) If $x[n]$ is real, $x[n] = x^*[n]$. Therefore,

$$a_{-k} = \sum_n x[n] e^{j2\pi kn/N} = a_k^*.$$

From this result, we get $b_{-k} = b_k$ and $c_{-k} = -c_k$.

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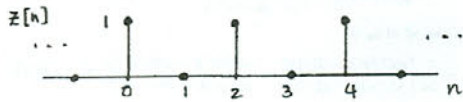


Figure S3.51

(b) If N is even, then

$$a_{N/2} = \frac{1}{N} \sum_n x[n] e^{-j\pi n} = \frac{1}{N} \sum_n (-1)^n x[n] = \text{real}.$$

(c) If N is odd, then

$$\begin{aligned} x[n] &= \sum_{k=-(N-1)/2}^{(N-1)/2} a_k e^{j(2\pi/N)kn} \\ &= \sum_{k=0}^{(N-1)/2} a_k e^{j(2\pi/N)kn} + \sum_{k=1}^{(N-1)/2} a_k^* e^{-j(2\pi/N)kn} \quad (\text{From (a)}) \\ &= a_0 + \sum_{k=1}^{(N-1)/2} (b_k + j c_k) e^{j(2\pi/N)kn} + \sum_{k=1}^{(N-1)/2} (b_k - j c_k) e^{-j(2\pi/N)kn} \\ &= a_0 + 2 \sum_{k=1}^{(N-1)/2} b_k \cos(2\pi k n / N) - c_k \sin(2\pi k n / N). \end{aligned}$$

If N is even, then

$$\begin{aligned} x[n] &= \sum_{k=0}^{N-1} a_k e^{j(2\pi/N)kn} \\ &= a_0 + (-1)^n a_{N/2} + 2 \sum_{k=1}^{(N-2)/2} a_k e^{j(2\pi/N)kn} + a_{N-k} e^{j(2\pi/N)(N-k)n} \\ &= a_0 + (-1)^n a_{N/2} + 2 \sum_{k=1}^{(N-2)/2} a_k e^{j(2\pi/N)kn} - a_k^* e^{-j(2\pi/N)kn} \quad (\text{From (a)}) \\ &= a_0 + (-1)^n a_{N/2} + 2 \sum_{k=1}^{(N-1)/2} b_k \cos(2\pi k n / N) - c_k \sin(2\pi k n / N). \end{aligned}$$

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(d) If $a_k = A_k e^{-j\theta_k}$, then $b_k = A_k \cos(\theta_k)$ and $c_k = A_k \sin(\theta_k)$. Substituting in the result of the previous part, we get for N odd:

$$\begin{aligned} x[n] &= a_0 + 2 \sum_{k=1}^{(N-1)/2} A_k \cos(\theta_k) \cos(2\pi k n / N) - c_k \sin(\theta_k) \sin(2\pi k n / N) \\ &= a_0 + 2 \sum_{k=1}^{(N-1)/2} A_k \cos\left\{\frac{2\pi k n}{N} + \theta_k\right\}. \end{aligned}$$

Similarly, for N even,

$$\begin{aligned} x[n] &= a_0 + (-1)^n a_{N/2} + 2 \sum_{k=1}^{(N-1)/2} A_k \cos(\theta_k) \cos(2\pi k n / N) - c_k \sin(\theta_k) \sin(2\pi k n / N) \\ &= a_0 + (-1)^n a_{N/2} + 2 \sum_{k=1}^{(N-2)/2} A_k \cos\left\{\frac{2\pi k n}{N} + \theta_k\right\}. \end{aligned}$$

(e) The signal is:

$$y[n] = d.c\{x[n]\} - d.c\{z[n]\} + \mathcal{E}v\{z\} + \mathcal{O}d\{z\} - 2\mathcal{O}d\{z\}.$$

This is as shown Figure S3.52.

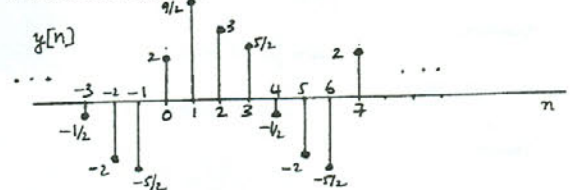


Figure S3.52

3.53. We have

$$a_k = \frac{1}{N} \sum_{n \in \mathbb{Z}} x[n] e^{-j(2\pi/N)kn}.$$

Note that

$$a_0 = \frac{1}{N} \sum_{n \in \mathbb{Z}} x[n]$$

which is real if $x[n]$ is real.

