Chapter 4 Answers

4.1. (a) Let $x(t) = e^{-2(t-1)}u(t-1)$. Then the Fourier transform $X(j\omega)$ of $x(t)$ is:

$$\begin{aligned} X(j\omega) &= \int_{-\infty}^{\infty} e^{-2(t-1)}u(t-1)e^{-j\omega t}dt \\ &= \int_{1}^{\infty} e^{-2(t-1)}e^{-j\omega t}dt \\ &= e^{-j\omega}/(2+j\omega) \end{aligned}$$

$|X(j\omega)|$ is as shown in Figure S4.1.

(b) Let $x(t) = e^{-2|t-1|}$. Then the Fourier transform $X(j\omega)$ of $x(t)$ is:

$$\begin{aligned} X(j\omega) &= \int_{-\infty}^{\infty} e^{-2|t-1|}e^{-j\omega t}dt \\ &= \int_{1}^{\infty} e^{-2(t-1)}e^{-j\omega t}dt + \int_{-\infty}^{1} e^{2(t-1)}e^{-j\omega t}dt \\ &= e^{-j\omega}/(2+j\omega) + e^{-j\omega}/(2-j\omega) \\ &= 4e^{-j\omega}/(4+\omega^2) \end{aligned}$$

$|X(j\omega)|$ is as shown in Figure S4.1.

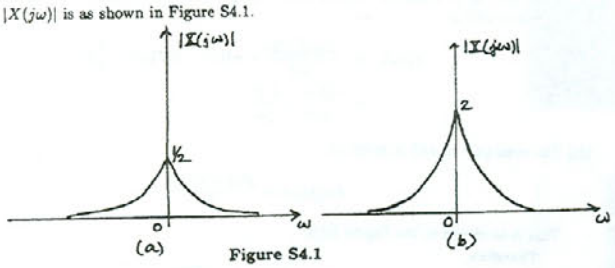

Figure S4.1

4.2. (a) Let $x_1(t) = \delta(t+1) + \delta(t-1)$. Then the Fourier transform $X_1(j\omega)$ of $x(t)$ is:

$$\begin{aligned} X_1(j\omega) &= \int_{-\infty}^{\infty} [\delta(t+1) + \delta(t-1)]e^{-j\omega t}dt \\ &= e^{j\omega} + e^{-j\omega} = 2\cos\omega \end{aligned}$$

$|X_1(j\omega)|$ is as sketched in Figure S4.2.

(b) The signal $x_2(t) = u(-2-t) + u(t-2)$ is as shown in the figure below. Clearly,

$$\frac{d}{dt}\{u(-2-t) + u(t-2)\} = \delta(t-2) - \delta(t+2)$$

Therefore,

$$\begin{aligned} X_2(j\omega) &= \int_{-\infty}^{\infty} [\delta(t-2) - \delta(t+2)]e^{-j\omega t}dt \\ &= e^{-2j\omega} - e^{2j\omega} = -2j\sin(2\omega) \end{aligned}$$

$|X_1(j\omega)|$ is as sketched in Figure S4.2.

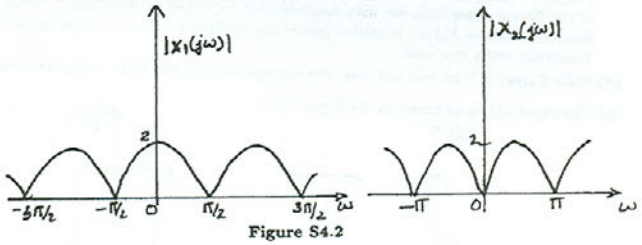

Figure S4.2

4.3. (a) The signal $x_1(t) = \sin(2\pi t + \pi/4)$ is periodic with a fundamental period of $T = 1$. This translates to a fundamental frequency of $\omega_0 = 2\pi$. The nonzero Fourier series coefficients of this signal may be found by writing it in the form

$$\begin{aligned} x_1(t) &= \frac{1}{2j}\left(e^{j(2\pi t+\pi/4)} - e^{-j(2\pi t+\pi/4)}\right) \\ &= \frac{1}{2j}e^{j\pi/4}e^{j2\pi t} - \frac{1}{2j}e^{-j\pi/4}e^{-j2\pi t} \end{aligned}$$

Therefore, the nonzero Fourier series coefficients of $x_1(t)$ are

$$a_1 = \frac{1}{2j}e^{j\pi/4}e^{j2\pi t}, \qquad a_{-1} = -\frac{1}{2j}e^{-j\pi/4}e^{-j2\pi t}$$

From Section 4.2, we know that for periodic signals, the Fourier transform consists of a train of impulses occurring at $k\omega_0$. Furthermore, the area under each impulse is 2π times the Fourier series coefficient a_k. Therefore, for $x_1(t)$, the corresponding Fourier transform $X_1(j\omega)$ is given by

$$\begin{aligned} X_1(j\omega) &= 2\pi a_1\delta(\omega-\omega_0) + 2\pi a_{-1}\delta(\omega+\omega_0) \\ &= (\pi/j)e^{j\pi/4}\delta(\omega-2\pi) - (\pi/j)e^{-j\pi/4}\delta(\omega+2\pi) \end{aligned}$$

(b) The signal $x_2(t) = 1 + \cos(6\pi t + \pi/8)$ is periodic with a fundamental period of $T = 1/3$. This translates to a fundamental frequency of $\omega_0 = 6\pi$. The nonzero Fourier series coefficients of this signal may be found by writing it in the form

$$\begin{aligned} x_2(t) &= 1 + \frac{1}{2}\left(e^{j(6\pi t+\pi/8)} - e^{-j(6\pi t+\pi/8)}\right) \\ &= 1 + \frac{1}{2}e^{j\pi/8}e^{j6\pi t} + \frac{1}{2}e^{-j\pi/8}e^{-j6\pi t} \end{aligned}$$

Therefore, the nonzero Fourier series coefficients of $x_2(t)$ are

$$a_0 = 1, \qquad a_1 = \frac{1}{2}e^{j\pi/8}e^{j6\pi t}, \qquad a_{-1} = \frac{1}{2}e^{-j\pi/8}e^{-j6\pi t}$$

From Section 4.2, we know that for periodic signals, the Fourier transform consists of a train of impulses occurring at $k\omega_0$. Furthermore, the area under each impulse is 2π times the Fourier series coefficient a_k. Therefore, for $x_2(t)$, the corresponding Fourier transform $X_2(j\omega)$ is given by

$$\begin{aligned} X_2(j\omega) &= 2\pi a_0\delta(\omega) + 2\pi a_1\delta(\omega-\omega_0) + 2\pi a_{-1}\delta(\omega+\omega_0) \\ &= 2\pi\delta(\omega) + \pi e^{j\pi/8}\delta(\omega-6\pi) + \pi e^{-j\pi/8}\delta(\omega+6\pi) \end{aligned}$$

4.4. (a) The inverse Fourier transform is

$$\begin{aligned} x_1(t) &= (1/2\pi)\int_{-\infty}^{\infty} [2\pi\delta(\omega) + \pi\delta(\omega-4\pi) + \pi\delta(\omega+4\pi)]e^{j\omega t}d\omega \\ &= (1/2\pi)[2\pi e^{j0t} + \pi e^{j4\pi t} + \pi e^{-j4\pi t}] \\ &= 1 + (1/2)e^{j4\pi t} + (1/2)e^{-j4\pi t} = 1 + \cos(4\pi t) \end{aligned}$$