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(a) If N is even, then

$$a_{N/2} = \frac{1}{N} \sum_{n \in \mathbb{Z}} x[n] e^{-j\pi n} = \frac{1}{N} \sum_{n \in \mathbb{Z}} x[n] (-1)^n.$$

Clearly, $a_{N/2}$ is also real if $x[n]$ is real.

(b) If N is odd, only a_0 is guaranteed to be real.

3.54. (a) Let $k = pN$, $p \in \mathbb{Z}$. Then,

$$a[pN] = \sum_{n=0}^{N-1} e^{j(2\pi/N)pNn} = \sum_{n=0}^{N-1} e^{j2\pi pn} = \sum_{n=0}^{N-1} 1 = N.$$

(b) Using the finite sum formula, we have

$$a[k] = \frac{1 - e^{j2\pi k}}{1 - e^{j(2\pi/N)k}} = 0, \quad \text{if } k \neq pN, p \in \mathbb{Z}.$$

(c) Let

$$a[k] = \sum_{n=q}^{q+N-1} e^{j(2\pi/N)kn},$$

where q is some arbitrary integer. By putting $k = pN$, we may again easily show that

$$a[pN] = \sum_{n=q}^{q+N-1} e^{j(2\pi/N)pNn} = \sum_{n=q}^{q+N-1} e^{j2\pi pn} = \sum_{n=q}^{q+N-1} 1 = N.$$

Now,

$$a[k] = e^{j(2\pi/N)kq} \sum_{n=0}^{N-1} e^{j(2\pi/N)kn}.$$

Using part (b), we may argue that $a[k] = 0$ for $k \neq pN$, $p \in \mathbb{Z}$.

3.55. (a) Note that

$$x_m[n+mN] = \begin{cases} x[\frac{n}{m} + N], & n = 0, \pm m, \dots \\ 0, & \text{otherwise} \end{cases} = \begin{cases} x[\frac{n}{m}], & n = 0, \pm m, \dots \\ 0, & \text{otherwise} \end{cases} = x_m[n].$$

Therefore, $x_m[n]$ is periodic with period mN .

(b) The time-scaling operation discussed in this problem is a linear operation. Therefore, if $x[n] = v[n] + w[n]$, then, $x_m[n] = v_m[n] + w_m[n]$.

(c) Let us consider

$$y[n] = \frac{1}{m} \sum_{l=0}^{m-1} e^{j(2\pi/mN)(k_0+lN)n} = \frac{1}{m} e^{j(2\pi/mN)k_0n} \sum_{l=0}^{m-1} e^{j(2\pi/m)ln}.$$

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This may be written as [From Problem 3.54]

$$y[n] = \begin{cases} e^{j(2\pi/mN)k_0n}, & n = 0, \pm N, \pm 2N, \dots \\ 0, & \text{otherwise.} \end{cases} \quad (\text{S3.55-1})$$

Now, also note that by applying time-scaling on $x[n]$, we get

$$x_{(m)}[n] = \begin{cases} e^{j(2\pi/mN)k_0n}, & n = 0, \pm N, \pm 2N, \dots \\ 0, & \text{otherwise.} \end{cases} \quad (\text{S3.55-2})$$

Comparing eqs. (S3.55-1) and (S3.55-2), we see that $y[n] = x_{(m)}[n]$.

(d) We have

$$b_k = \frac{1}{mN} \sum_{n=0}^{mN-1} x_{(m)}[n] e^{-j(2\pi/mN)kn}.$$

We know that only every m th value in the above summation is nonzero. Therefore,

$$\begin{aligned} b_k &= \frac{1}{mN} \sum_{n=0}^{N-1} x_{(m)}[nm] e^{-j(2\pi/mN)kmn} \\ &= \frac{1}{mN} \sum_{n=0}^{N-1} x_{(m)}[nm] e^{-j(2\pi/N)kn} \end{aligned}$$

Note that $x_{(m)}[nm] = x[n]$. Therefore,

$$b_k = \frac{1}{mN} \sum_{n=0}^{N-1} x[n] e^{-j(2\pi/N)kn} = \frac{a_k}{m}.$$

3.56. (a) We have

$$x[n] \xrightarrow{FS} a_k \quad \text{and} \quad x^*[n] \xrightarrow{FS} a_k^*.$$

Using the multiplication property,

$$x[n]x^*[n] = |x[n]|^2 \xrightarrow{FS} \sum_{l \in \mathbb{Z}} a_l a_{l+k}^*.$$

(b) From above, it is clear that the answer is yes.