(b) The inverse Fourier transform is

$$\begin{aligned} x_2(t) &= (1/2\pi)\int_{-\infty}^{\infty} X_2(j\omega)e^{j\omega t}d\omega \\ &= (1/2\pi)\int_{0}^{2} 2e^{j\omega t}d\omega + (1/2\pi)\int_{-2}^{0}(-2)e^{j\omega t}d\omega \\ &= (e^{j2t}-1)/(\pi jt) - (1-e^{-j2t})/(\pi jt) \\ &= -(4j\sin^2 t)/(\pi t) \end{aligned}$$

4.5. From the given information,

$$\begin{aligned} x(t) &= (1/2\pi)\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega \\ &= (1/2\pi)\int_{-\infty}^{\infty} |X(j\omega)|e^{j\sphericalangle\{X(j\omega)\}}e^{j\omega t}d\omega \\ &= (1/2\pi)\int_{-3}^{3} 2e^{-\frac{3}{2}\omega+\pi}e^{j\omega t}d\omega \\ &= \frac{-2}{\pi(t-3/2)}\sin[3(t-3/2)] \end{aligned}$$

The signal $x(t)$ is zero when $3(t-3/2)$ is a nonzero integer multiple of π. This gives

$$t = \frac{k\pi}{2} + \frac{3}{2}, \qquad \text{for } k \in \mathcal{I}, \text{ and } k \neq 0.$$

4.6. Throughout this problem, we assume that

$$x(t) \overset{FT}{\longleftrightarrow} X_1(j\omega).$$

(a) Using the time reversal property (Sec. 4.3.5), we have

$$x(-t) \overset{FT}{\longleftrightarrow} X(-j\omega)$$

Using the time shifting property (Sec. 4.3.2) on this, we have

$$x(-t+1) \overset{FT}{\longleftrightarrow} e^{-j\omega t}X(-j\omega) \quad \text{and} \quad x(-t-1) \overset{FT}{\longleftrightarrow} e^{j\omega t}X(-j\omega)$$

Therefore,

$$\begin{aligned} x_1(t) = x(-t+1) + x(-t-1) &\overset{FT}{\longleftrightarrow} e^{-j\omega t}X(-j\omega) + e^{j\omega t}X(-j\omega) \\ &\overset{FT}{\longleftrightarrow} 2X(-j\omega)\cos\omega \end{aligned}$$

(b) Using the time scaling property (Sec. 4.3.5), we have

$$x(3t) \overset{FT}{\longleftrightarrow} \frac{1}{3}X\left(j\frac{\omega}{3}\right)$$

Using the time shifting property on this, we have

$$x_2(t) = x(3(t-2)) \overset{FT}{\longleftrightarrow} e^{-2j\omega}\frac{1}{3}X\left(j\frac{\omega}{3}\right)$$

(c) Using the differentiation in time property (Sec. 4.3.4), we have

$$\frac{dx(t)}{dt} \overset{FT}{\longleftrightarrow} j\omega X(j\omega)$$

Applying this property again, we have

$$\frac{d^2x(t)}{dt^2} \overset{FT}{\longleftrightarrow} -\omega^2 X(j\omega).$$

Using the time shifting property, we have

$$x_3(t) = \frac{d^2x(t-1)}{dt^2} \overset{FT}{\longleftrightarrow} -\omega^2 X(j\omega)e^{-j\omega t}.$$

4.7. (a) Since $X_1(j\omega)$ is not conjugate symmetric, the corresponding signal $x_1(t)$ is **not real**. Since $X_1(j\omega)$ is neither even nor odd, the corresponding signal $x_1(t)$ is **neither even nor odd**.

(b) The Fourier transform of a real and odd signal is purely imaginary and odd. Therefore, we may conclude that the Fourier transform of a purely imaginary and odd signal is real and odd. Since $X_2(j\omega)$ is real and odd, we may therefore conclude that the corresponding signal $x_2(t)$ is **purely imaginary and odd**.

(c) Consider a signal $y_3(t)$ whose magnitude of the Fourier transform is $|Y_3(j\omega)| = A(\omega)$, and whose phase of the Fourier transform is $\sphericalangle\{Y_3(j\omega)\} = 2\omega$. Since $|Y_3(j\omega)| = |Y_3(-j\omega)|$ and $\sphericalangle\{Y_3(j\omega)\} = -\sphericalangle\{Y_3(-j\omega)\}$, we may conclude that the signal $y_3(t)$ is real (See Table 4.1, Property 4.3.3).

Now, consider the signal $x_3(t)$ with Fourier transform $X_3(j\omega) = Y_3(j\omega)e^{j\pi/2} = jY_3(j\omega)$. Using the result from the previous paragraph and the linearity property of the Fourier transform, we may conclude that $x_3(t)$ has to **imaginary**. Since the Fourier transform $X_3(j\omega)$ is neither purely imaginary nor purely real, the signal $x_3(t)$ is **neither even nor odd**.

(d) Since $X_4(j\omega)$ is both real and even, the corresponding signal $x_4(t)$ is **real and even**.

4.8. (a) The signal $x(t)$ is as shown in the Figure S4.8.

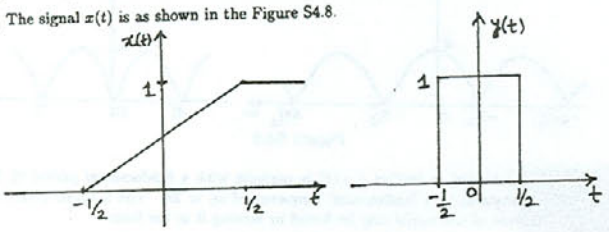

Figure S4.8

We may express this signal as

$$x(t) = \int_{-\infty}^{t} y(t)dt,$$

where $y(t)$ is the rectangular pulse shown in Figure S4.8. Using the integration property of the Fourier transform, we have

$$x(t) \overset{FT}{\longleftrightarrow} X(j\omega) = \frac{1}{j\omega}Y(j\omega) + \pi Y(j0)\delta(\omega)$$

We know from Table 4.2 that

$$Y(j\omega) = \frac{2\sin(\omega/2)}{\omega}$$

Therefore,

$$X(j\omega) = \frac{2\sin(\omega/2)}{j\omega^2} + \pi\delta(\omega)$$

(b) If $g(t) = x(t) - 1/2$, then the Fourier transform $G(j\omega)$ of $g(t)$ is given by

$$G(j\omega) = X(j\omega) - (1/2)2\pi\delta(\omega) = \frac{2\sin(\omega/2)}{j\omega^2}.$$

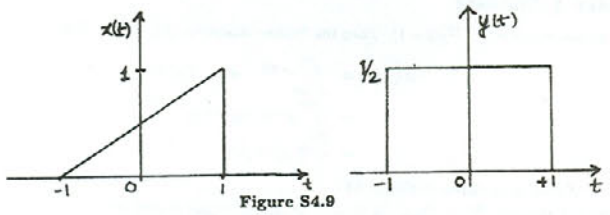

Figure S4.9

4.9. (a) The signal $x(t)$ is plotted in Figure S4.9.

We see that this signal is very similar to the one considered in the previous problem. In fact we may again express the signal $x(t)$ in terms of the rectangular pulse $y(t)$ shown above as follows

$$x(t) = \int_{-\infty}^{t} y(t)dt - u(t - \tfrac{1}{2}).$$

Using the result obtained in part (a) of the previous problem, the Fourier transform $X(j\omega)$ of $x(t)$ is

$$\begin{aligned} X(j\omega) &= \frac{2\sin(\omega/2)}{j\omega^2} + \pi\delta(\omega) - \mathcal{FT}\{u(t - \tfrac{1}{2})\} \\ &= \frac{\sin\omega}{j\omega^2} - \frac{e^{-j\omega}}{j\omega} \end{aligned}$$

(b) The even part of $x(t)$ is given by

$$\mathcal{E}v\{x(t)\} = \frac{x(t) + x(-t)}{2}.$$

This is as shown in the Figure S4.9.

Therefore,

$$\mathcal{FT}\{\mathcal{E}v\{x(t)\}\} = \frac{\sin\omega}{\omega}.$$

Now the real part of the answer to part (a) is

$$\mathcal{R}e\left\{-\frac{e^{-j\omega}}{j\omega}\right\} = (1/\omega)\mathcal{R}e\left\{j(\cos\omega - j\sin\omega)\right\} = \frac{\sin\omega}{\omega}$$

(c) The Fourier transform of the odd part of $x(t)$ is same as j times imaginary part of the answer to part (a). We have

$$\mathcal{I}m\left\{\frac{\sin\omega}{j\omega^2} - \frac{e^{-j\omega}}{j\omega}\right\} = -\frac{\sin\omega}{\omega^2} + \frac{\cos\omega}{\omega}$$

Therefore, the desired result is

$$\mathcal{FT}\{\text{Odd part of} x(t)\} = \frac{\sin\omega}{j\omega^2} - \frac{\cos\omega}{j\omega}$$

4.10. (a) We know from Table 4.2 that

$$\frac{\sin t}{\pi t} \overset{FT}{\longleftrightarrow} \text{Rectangular function } Y(j\omega) \text{ [See Figure S4.10]}$$

Therefore

$$\left(\frac{\sin t}{\pi t}\right)^2 \overset{FT}{\longleftrightarrow} (1/2\pi)[\text{Rectangular function } Y(j\omega) * \text{Rectangular function } Y(j\omega)]$$

This is a triangular function $Y_1(j\omega)$ as shown in the Figure S4.10.

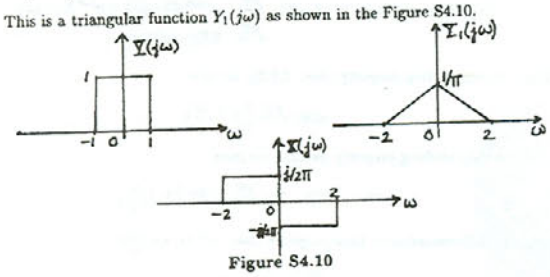

Figure S4.10

Using Table 4.1, we may write

$$t\left(\frac{\sin t}{\pi t}\right)^2 \overset{FT}{\longleftrightarrow} X(j\omega) = j\frac{d}{d\omega}Y_1(j\omega)$$

This is as shown in the figure above. $X(j\omega)$ may be expressed mathematically as

$$X(j\omega) = \begin{cases} j/2\pi, & -2 \le \omega < 0 \\ -j/2\pi, & 0 \le \omega 2 \\ 0. & \text{otherwise} \end{cases}$$

(b) Using Parseval's relation,

$$\int_{-\infty}^{\infty} t^2\left(\frac{\sin t}{\pi t}\right)^4 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}|X(j\omega)|^2 d\omega = \frac{1}{2\pi^3}$$

4.11. We know that

$$x(3t) \overset{FT}{\longleftrightarrow} \frac{1}{3}X(j\tfrac{\omega}{3}), \qquad h(3t) \overset{FT}{\longleftrightarrow} \frac{1}{3}H(j\tfrac{\omega}{3})$$

Therefore,

$$G(j\omega) = \mathcal{FT}\{x(3t) * h(3t)\} = \frac{1}{9}X(j\tfrac{\omega}{3})H(j\tfrac{\omega}{3})$$

Now note that

$$Y(j\omega) = \mathcal{FT}\{x(t) * h(t)\} = X(j\omega)H(j\omega)$$

From this, we may write

$$Y(j\tfrac{\omega}{3}) = X\left(j\tfrac{\omega}{3}\right)H\left(j\tfrac{\omega}{3}\right)$$

Using this in eq. (**), we have

$$G(j\omega) = \frac{1}{9}Y(j\tfrac{\omega}{3})$$

and

$$g(t) = \frac{1}{3}y(3t).$$

Therefore, $A = \frac{1}{3}$ and $B = 3$.

4.12. (a) From Example 4.2 we know that

$$e^{-|t|} \overset{FT}{\longleftrightarrow} \frac{2}{1+\omega^2}.$$

Using the differentiation in frequency property, we have

$$te^{-|t|} \overset{FT}{\longleftrightarrow} j\frac{d}{d\omega}\left\{\frac{2}{1+\omega^2}\right\} = -\frac{4j\omega}{(1+\omega^2)^2}.$$

(b) The duality property states that if

$$g(t) \overset{FT}{\longleftrightarrow} G(j\omega)$$

then

$$G(t) \overset{FT}{\longleftrightarrow} 2\pi g(j\omega).$$

Now, since

$$te^{-|t|} \overset{FT}{\longleftrightarrow} -\frac{4j\omega}{(1+\omega^2)^2}$$

we may use duality to write

$$-\frac{4jt}{(1+t^2)^2} \overset{FT}{\longleftrightarrow} 2\pi\omega e^{-|\omega|}$$

Multiplying both sides by j, we obtain

$$\frac{4t}{(1+t^2)^2} \overset{FT}{\longleftrightarrow} j2\pi\omega e^{-|\omega|}.$$

4.13. (a) Taking the inverse Fourier transform of $X(j\omega)$, we obtain

$$x(t) = \frac{1}{2\pi} + \frac{1}{2\pi}e^{j\pi t} + \frac{1}{2\pi}e^{j5t}$$

The signal $x(t)$ is therefore a constant summed with two complex exponentials whose fundamental frequencies are $2\pi/5$ rad/sec and 2 rad/sec. These two complex exponentials are not harmonically related. That is, the fundamental frequencies of these complex complex exponentials can never be integral multiples of a common fundamental frequency. Therefore, the signal is **not periodic**.

(b) Consider the signal $y(t) = x(t) * h(t)$. From the convolution property, we know that $Y(j\omega) = X(j\omega)H(j\omega)$. Also, from $h(t)$, we know that

$$H(j\omega) = e^{-j\omega}\frac{2\sin\omega}{\omega}.$$

The function $H(j\omega)$ is zero when $\omega = k\pi$, where k is a nonzero integer. Therefore,

$$Y(j\omega) = X(j\omega)H(j\omega) = \delta(\omega) + \delta(\omega - 5)$$

This gives

$$y(t) = \frac{1}{2\pi} + \frac{1}{2\pi}e^{j5t}$$

Therefore, $y(t)$ is a complex exponential summed with a constant. We know that a complex exponential is periodic. Adding a constant to a complex exponential does not affect its periodicity. Therefore, $y(t)$ will be a signal with a fundamental frequency of $2\pi/5$.

(c) From the results of parts (a) and (b), we see that the answer is yes.