3.57. (a) We have

$$x[n]y[n] = \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} a_k b_l e^{j(2\pi/N)(k+l)n}.$$

Putting $l' = k + l$, we get

$$x[n]y[n] = \sum_{k=0}^{N-1} \sum_{l'=k}^{(k+N-1)} a_k b_{l'-k} e^{j(2\pi/N)l'n}.$$

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But since both b_{r-k} and $e^{j(2\pi/N)kn}$ are periodic with period N , we may rewrite this as

$$x[n]y[n] = \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} a_k b_{l-k} e^{j(2\pi/N)kn} = \sum_{l=0}^{N-1} \left[\sum_{k=0}^{N-1} a_k b_{l-k} \right] e^{j(2\pi/N)ln}.$$

Therefore,

$$c_k = \sum_{l=0}^{N-1} a_l b_{l-k}.$$

By interchanging a_k and b_k , we may show that

$$c_k = \sum_{l=0}^{N-1} b_l a_{l-k}.$$

(b) Note that since both a_k and b_k are periodic with period N , we may rewrite the above summation as

$$c_k = \sum_{l=0}^{N-1} a_l b_{l-k} = \sum_{l=0}^{N-1} b_l a_{l-k}.$$

(c) (i) Here,

$$c_k = \sum_{l=0}^{N-1} \frac{1}{2} [\delta[l-3] + \delta[l-N+3]] a_{l-k}.$$

Therefore,

$$c_k = \frac{1}{2} a_{k-3} + \frac{1}{2} a_{k+3-N}.$$

(ii) Period = N . Also,

$$b_k = \frac{1}{N}, \quad \text{for all } k.$$

Therefore,

$$c_k = \frac{1}{N} \sum_{l=0}^{N-1} a_l.$$

(iii) Here,

$$b_k = \frac{1}{N} [1 + e^{-j2\pi k/3} + e^{-j4\pi k/3}].$$

Therefore,

$$c_k = \frac{1}{N} \sum_{l=0}^{N-1} [1 + e^{-j2\pi l/3} + e^{-j4\pi l/3}] a_{l-k}.$$

(d) Period = 12. Also,

$$x[n] \xrightarrow{FS} a_k = a_{10} = 1/2, \quad \text{All other } a_k = 0, \quad 0 \leq k \leq 11$$

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(c) Here, $n = 8$. The nonzero FS coefficients in the range $0 \leq k \leq 7$ for $x[n]$ are $a_3 = a_5^* = 1/2j$. Note that for $y[n]$, we need only evaluate b_3 and b_5 . We have

$$b_3 = b_5^* = \frac{1}{4(1 - e^{-j3\pi/4})}.$$

Therefore, the only nonzero FS coefficients in the range $0 \leq k \leq 7$ for the periodic convolution of these signals are $c_3 = 8a_3b_3$ and $c_5 = 8a_5b_5$.

(d) Here,

$$x[n] \xrightarrow{FS} a_k = \frac{1}{16j} \left[\frac{1 - e^{j(3\pi/7 - \pi k/4)}}{1 - e^{-j(3\pi/7 - \pi k/4)}} - \frac{1 - e^{j(3\pi/7 + \pi k/4)}}{1 - e^{-j(3\pi/7 + \pi k/4)}} \right]$$

and

$$y[n] \xrightarrow{FS} b_k = \frac{1}{8} \left[\frac{1 - (1/2)^8}{1 - (1/2)e^{-j\pi k/4}} \right].$$

Therefore,

$$z[n] = x[n]y[n] \xrightarrow{FS} 8a_k b_k.$$

3.59. (a) Note that the signal $x(t)$ is periodic with period NT . The FS coefficients of $x(t)$ are

$$a_k = \frac{1}{NT} \int_0^{NT} \sum_{p=-\infty}^{\infty} x[p] \delta(t - pT) e^{-j(2\pi/NT)kt} dt.$$

Note that the limits of the summation may be changed in accordance with the limits of the integration so that we get

$$a_k = \frac{1}{NT} \int_0^{NT} \sum_{p=0}^{N-1} x[p] \delta(t - pT) e^{-j(2\pi/NT)kt} dt.$$

Interchanging the summation and the integration and simplifying

$$\begin{aligned} a_k &= (1/NT) \sum_{p=0}^{N-1} x[p] \int_0^{NT} \delta(t - pT) e^{-j(2\pi/NT)kt} dt \\ &= (1/NT) \sum_{p=0}^{N-1} x[p] e^{-j(2\pi/N)pk} \\ &= (1/T) \left[(1/N) \sum_{p=0}^{N-1} x[p] e^{-j(2\pi/N)pk} \right]. \end{aligned}$$

Note that the term within brackets on the RHS of the above equation constitutes the FS coefficients of the signal $x[n]$. Since, this is periodic with period N , a_k must also be periodic with period N .

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and

$$y[n] \xrightarrow{FS} b_k = \left(\frac{1}{12} \right) \frac{\sin 7\pi k/12}{\sin \pi k/12}, \quad 0 \leq k \leq 11.$$

Therefore one period of c_k is,

$$c_k = \frac{1}{24} \left[\frac{\sin\{7\pi(k-2)/12\}}{\sin\{\pi(k-2)/12\}} + \frac{\sin\{7\pi(k-10)/12\}}{\sin\{\pi(k-10)/12\}} \right], \quad 0 \leq k \leq 11$$

(e) Using the FS analysis equation, we have

$$N \sum_{l=0}^{N-1} a_l b_{k-l} = \sum_{l=0}^{N-1} x[l] y[l] e^{-j(2\pi/N)kn}.$$

Putting $k = 0$ in this, we get

$$N \sum_{l=0}^{N-1} a_l b_{-l} = \sum_{l=0}^{N-1} x[l] y[l].$$

Now let $y[n] = x^*[n]$. Then $b_l = a_{-l}^*$. Therefore,

$$N \sum_{l=0}^{N-1} a_l a_l^* = \sum_{l=0}^{N-1} x[l] x^*[l].$$

Therefore,

$$N \sum_{l=0}^{N-1} |a_l|^2 = \sum_{l=0}^{N-1} |x[l]|^2.$$

3.58. (a) We have

$$z[n+N] = \sum_{l=0}^{N-1} x[l] y[n+N-l].$$

Since $y[n]$ is periodic with period N , $y[n+N-l] = y[n-l]$. Therefore,

$$z[n+N] = \sum_{l=0}^{N-1} x[l] y[n-l] = z[n].$$

Therefore, $z[n]$ is also periodic with period N .

(b) The FS coefficients of $z[n]$ are

$$\begin{aligned} c_k &= \frac{1}{N} \sum_{n=0}^{N-1} \sum_{l=0}^{N-1} a_l b_{n-l} e^{-j2\pi kn/N} \\ &= \frac{1}{N} \sum_{k=0}^{N-1} a_k e^{-j2\pi kn/N} \sum_{n=0}^{N-1} b_{n-k} e^{-j2\pi(n-k)l/N} \\ &= \frac{1}{N} N a_k b_l \\ &= N a_l b_l. \end{aligned}$$

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(b) If the FS coefficients of $x(t)$ are periodic with period N , then

$$a_k = a_{k-N}.$$

This implies that

$$x(t) = x(t) e^{j(2\pi/T)Nt}.$$

This is possible only if $x(t)$ is zero for all t other than when $(2\pi/T)Nt = 2\pi k$, where $k \in \mathbb{Z}$. Therefore, $x(t)$ is of the form

$$x(t) = \sum_{k=-\infty}^{\infty} g[k] \delta(t - kT/N).$$

(c) A simple example would be $x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$.

3.60. (a) The system is not LTI. $(1/2)^n$ is an eigen function of LTI systems. Therefore, the output should have been of the form $K(1/2)^n$, where K is a complex constant.

(b) It is possible to find an LTI system with this input-output relationship. The frequency response of this system would be $H(e^{j\omega}) = (1 - (1/2)e^{-j\omega}) / (1 - (1/4)e^{-j\omega})$. The system is unique.

(c) It is possible to find an LTI system with this input-output relationship. The frequency response of this system would be $H(e^{j\omega}) = (1 - (1/2)e^{-j\omega}) / (1 - (1/4)e^{-j\omega})$. The system is unique.

(d) It is possible to find an LTI system with this input-output relationship. The system is not unique because we only require that $H(e^{j\pi/6}) = 2$.

(e) It is possible to find an LTI system with this input-output relationship. The frequency response of this system would be $H(e^{j\omega}) = 2$. The system is unique.

(f) It is possible to find an LTI system with this input-output relationship. The system is not unique because we only require that $H(e^{j\pi/2}) = 2(1 - e^{j\pi/2})$.

(g) It is possible to find an LTI system with this input-output relationship. The system is not unique because we only require that $H(e^{j\pi/3}) = 1 - j\sqrt{3}$.

(h) Note that $x[n]$ and $y_1[n]$ are periodic with the same fundamental frequency. Therefore, it is possible to find an LTI system with this input-output relationship without violating the Eigen function property. The system is not unique because $H(e^{j\omega})$ needs to be have specific values only for $H(e^{j(2\pi/12)k})$. The rest of $H(e^{j\omega})$ may be chosen arbitrarily.