4.14. Taking the Fourier transform of both sides of the equation

$$\mathcal{F}^{-1}\{(1+j\omega)X(j\omega)\} = A2^{-2t}u(t),$$

we obtain

$$X(j\omega) = \frac{A}{(1+j\omega)(2+j\omega)} = A\left\{\frac{1}{1+j\omega} - \frac{1}{2+j\omega}\right\}$$

Taking the inverse Fourier transform of the above equation

$$x(t) = Ae^{-t}u(t) - Ae^{-2t}u(t)$$

Using Parseval's relation, we have

$$\int_{-\infty}^{\infty}|X(j\omega)|^2 d\omega = 2\pi\int_{-\infty}^{\infty}|x(t)|^2 dt$$

Using the fact that $\int_{-\infty}^{\infty}|X(j\omega)|^2 d\omega = 2\pi$, we have

$$\int_{-\infty}^{\infty}|x(t)|^2 dt = 1$$

Substituting the previously obtained expression for $x(t)$ in the above equation, we have

$$\int_{-\infty}^{\infty}[A^2e^{-2t} + A^2e^{-4t} - 2A^2e^{-3t}]u(t)dt = 1$$
$$\int_{0}^{\infty}[A^2e^{-2t} + A^2e^{-4t} - 2A^2e^{-3t}]dt = 1$$
$$A^2/12 = 1$$
$$\Rightarrow A = \sqrt{12}$$

We choose A to be $\sqrt{12}$ instead of $-\sqrt{12}$ because we know that $x(t)$ is non negative.

4.15. Since $x(t)$ is real,

$$\mathcal{E}v\{x(t)\} = \frac{x(t)+x(-t)}{2} \overset{FT}{\longleftrightarrow} \mathcal{R}e\{X(j\omega)\}.$$

We are given that

$$\mathcal{IFT}\{\mathcal{R}e\{X(j\omega)\}\} = |t|e^{-|t|}.$$

Therefore,

$$\mathcal{E}v\{x(t)\} = \frac{x(t)+x(-t)}{2} = |t|e^{-|t|}.$$

We also know that $x(t) = 0$ for $t \le 0$. This implies that $x(-t)$ is zero for $t > 0$. We may conclude that

$$x(t) = 2|t|e^{-|t|} \qquad \text{for } t \ge 0$$

Therefore,

$$x(t) = 2te^{-t}u(t)$$

4.16. (a) We may write

$$\begin{aligned} x(t) &= \sum_{k=-\infty}^{\infty}\frac{\sin(k\pi/4)}{k\pi/4}\delta(t-k\pi/4) \\ &= \frac{\sin t}{\pi t}\sum_{k=-\infty}^{\infty}\pi\delta(t-k\pi/4) \end{aligned}$$

Therefore, $g(t) = \sum_{k=-\infty}^{\infty}\pi\delta(t-k\pi/4)$.

(b) Since $g(t)$ is an impulse train, its Fourier transform $G(j\omega)$ is also an impulse train. From Table 4.2,

$$\begin{aligned} G(j\omega) &= \pi\frac{2\pi}{\pi/4}\sum_{k=-\infty}^{\infty}\delta\left(\omega - \frac{2\pi k}{\pi/4}\right) \\ &= 8\pi\sum_{k=-\infty}^{\infty}\delta(\omega - 8k) \end{aligned}$$

We see that $G(j\omega)$ is periodic with a period of 8. Using the multiplication property, we know that

$$X(j\omega) = \frac{1}{2\pi}\left[\mathcal{FT}\left\{\frac{\sin t}{\pi t}\right\} * G(j\omega)\right]$$

If we denote $\mathcal{FT}\left\{\frac{\sin t}{\pi t}\right\}$ by $A(j\omega)$, then

$$\begin{aligned} X(j\omega) &= (1/2\pi)[A(j\omega) * 8\pi\sum_{k=-\infty}^{\infty}\delta(\omega - 8k) \\ &= 4\sum_{k=-\infty}^{\infty}A(j\omega - 8k) \end{aligned}$$

$X(j\omega)$ may thus be viewed as a replication of $4A(j\omega)$ every 8 rad/sec. This is obviously periodic.

Using Table 4.2, we obtain

$$A(j\omega) = \begin{cases} 1, & |\omega| \le 1 \\ 0, & \text{otherwise} \end{cases}$$

Therefore, we may specify $X(j\omega)$ over one period as

$$X(j\omega) = \begin{cases} 4, & |\omega| \le 1 \\ 0, & 1 < |\omega \le 4 \end{cases}$$

4.17. (a) From Table 4.1, we know that a real and odd signal signal $x(t)$ has a purely imaginary and odd Fourier transform $X(j\omega)$. Let us now consider the purely imaginary and odd signal $jx(t)$. Using linearity, we obtain the Fourier transform of this signal to be $jX(j\omega)$. The function $jX(j\omega)$ will clearly be real and odd. Therefore the given statement is **false**.

(b) An odd Fourier transform corresponds to an odd signal, while an even Fourier transform corresponds to an even signal. The convolution of an even Fourier transform with an odd Fourier may be viewed in the time domain as a multiplication of an even and odd signal. Such a multiplication will always result in an odd time signal. The Fourier transform of this odd signal will always be odd. Therefore, the given statement is **true**.

4.18. Using Table 4.2, we see that the rectangular pulse $x_1(t)$ shown in Figure S4.18 has a Fourier transform $X_1(j\omega) = \sin(3\omega)/\omega$. Using the convolution property of the Fourier transform, we may write

$$x_2(t) = x_1(t) * x_1(t) \overset{FT}{\longleftrightarrow} X_2(j\omega) = X_1(j\omega)X_1(j\omega) = \left(\frac{\sin(3\omega)}{\omega}\right)^2$$

The signal $x_2(t)$ is shown in Figure S4.18. Using the shifting property, we also note that

$$\frac{1}{2}x_2(t+1) \overset{FT}{\longleftrightarrow} \frac{1}{2}e^{j\omega}\left(\frac{\sin(3\omega)}{\omega}\right)^2$$

and

$$\frac{1}{2}x_2(t-1) \overset{FT}{\longleftrightarrow} \frac{1}{2}e^{-j\omega}\left(\frac{\sin(3\omega)}{\omega}\right)^2.$$

Adding the two above equations, we obtain

$$h(t) = \frac{1}{2}x_2(t+1) + \frac{1}{2}x_2(t-1) \overset{FT}{\longleftrightarrow} \cos(\omega)\left(\frac{\sin(3\omega)}{\omega}\right)^2.$$

The signal $h(t)$ is as shown in Figure S4.18. We note that $h(t)$ has the given Fourier transform $H(j\omega)$.