(i) Note that $x[n]$ and $y_1[n]$ are not periodic with the same fundamental frequency. Furthermore, note that $y_2[n]$ has $2/3$ the period of $x[n]$. Therefore, $y[n]$ will be made up of complex exponentials which are not present in $x[n]$. This violates the eigen function property of LTI systems. Therefore, the system cannot be LTI.

3.61. (a) For this system,

$$x(t) \rightarrow \boxed{\delta(t)} \rightarrow x(t).$$

Therefore, all functions are eigenfunctions with an eigenvalue of one.

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(b) The following is an eigen function with an eigen value of 1:

$$x(t) = \sum_k \delta(t - kT).$$

The following is an eigen function with an eigen value of 1/2:

$$x(t) = \sum_k \left(\frac{1}{2}\right)^k \delta(t - kT).$$

The following is an eigen function with an eigen value of 2:

$$x(t) = \sum_k (2)^k \delta(t - kT).$$

(c) If $h(t)$ is real and even then $H(\omega)$ is real and even.

$$e^{j\omega t} \rightarrow [H(j\omega)] \rightarrow H(j\omega)e^{j\omega t}$$

and

$$e^{-j\omega t} \rightarrow [H(j\omega)] \rightarrow H(-j\omega)e^{-j\omega t} = H(j\omega)e^{-j\omega t}.$$

From these two statements, we may argue that

$$\cos(\omega t) = \frac{1}{2}[e^{j\omega t} + e^{-j\omega t}] \rightarrow [H(j\omega)] \rightarrow H(j\omega)\cos(\omega t).$$

Therefore, $\cos(\omega t)$ is an eigenfunction. We may similarly show that $\sin(\omega t)$ is an eigenfunction.

(d) We have

$$\phi(t) \rightarrow [u(t)] \rightarrow \lambda\phi(t).$$

Therefore,

$$\lambda\phi(t) = \int_{-\infty}^t \phi(\tau) d\tau.$$

Differentiating both sides wrt t , we get

$$\lambda\phi'(t) = \phi(t).$$

Let $\phi(0) = \phi_0$. Then

$$\phi(t) = \phi_0 e^{t/\lambda}.$$

3.62. (a) The fundamental period of the input is $T = 2\pi$. The fundamental period of the input is $T = \pi$. The signals are as shown in Figure S3.62.

(b) The Fourier series coefficients of the output are

$$b_k = \frac{2(-1)^k}{\pi(1 - 4k^2)}.$$

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Therefore, the system is linear.

Now consider

$$x_4(t) = x(t - t_0) \rightarrow y_4(t).$$

We have

$$y_4(t) = t^2 \frac{d^2 x(t - t_0)}{dt^2} + t \frac{dx(t - t_0)}{dt} \neq y(t - t_0)$$

Therefore, the system is not time invariant.

(c) For inputs of the form $\phi_k(t) = t^k$, the output is

$$y(t) = k^2 t^k = k^2 \phi_k(t).$$

Therefore, $\phi_k(t)$ are eigenfunctions with eigenvalue $\lambda_k = k^2$.

(d) The output is

$$y(t) = 10^3 t^{-10} + 3t + 8t^4.$$

3.65. (a) Pairs (a) and (b) are orthogonal. Pairs (c) and (d) are not orthogonal.

(b) Orthogonal, but not orthonormal. $A_m = 1/\omega_0$.

(c) Orthonormal.

(d) We have

$$\int_{t_0}^{t_0+T} e^{jm\omega_0\tau} e^{-jn\omega_0\tau} d\tau = e^{j(m-n)\omega_0 t_0} \frac{[e^{j(m-n)2\pi} - 1]}{(m-n)\omega_0}$$

This evaluates to 0 when $m \neq n$ and to jT when $m = n$. Therefore, the functions are orthogonal but not orthonormal.

(e) We have

$$\begin{aligned} \int_{-T}^T x_e(t)x_o(t)dt &= \frac{1}{4} \int_{-T}^T [x(t) + x(-t)][x(t) - x(-t)]dt \\ &= \frac{1}{4} \int_{-T}^T x^2(t)dt - \frac{1}{4} \int_{-T}^T x^2(-t)dt \\ &= 0. \end{aligned}$$

(f) Consider

$$\int_a^b \frac{1}{\sqrt{A_k}} \phi_k(t) \frac{1}{\sqrt{A_l}} \phi_l^*(t) dt = \frac{1}{\sqrt{A_k A_l}} \int_a^b \phi_k(t) \phi_l^*(t) dt.$$

This evaluates to zero for $k \neq l$. For $k = l$, it evaluates to $A_k/A_k = 1$. Therefore, the functions are orthonormal.

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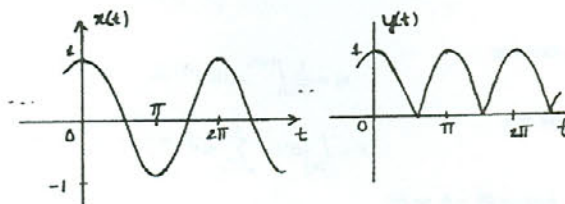


Figure S3.62

(c) The dc component of the input is 0. The dc component of the output is $2/\pi$.

3.63. The average energy per period is

$$\frac{1}{T} \int_T |x(t)|^2 dt = \sum_k |\alpha_k|^2 = \sum_k \alpha^{2|k|} = \frac{1 + \alpha^2}{1 - \alpha^2}.$$

We want N such that

$$\sum_{k=-N+1}^{N-1} |\alpha_k|^2 = 0.9 \frac{1 + \alpha^2}{1 - \alpha^2}.$$

This implies that

$$\frac{1 - 2\alpha^{2N} + 2\alpha^2}{1 - \alpha^2} = \frac{1 + \alpha^2}{1 - \alpha^2}.$$

Solving,

$$N = \frac{\log[1.45\alpha^2 + 0.95]}{2 \log \alpha},$$

and

$$\frac{\pi N}{4} < W < \frac{(N-1)\pi}{4}.$$

3.64. (a) Due to linearity, we have

$$y(t) = \sum_k c_k \lambda_k \phi_k(t).$$

(b) Let

$$x_1(t) \rightarrow y_1(t) \quad \text{and} \quad x_2(t) \rightarrow y_2(t).$$

Also, let

$$x_3(t) = ax_1(t) + bx_2(t) \rightarrow y_3(t).$$

Then,

$$\begin{aligned} y_3(t) &= t^2[ax_1''(t) + bx_2''(t)] + t[ax_1'(t) + bx_2'(t)] \\ &= ay_1(t) + by_2(t) \end{aligned}$$

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(g) We have

$$\begin{aligned} \int_a^b |x(t)|^2 dt &= \int_a^b x(t)x^*(t) dt \\ &= \int_a^b \sum_i a_i \phi_i(t) \sum_j a_j^* \phi_j^*(t) dt \\ &= \sum_i \sum_j a_i a_j^* \int_a^b \phi_i(t) \phi_j^*(t) dt \\ &= \sum_i |a_i|^2. \end{aligned}$$

(h) We have

$$\begin{aligned} y(T) &= \int_{-\infty}^{\infty} h_1(T - \tau) \phi_j(\tau) d\tau \\ &= \int_{-\infty}^{\infty} \phi_1(\tau) \phi_j(\tau) d\tau \\ &= \delta_{ij} = 1 \text{ for } i = j \text{ and } 0 \text{ for } i \neq j. \end{aligned}$$

3.66. (a) We have

$$E = \int_a^b \left[x(t) - \sum_{k=-N}^N a_k \phi_k(t) \right] \left[x^*(t) - \sum_{k=-N}^N a_k^* \phi_k^*(t) \right] dt$$

Now, let $a_i = b_i + jc_i$. Then

$$\frac{\partial E}{\partial b_i} = 0 = - \int_a^b \phi_i^*(t) x(t) dt + 2b_i - \int_a^b \phi_i(t) x^*(t) dt$$

and

$$\frac{\partial E}{\partial c_i} = 0 = j \int_a^b \phi_i(t) x^*(t) dt + 2c_i - j \int_a^b \phi_i^*(t) x(t) dt.$$

Multiplying the last equation by j and adding to the one before, we get

$$2b_i + 2jc_i = 2 \int_a^b x(t) \phi_i^*(t) dt.$$

This implies that

$$a_i = \int_a^b x(t) \phi_i^*(t) dt.$$

(b) In this case, a_i would be

$$a_i = \frac{1}{A_i} \int_a^b x(t) \phi_i^*(t) dt.$$

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(c) Choosing

$$a_k = \frac{1}{T_0} \int_b^{b+T_0} x(t) e^{-jk\omega_0 t} dt,$$

we have

$$E = \int_{T_0} \left| x(t) - \sum_{k=-N}^N a_k e^{jk\omega_0 t} \right|^2 dt.$$

Putting $\frac{\partial E}{\partial a_k} = 0$, we get

$$a_k = \frac{1}{T_0} \int_{T_0} x(t) e^{-jk\omega_0 t} dt.$$

(d) $a_0 = 2/\pi$, $a_1 = a_3 = 0$, $a_2 = 2(1 - 2\sqrt{2})/\pi$, $a_4 = (1/\pi)[2 - 4\cos(\pi/8) + 4\cos(3\pi/8)]$.