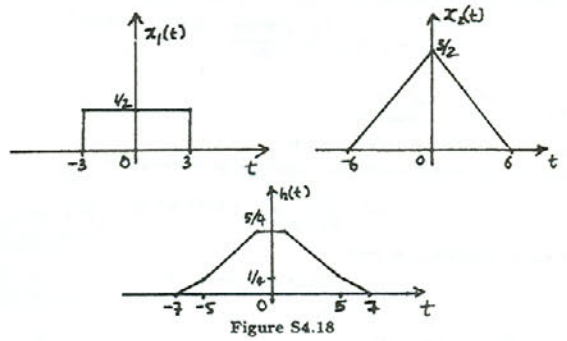

Figure S4.18

Mathematically $h(t)$ may be expressed as

$$h(t) = \begin{cases} \frac{5}{4}, & |t| < 1 \\ -\frac{|t|}{4} + \frac{3}{2}, & 1 \le |t| \le 5 \\ -\frac{|t|}{8} + \frac{7}{8}, & 5 < |t| \le 7 \\ 0, & \text{otherwise} \end{cases}$$

4.19. We know that

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)}.$$

Since it is given that $y(t) = e^{-3t}u(t) - e^{-4t}u(t)$, we can compute $Y(j\omega)$ to be

$$Y(j\omega) = \frac{1}{3+j\omega} - \frac{1}{4+j\omega} = \frac{1}{(3+j\omega)(4+j\omega)}.$$

Since, $H(j\omega) = 1/(3 + j\omega)$, we have

$$X(j\omega) = \frac{Y(j\omega)}{H(j\omega)} = 1/(4 + j\omega)$$

Taking the inverse Fourier transform of $X(j\omega)$, we have

$$x(t) = e^{-4t}u(t).$$

4.20. From the answer to Problem 3.20, we know that the frequency response of the circuit is

$$H(j\omega) = \frac{1}{-\omega^2 + j\omega + 1}.$$

Breaking this up into partial fractions, we may write

$$H(j\omega) = -\frac{1}{j\sqrt{3}}\left[\frac{-1}{\frac{1}{2} - \frac{\sqrt{3}}{2}j + j\omega} + \frac{-1}{\frac{1}{2} + \frac{\sqrt{3}}{2}j + j\omega}\right]$$

Using the Fourier transform pairs provided in Table 4.2, we obtain the Fourier transform of $H(j\omega)$ to be

$$h(t) = -\frac{1}{j\sqrt{3}}\left[-e^{(-\frac{1}{2}+\frac{\sqrt{3}}{2}j)t} + +e^{(-\frac{1}{2}-\frac{\sqrt{3}}{2}j)t}\right]u(t).$$

Simplifying,

$$h(t) = \frac{2}{\sqrt{3}}e^{-\frac{1}{2}t}\sin(\frac{\sqrt{3}}{2}t)u(t).$$

4.21. (a) The given signal is

$$e^{-\alpha t}\cos(\omega_0 t)u(t) = \frac{1}{2}e^{-\alpha t}e^{j\omega_0 t}u(t) + \frac{1}{2}e^{-\alpha t}e^{-j\omega_0 t}u(t).$$

Therefore,

$$X(j\omega) = \frac{1}{2(\alpha - j\omega_0 + j\omega)} - \frac{1}{2(\alpha - j\omega_0 + j\omega)}.$$

(b) The given signal is

$$x(t) = e^{-3t}\sin(2t)u(t) + e^{3t}\sin(2t)u(-t).$$

We have

$$x_1(t) = e^{-3t}\sin(2t)u(t) \overset{FT}{\longleftrightarrow} X_1(j\omega) = \frac{1/2j}{3 - j2 + j\omega} - \frac{1/2j}{3 + j2 + j\omega}.$$

Also,

$$x_2(t) = e^{3t}\sin(2t)u(-t) = -x_1(-t) \overset{FT}{\longleftrightarrow} X_2(j\omega) = -X_1(-j\omega) = \frac{1/2j}{3 - j2 - j\omega} - \frac{1/2j}{3 + j2 - j\omega}.$$

Therefore,

$$X(j\omega) = X_1(j\omega + X_2(j\omega) = \frac{3j}{9 + (\omega + 2)^2} - \frac{3j}{9 + (\omega - 2)^2}.$$

(c) Using the Fourier transform analysis equation (4.9) we have

$$X(j\omega) = \frac{2\sin\omega}{\omega} + \frac{\sin\omega}{\pi - \omega} - \frac{\sin\omega}{\pi + \omega}.$$

(d) Using the Fourier transform analysis equation (4.9) we have

$$X(j\omega) = \frac{1}{1 - \alpha e^{-j\omega T}}.$$

(e) We have

$$x(t) = (1/2j)te^{-2t}e^{j4t}u(t) - (1/2j)te^{-2t}e^{-j4t}u(t).$$

Therefore,

$$X(j\omega) = \frac{1/2j}{(2 - j4 + j\omega)^2} - \frac{1/2j}{(2 + j4 - j\omega)^2}.$$

(f) We have

$$x_1(t) = \frac{\sin\pi t}{\pi t} \overset{FT}{\longleftrightarrow} X_1(j\omega) = \begin{cases} 1, & |\omega| < \pi \\ 0, & \text{otherwise} \end{cases}$$

Also

$$x_2(t) = \frac{\sin 2\pi(t-1)}{\pi(t-1)} \overset{FT}{\longleftrightarrow} X_2(j\omega) = \begin{cases} e^{-2\omega}, & |\omega| < 2\pi \\ 0, & \text{otherwise} \end{cases}$$

$$x(t) = x_1(t)x_2(t) \overset{FT}{\longleftrightarrow} X(j\omega) = \frac{1}{2\pi}\{X_1(j\omega) * X_2(j\omega)\}.$$

Therefore,

$$X(j\omega) = \begin{cases} e^{-j\omega}, & |\omega| < \pi \\ (1/2\pi)(3\pi + \omega)e^{-j\omega}, & -3\pi < \omega < -\pi \\ (1/2\pi)(3\pi - \omega)e^{-j\omega}, & \pi < \omega < 3\pi \\ 0, & \text{otherwise} \end{cases}.$$

(g) Using the Fourier transform analysis eq. (4.9) we obtain

$$X(j\omega) = \frac{2j}{\omega}\left[\cos 2\omega - \frac{\sin\omega}{\omega}\right].$$

(h) If

$$x_1(t) = \sum_{k=-\infty}^{\infty}\delta(t - 2k),$$

then

$$x(t) = 2x_1(t) + x_1(t - 1).$$

Therefore,

$$X(j\omega) = X_1(j\omega)[2 + e^{-\omega}] = \pi\sum_{k=-\infty}^{\infty}\delta(\omega - k\pi)[2 + (-1)^k].$$

(i) Using the Fourier transform analysis eq. (4.9) we obtain

$$X(j\omega) = \frac{1}{j\omega} + \frac{2e^{-j\omega}}{-\omega^2} - \frac{2e^{-j\omega} - 2}{j\omega^2}.$$

(j) $x(t)$ is periodic with period 2. Therefore,

$$X(j\omega) = \pi\sum_{k=-\infty}^{\infty}\bar{X}(jk\pi)\delta(\omega - k\pi),$$

where $\bar{X}(j\omega)$ is the Fourier transform of one period of $x(t)$. That is,

$$\bar{X}(j\omega) = \frac{1}{1 - e^{-2}}\left[\frac{1 - e^{-2(1+j\omega)}}{1 + j\omega} - \frac{e^{-2}[1 - e^{-2(1+j\omega)}]}{1 - j\omega}\right].$$

4.22. (a) $x(t) = \begin{cases} e^{j2\pi t}, & |t| < 3 \\ 0, & \text{otherwise} \end{cases}$

(b) $x(t) = \frac{1}{2}e^{-j\pi/3}\delta(t - 4) + \frac{1}{2}e^{j\pi/3}\delta(t + 4)$.