(e) We have

$$\begin{aligned} \int_0^1 \sum_i (a_i \phi_i(t))^* [x(t) - \sum_i a_i \phi_i(t)] dt &= \sum_i a_i^* \int_0^1 x(t) \phi_i^*(t) dt \\ &\quad - \sum_i \sum_j a_i^* a_j \int_0^1 \phi_i^*(t) \phi_j(t) dt \\ &= \sum_i a_i^* a_i - \sum_i a_i^* a_i = 0 \end{aligned}$$

(f) Not orthogonal. Example: $\int_0^1 \phi_0(t) \phi_1(t) dt = \int_0^1 t dt = 1/2 \neq 0$.

(g) Here,

$$a_0 = \int_0^1 e^t \phi_0^*(t) dt = e - 1.$$

(h) Here, $\hat{x}(t) = a_0 + a_1 t$. Therefore,

$$E = \int_0^1 (e^t - a_0 - a_1 t)(e^t - a_0 - a_1 t) dt.$$

Setting $\partial E / \partial a_0 = 0 = \partial E / \partial a_1$, we get $a_0 = 2(2e - 5)$ and $a_1 = 6(3 - e)$.

3.67. (a) From eq. (P3.67-1) and (P3.67-4), we get

$$\sum_{n=-\infty}^{\infty} j 2\pi n b_n(x) e^{j 2\pi n t} = \frac{1}{2} k^2 \sum_{n=-\infty}^{\infty} \frac{\partial^2 b_n(x)}{\partial x^2} e^{j 2\pi n t}.$$

Equating coefficients of $e^{j 2\pi n t}$ on both sides, we get

$$\frac{\partial^2 b_n(x)}{\partial x^2} = \frac{j 4\pi n}{k^2} b_n(x).$$

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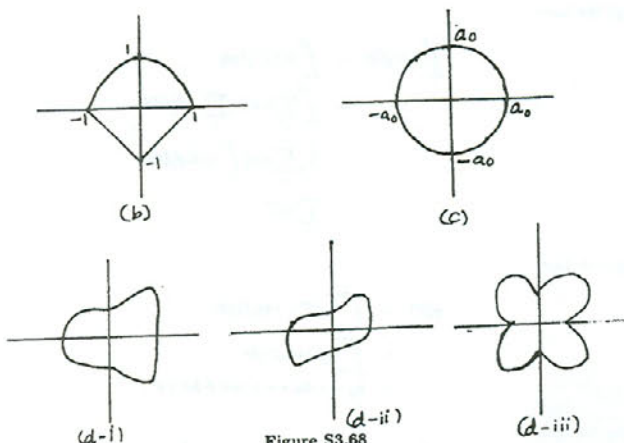


Figure S3.68

(c) We have

$$\begin{aligned} \sum_{n=N_1}^{N_2} |x[n]|^2 &= \sum_{n=N_1}^{N_2} \sum_{i=1}^M a_i \phi_i[n] \sum_{k=1}^M a_k^* \phi_k^*[n] \\ &= \sum_{k=1}^M \sum_{i=1}^M a_i a_k^* \sum_{n=N_1}^{N_2} \phi_k^*[n] \phi_i[n] \\ &= \sum_{k=1}^M \sum_{i=1}^M a_i a_k^* A_i \delta[i - k] = \sum_{i=1}^M |a_i|^2 A_i \end{aligned}$$

(d) Let $a_i = b_i + j c_i$. Then

$$\begin{aligned} E &= \sum_{n=N_1}^{N_2} |x[n]|^2 + \sum_{i=1}^M (b_i^2 + c_i^2) A_i - \sum_{n=N_1}^{N_2} x[n] \sum_{i=1}^M (b_i - j c_i) \phi_i^*[n] \\ &\quad - \sum_{n=N_1}^{N_2} x^*[n] \sum_{i=1}^M (b_i + j c_i) \phi_i[n] \end{aligned}$$

(b) Since $s^2 = 4\pi j n / k^2$,

$$s = \pm \frac{2\sqrt{\pi n} e^{j\pi/4}}{k}.$$

For $n > 0$,

$$s = \frac{\sqrt{2\pi n(1+j)}}{k}$$

is a stable solution. For $n < 0$,

$$s = -\frac{\sqrt{2\pi |n|(1-j)}}{k}$$

is a stable solution. Also, $b_n(0) = a_n$ and

$$b_n(x) = \begin{cases} a_n e^{-\sqrt{2\pi n(1+j)}x/k}, & n > 0 \\ a_n e^{-\sqrt{2\pi |n|(1-j)}x/k}, & n < 0 \end{cases}$$

(c) $b_0 = 2$, $b_1 = (1/2j)e^{-(1+j)\pi}$, $b_{-1} = -(1/2j)e^{-(1-j)\pi}$.

$$T(k\sqrt{\pi/2}, t) = 2 + e^{-\pi} \sin(2\pi t - \pi).$$

Phase reversed.

3.68. (a) $x(\theta) = r(\theta) \cos(\theta) = \frac{1}{2} r(\theta) e^{j\theta} + \frac{1}{2} r(\theta) e^{-j\theta}$. If

$$x(\theta) = \sum_{k=-\infty}^{\infty} b_k e^{jk\theta},$$

then $b_k = (1/2)a_{k+1} + (1/2)a_{k-1}$.

(b) $x(\theta) \xrightarrow{FS} b_k$. Then $x(\theta) = r(\theta + \pi/4)$. The sketch is as shown in Figure S3.68.

(c) $b_0 = a_0$. Rest of b_k is all zero. Therefore, the sketch will be a circle of radius a_0 as shown in Figure S3.68.

(d) (i) $r(\theta) = r(-\theta)$. Even. Sketch as shown in Figure S3.68.

(ii) $r(\theta + k\pi) = r(\theta)$. Sketch as shown in Figure S3.68.

(iii) $r(\theta + k\pi/2) = r(\theta)$. Sketch as shown in Figure S3.68.

3.69. (a) $\sum_{n=-N}^N \phi_k[n] \phi_k^*[m] = \sum_{n=-N}^N \delta[n-k] \delta[n-m]$. This is 1 for $k = m$ and 0 for $k \neq m$. Therefore, orthogonal.

(b) We have

$$\sum_{n=r}^{r+N-1} \phi_k[n] \phi_m^*[n] = e^{j(2\pi/N)r(k-m)} \left[\frac{1 - e^{j2\pi(k-m)}}{1 - e^{j2\pi(k-m)/N}} \right] = \begin{cases} 0, & k \neq m \\ N, & k = m \end{cases}$$

Therefore, orthogonal.

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Set $\partial E / \partial b_i = 0$. Then

$$b_i = [2A_i]^{-1} \left[\sum_{n=N_1}^{N_2} \{ x[n] \phi_i^*[n] + x^*[n] \phi_i[n] \} \right] = \frac{1}{A_i} \operatorname{Re} \left\{ \sum_{n=N_1}^{N_2} x[n] \phi_i^*[n] \right\}.$$

Similarly,

$$c_i = \frac{1}{A_i} \operatorname{Im} \left\{ \sum_{n=N_1}^{N_2} x[n] \phi_i^*[n] \right\}.$$

Therefore,

$$a_i = b_i + j c_i = \frac{1}{A_i} \sum_{n=N_1}^{N_2} x[n] \phi_i^*[n].$$

(e) $\phi_i[n] = \delta[n - i]$. Then,

$$a_i = \sum_{n=N_1}^{N_2} x[n] \delta[n - i] = x[i].$$

3.70. (a) We get

$$a_{mn} = \frac{1}{T_1 T_2} \int_0^{T_1} \int_0^{T_2} x(t_1, t_2) e^{-jm\omega_1 t_1} e^{-jn\omega_2 t_2} dt_1 dt_2.$$

(b) (i) $T_1 = 1$, $T_2 = \pi$. $a_{11} = 1/2$, $a_{-1,-1} = 1/2$. Rest of the coefficients are all zero.

(ii) Here,

$$a_{mn} = \begin{cases} 1/(\pi^2 mn), & m, n \text{ odd} \\ 0, & \text{otherwise} \end{cases}$$

3.71. (a) The differential equation $f_s(t)$ and $f(t)$ is

$$\frac{B}{K} \frac{df_s(t)}{dt} + f_s(t) = f(t).$$

The frequency response of this system may be easily shown to be

$$H(j\omega) = \frac{1}{1 + (B/K)j\omega}.$$

Note that for $\omega = 0$, $H(j\omega) = 1$ and for $\omega \rightarrow \infty$, $H(j\omega) = 0$. Therefore, the system approximates a lowpass filter.

(b) The differential equation $f_d(t)$ and $f(t)$ is

$$\frac{df_d(t)}{dt} + \frac{K}{B} f_d(t) = \frac{df(t)}{dt}.$$

The frequency response of this system may be easily shown to be

$$H(j\omega) = \frac{j\omega}{j\omega + (K/B)}.$$

Note that for $\omega = 0$, $H(j\omega) = 0$ and for $\omega \rightarrow \infty$, $H(j\omega) = 1$. Therefore, the system approximates a highpass filter.

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