(c) The Fourier transform synthesis eq. (4.8) may be written as

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}|X(j\omega)|e^{j\angle X(j\omega)}e^{j\omega t}d\omega.$$

From the given figure we have

$$x(t) = \frac{1}{\pi}\left[\frac{\sin(t-3)}{t-3} + \frac{\cos(t-3) - 1}{(t-3)^2}\right].$$

(d) $x(t) = \frac{2j}{\pi}\sin t + \frac{3}{\pi}\cos(2\pi t)$

(e) Using the Fourier transform synthesis equation (4.8),

$$x(t) = \frac{\cos 3t}{j\pi t} + \frac{\sin t - \sin 2t}{j\pi t^2}.$$

4.23. For the given signal $x_0(t)$, we use the Fourier transform analysis eq. (4.8) to evaluate the corresponding Fourier transform

$$X_0(j\omega) = \frac{1 - e^{-(1+j\omega)}}{1 + j\omega}.$$

(i) We know that

$$x_1(t) = x_0(t) + x_0(-t).$$

Using the linearity and time reversal properties of the Fourier transform we have

$$X_1(j\omega) = X_0(j\omega) + X_0(-j\omega) = \frac{2 - 2e^{-1}\cos\omega - 2\omega e^{-1}\sin\omega}{1 + \omega^2}.$$

(ii) We know that

$$x_2(t) = x_0(t) - x_0(-t).$$

Using the linearity and time reversal properties of the Fourier transform we have

$$X_2(j\omega) = X_0(j\omega) - X_0(-j\omega) = j\left[\frac{-2\omega + 2e^{-1}\sin\omega + 2\omega e^{-1}\cos\omega}{1 + \omega^2}\right].$$

(iii) We know that

$$x_3(t) = x_0(t) + x_0(t + 1).$$

Using the linearity and time shifting properties of the Fourier transform we have

$$X_3(j\omega) = X_0(j\omega) + e^{j\omega}X_0(-j\omega) = \frac{1 + e^{j\omega} - e^{-1}(1 + e^{-j\omega})}{1 + j\omega}.$$

(iv) We know that

$$x_4(t) = tx_0(t).$$

Using the differentiation in frequency property

$$X_4(j\omega) = j\frac{d}{d\omega}X_0(j\omega).$$

Therefore,

$$X_4(j\omega) = \frac{1 - 2e^{-1}e^{-j\omega} - j\omega e^{-1}e^{-j\omega}}{(1 + j\omega)^2}.$$

4.24. (a) (i) For $\mathcal{R}e\{X(j\omega)\}$ to be 0, the signal $x(t)$ must be real and odd. Therefore, signals in figures (a) and (c) have this property.

(ii) For $\mathcal{I}m\{X(j\omega)\}$ to be 0, the signal $x(t)$ must be real and even. Therefore, signals in figures (e) and (f) have this property.

(iii) For there to exist a real α such that $e^{j\alpha\omega}X(j\omega)$ is real, we require that $x(t + \alpha)$ be a real and even signal. Therefore, signals in figures (a), (b), (e), and (f) have this property.

(iv) For this condition to be true, $x(0) = 0$. Therefore, signals in figures (a), (b), (c), (d), and (f) have this property.

(v) For this condition to be true the derivative of $x(t)$ has to be zero at $t = 0$. Therefore, signals in figures (b), (c), (e), and (f) have this property.

(vi) For this to be true, the signal $x(t)$ has to be periodic. Only the signal in figure (a) has this property.

(b) For a signal to satisfy only properties (i), (iv), and (v), it must be real and odd, and

$$x(t) = 0, \quad x'(0) = 0.$$

The signal shown below is an example of that.

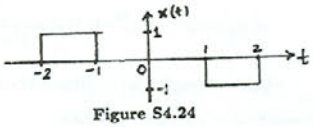

Figure S4.24

4.25. (a) Note that $y(t) = x(t+1)$ is a real and even signal. Therefore, $Y(j\omega)$ is also real and even. This implies that $\sphericalangle Y(j\omega) = 0$. Also, since $Y(j\omega) = e^{j\omega}X(j\omega)$, we know that $\sphericalangle X(j\omega) = -\omega$.

(b) We have

$$X(j0) = \int_{-\infty}^{\infty} x(t)dt = 7.$$

(c) We have

$$\int_{-\infty}^{\infty} X(j\omega)d\omega = 2\pi x(0) = 4\pi.$$

(d) Let $Y(j\omega) = \frac{2\sin\omega}{\omega}e^{2j\omega}$. The corresponding signal $y(t)$ is

$$y(t) = \begin{cases} 1, & -3 < t < -1 \\ 0, & \text{otherwise} \end{cases}.$$

Then the given integral is

$$\int_{-\infty}^{\infty} X(j\omega)Y(j\omega)d\omega = 2\pi\{x(t) * y(t)\}_{t=0} = 7\pi.$$

(e) We have

$$\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega = 2\pi\int_{-\infty}^{\infty} |x(t)|^2 dt = 26\pi.$$

(f) The inverse Fourier transform of $\mathcal{R}e\{X(j\omega)\}$ is the $\mathcal{E}v\{x(t)\}$ which is $[x(t) + x(-t)]/2$. This is as shown in the figure below.

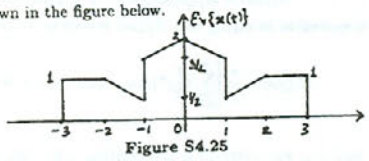

Figure S4.25

4.26. (a) (i) We have

$$\begin{aligned} Y(j\omega) &= X(j\omega)H(j\omega) = \left[\frac{1}{(2+j\omega)^2}\right]\left[\frac{1}{4+j\omega}\right] \\ &= \frac{(1/4)}{4+j\omega} - \frac{(1/4)}{2+j\omega} + \frac{(1/2)}{(2+j\omega)^2} \end{aligned}$$

Taking the inverse Fourier transform we obtain

$$y(t) = \frac{1}{4}e^{-4t}u(t) - \frac{1}{4}e^{-2t}u(t) + \frac{1}{2}te^{-2t}u(t).$$

(ii) We have

$$\begin{aligned} Y(j\omega) &= X(j\omega)H(j\omega) = \left[\frac{1}{(2+j\omega)^2}\right]\left[\frac{1}{(4+j\omega)^2}\right] \\ &= \frac{(1/4)}{2+j\omega} + \frac{(1/4)}{(2+j\omega)^2} - \frac{(1/4)}{4+j\omega} + \frac{(1/4)}{(4+j\omega)^2} \end{aligned}$$

Taking the inverse Fourier transform we obtain

$$y(t) = \frac{1}{4}e^{-2t}u(t) + \frac{1}{4}te^{-2t}u(t) - \frac{1}{4}e^{-4t}u(t) + \frac{1}{4}te^{-4t}u(t).$$

(iii) We have

$$\begin{aligned} Y(j\omega) &= X(j\omega)H(j\omega) \\ &= \left[\frac{1}{1+j\omega}\right]\left[\frac{1}{1-j\omega}\right] \\ &= \frac{1/2}{1+j\omega} + \frac{1/2}{1-j\omega} \end{aligned}$$

Taking the inverse Fourier transform, we obtain

$$y(t) = \frac{1}{2}e^{-|t|}.$$

(b) By direct convolution of $x(t)$ with $h(t)$ we obtain

$$y(t) = \begin{cases} 0, & t < 1 \\ 1 - e^{-(t-1)}, & 1 < t \le 5 \\ e^{-(t-5)} - e^{-(t-1)}, & t > 5 \end{cases}.$$

Taking the Fourier transform of $y(t)$,

$$\begin{aligned} Y(j\omega) &= \frac{2e^{-j3\omega}\sin(2\omega)}{\omega(1+j\omega)} \\ &= \left[\frac{e^{-j2\omega}}{1+j\omega}\right]\frac{e^{-j\omega}2\sin(2\omega)}{\omega} \\ &= X(j\omega)H(j\omega) \end{aligned}$$

4.27. (a) The Fourier transform $X(j\omega)$ is

$$\begin{aligned} X(j\omega) &= \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = \int_1^2 e^{-j\omega t}dt - \int_2^3 e^{-j\omega t}dt \\ &= 2\frac{\sin(\omega/2)}{\omega}\{1 - e^{-j\omega}\}e^{-j3\omega/2} \end{aligned}$$

(b) The Fourier series coefficients a_k are

$$\begin{aligned} a_k &= \frac{1}{T}\int_{<T>} \tilde{x}(t)e^{-j\frac{2\pi}{T}kt} \\ &= \frac{1}{2}\{\int_1^2 e^{-j\frac{2\pi}{T}kt}dt - \int_2^3 e^{-j\frac{2\pi}{T}kt}dt\} \\ &= \frac{\sin(k\pi/2)}{k\pi}\{1 - e^{-jk\pi}\}e^{-j3k\pi/2} \end{aligned}$$

Comparing the answers to parts (a) and (b), it is clear that

$$a_k = \frac{1}{T}X(j\frac{2\pi k}{T}),$$

where $T = 2$.

4.28. (a) From Table 4.2 we know that

$$p(t) = \sum_{n=-\infty}^{\infty} a_n e^{jn\omega_0 t} \overset{FT}{\longleftrightarrow} P(j\omega) = 2\pi\sum_{k=-\infty}^{\infty} a_k\delta(\omega - k\omega_0).$$

From this,

$$Y(j\omega) = \frac{1}{2\pi}\{X(j\omega) * H(j\omega)\} = \sum_{k=-\infty}^{\infty} a_k X(j(\omega - k\omega_0)).$$

(b) The spectra are sketched in Figure S4.28.

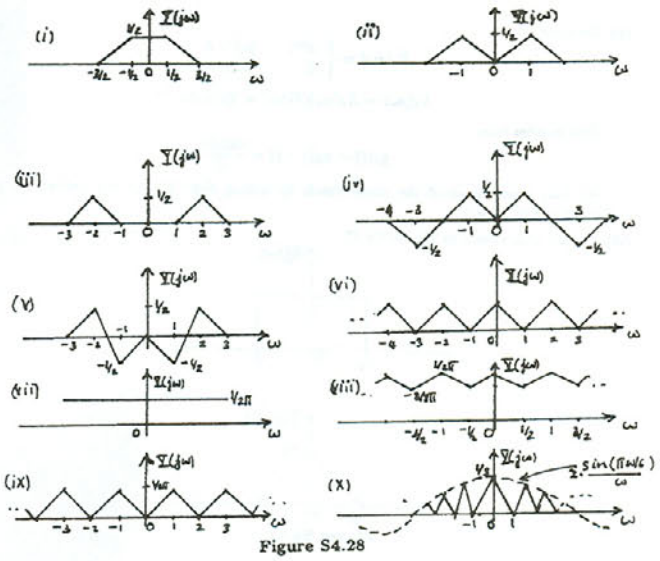

Figure S4.28

4.29. (i) We have

$$X_a(j\omega) = |X(j\omega)|e^{j\sphericalangle X(j\omega) - ja\omega} = X(j\omega)e^{-ja\omega}.$$

From the time shifting property we know that

$$x_a(t) = x(t-a).$$

(ii) We have

$$X_b(j\omega) = |X(j\omega)|e^{j\sphericalangle X(j\omega) + jb\omega} = X(j\omega)e^{jb\omega}.$$

From the time shifting property we know that

$$x_b(t) = x(t+b).$$

(iii) We have

$$X_c(j\omega) = |X(j\omega)|e^{-j\sphericalangle X(j\omega)} = X^*(j\omega).$$

From the conjugation and time reversal properties we know that

$$x_c(t) = x^*(-t).$$

Since $x(t)$ is real, $x_c(t) = x(-t)$.

(iv) We have

$$X_d(j\omega) = |X(j\omega)|e^{-j\sphericalangle X(j\omega) + jd\omega} = X^*(j\omega)e^{jd\omega}.$$

From the conjugation, time reversal, and time shifting properties, we know that

$$x_d(t) = x^*(-t-d).$$

Since $x(t)$ is real, $x_d(t) = x(-t-d)$.

4.30. (a) We know that

$$w(t) = \cos t \overset{FT}{\longleftrightarrow} W(j\omega) = \pi[\delta(\omega - 1) + \delta(\omega + 1)$$

and

$$g(t) = x(t)\cos t \overset{FT}{\longleftrightarrow} G(j\omega) = \frac{1}{2\pi}\{X(j\omega) * W(j\omega)\}.$$

Therefore,

$$G(j\omega) = \frac{1}{2}X(j(\omega - 1)) + \frac{1}{2}X(j(\omega + 1)).$$

Since $G(j\omega)$ is as shown in Figure S4.30, it is clear from the above equation that $X(j\omega)$ is as shown in the Figure S4.30.

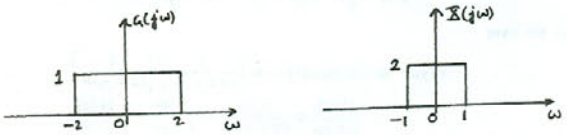

Figure S4.30

Therefore,

$$x(t) = \frac{2\sin t}{\pi t}.$$

(b) $X_1(j\omega)$ is as shown in Figure S4.30.

4.31. (a) We have

$$x(t) = \cos t \overset{FT}{\longleftrightarrow} X(j\omega) = \pi[\delta(\omega+1) + \delta(\omega-1)].$$

(i) We have

$$h_1(t) = u(t) \overset{FT}{\longleftrightarrow} H_1(j\omega) = \frac{1}{j\omega} + \pi\delta(\omega).$$

Therefore,

$$Y(j\omega) = X(j\omega)H_1(j\omega) = \frac{\pi}{j}[\delta(\omega+1) - \delta(\omega-1)].$$

Taking the inverse Fourier transform, we obtain

$$y(t) = \sin(t).$$

(ii) We have

$$h_2(t) = -2\delta(t) + 5e^{-2t}u(t) \overset{FT}{\longleftrightarrow} H_2(j\omega) = -2 + \frac{5}{2+j\omega}.$$

Therefore,

$$Y(j\omega) = X(j\omega)H_1(j\omega) = \frac{\pi}{j}[\delta(\omega+1) - \delta(\omega-1)].$$

Taking the inverse Fourier transform, we obtain

$$y(t) = \sin(t).$$

(iii) We have

$$h_3(t) = 2te^{-t}u(t) \overset{FT}{\longleftrightarrow} H_2(j\omega) = \frac{2}{(1+j\omega)^2}.$$

Therefore,

$$Y(j\omega) = X(j\omega)H_1(j\omega) = \frac{\pi}{j}[\delta(\omega+1) - \delta(\omega-1)].$$

Taking the inverse Fourier transform, we obtain

$$y(t) = \sin(t).$$

(b) An LTI system with impulse response

$$h_4(t) = \frac{1}{2}[h_1(t) + h_2(t)]$$

will have the same response to $x(t) = \cos(t)$. We can find other such impulse responses by suitably scaling and linearly combining $h_1(t)$, $h_2(t)$, and $h_3(t)$.

4.32. Note that $h(t) = h_1(t-1)$, where

$$h_1(t) = \frac{\sin 4t}{\pi t}.$$

The Fourier transform $H_1(j\omega)$ of $h_1(t)$ is as shown in Figure S4.32.

From the above figure it is clear that $h_1(t)$ is the impulse response of an ideal lowpass filter whose passband is in the range $|\omega| < 4$. Therefore, $h(t)$ is the impulse response of an ideal lowpass filter shifted by one to the right. Using the shift property,

$$H(j\omega) = \begin{cases} e^{-j\omega}, & |\omega| < 4 \\ 0, & \text{otherwise} \end{cases}.$$

(a) We have

$$X_1(j\omega) = \pi e^{j\frac{\pi}{12}}\delta(\omega-6) + \pi e^{j\frac{\pi}{12}}\delta(\omega+6).$$

It is clear that

$$Y_1(j\omega) = X_1(j\omega)H(j\omega) = 0 \Rightarrow y_1(t) = 0.$$

This result is equivalent to saying that $X_1(j\omega)$ is zero in the passband of $H(j\omega)$.

(b) We have

$$X_2(j\omega) = \frac{\pi}{j}\left[\sum_{k=0}^{\infty}(\frac{1}{2})^k\{\delta(\omega-3k) - \delta(\omega+3k)\}\right].$$

Therefore,

$$Y_2(j\omega) = X_2(j\omega)H(j\omega) = \frac{\pi}{j}\left[(1/2)\{\delta(\omega-3) - \delta(\omega+3)\}e^{-j\omega}\right].$$

This implies that

$$y_2(t) = \frac{1}{2}\sin(3t-1).$$

We may have obtained the same result by noting that only the sinusoid with frequency 3 in $X_2(j\omega)$ lies in the passband of $H(j\omega)$.

(c) We have

$$X_3(j\omega) = \begin{cases} e^{j\omega}, & |\omega| < 4 \\ 0, & \text{otherwise} \end{cases}.$$

$$Y_3(j\omega) = X_3(j\omega)H(j\omega) = X_3(j\omega)e^{-j\omega}.$$

This implies that

$$y_3(t) = x_3(t-1) = \frac{\sin(4t)}{\pi t}.$$

We may have obtained the same result by noting that $X_3(j\omega)$ lies entirely in the passband of $H(j\omega)$.

(d) $X_4(j\omega)$ is as shown in Figure S4.32.

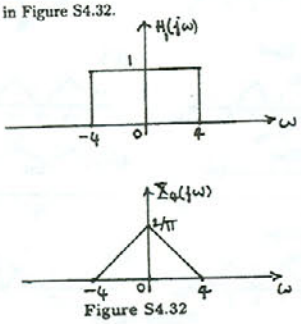

Figure S4.32

Therefore,

$$Y_4(j\omega) = X_4(j\omega)H(j\omega) = X_4(j\omega)e^{-j\omega}.$$

This implies that

$$y_4(t) = x_4(t-1) = \left(\frac{\sin(2(t-1))}{\pi(t-1)}\right)^2.$$

We may have obtained the same result by noting that $X_4(j\omega)$ lies entirely in the passband of $H(j\omega)$.

4.33. (a) Taking the Fourier transform of both sides of the given differential equation, we obtain

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{2}{-\omega^2 + 2j\omega + 8}.$$

Using partial fraction expansion, we obtain

$$H(j\omega) = \frac{1}{j\omega+2} - \frac{1}{j\omega+4}.$$

Taking the inverse Fourier transform,

$$h(t) = e^{-2t}u(t) - e^{-4t}u(t).$$

(b) For the given signal $x(t)$, we have

$$X(j\omega) = \frac{1}{(2+j\omega)^2}.$$

Therefore,

$$Y(j\omega) = X(j\omega)H(j\omega) = \frac{2}{(-\omega^2 + 2j\omega + 8)}\frac{1}{(2+j\omega)^2}.$$

Using partial fraction expansion, we obtain

$$Y(j\omega) = \frac{1/4}{j\omega+2} - \frac{1/2}{(j\omega+2)^2} + \frac{1}{(j\omega+2)^3} - \frac{1/4}{j\omega+4}.$$

Taking the inverse Fourier transform,

$$y(t) = \frac{1}{4}e^{-2t}u(t) - \frac{1}{2}te^{-2t}u(t) + t^2e^{-2t}u(t) - \frac{1}{4}e^{-4t}u(t).$$

(c) Taking the Fourier transform of both sides of the given differential equation, we obtain

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{2(-\omega^2-1)}{-\omega^2 + \sqrt{2}j\omega + 1}.$$

Using partial fraction expansion, we obtain

$$H(j\omega) = 2 + \frac{-\sqrt{2} - 2\sqrt{2}j}{j\omega - \frac{-\sqrt{2}+j\sqrt{2}}{2}} + \frac{-\sqrt{2} + 2\sqrt{2}j}{j\omega - \frac{-\sqrt{2}-j\sqrt{2}}{2}}.$$

Taking the inverse Fourier transform,

$$h(t) = 2\delta(t) - \sqrt{2}(1+2j)e^{-(1+j)t/\sqrt{2}}u(t) - \sqrt{2}(1-2j)e^{-(1-j)t/\sqrt{2}}u(t).$$

4.34. (a) We have

$$\frac{Y(j\omega}{X(j\omega)} = \frac{j\omega+4}{6-\omega^2+5j\omega}.$$

Cross-multiplying and taking the inverse Fourier transform, we obtain

$$\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 6y(t) = \frac{dx(t)}{dt} + 4x(t).$$

(b) We have

$$H(j\omega) = \frac{2}{2+j\omega} - \frac{1}{3+j\omega}.$$

Taking the inverse Fourier transform we obtain,

$$h(t) = 2e^{-2t}u(t) - e^{-3t}u(t).$$

(c) We have

$$X(j\omega) = \frac{1}{4+j\omega} - \frac{1}{(4+j\omega)^2}.$$

Therefore,

$$Y(j\omega) = X(j\omega)H(j\omega) = \frac{1}{(4+j\omega)(2+j\omega)}.$$

Finding the partial fraction expansion of $Y(j\omega)$ and taking the inverse Fourier transform,

$$y(t) = \frac{1}{2}e^{-2t}u(t) - \frac{1}{2}e^{-4t}u(t).$$

4.35. (a) From the given information,

$$|H(j\omega)| = \frac{\sqrt{a^2+\omega^2}}{\sqrt{a^2+\omega^2}} = 1.$$

Also,

$$\sphericalangle H(j\omega) = -\tan^{-1}\frac{\omega}{a} - \tan^{-1}\frac{\omega}{a} = -2\tan^{-1}\frac{\omega}{a}.$$

Also,

$$H(j\omega) = -1 + \frac{2a}{a+j\omega} \quad \Rightarrow \quad h(t) = -\delta(t) + 2ae^{-at}u(t).$$

(b) If $a = 1$, we have

$$|H(j\omega)| = 1, \quad \sphericalangle H(j\omega) = -2\tan^{-1}\omega.$$

Therefore,

$$y(t) = \cos(\frac{t}{\sqrt{3}} - \frac{\pi}{3}) - \cos(t - \frac{\pi}{2}) + \cos(\sqrt{3}t - \frac{2\pi}{3}).$$

4.36. (a) The frequency response is

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{3(3+j\omega)}{(4+j\omega)(2+j\omega)}.$$

(b) Finding the partial fraction expansion of answer in part (a) and taking its inverse Fourier transform, we obtain

$$h(t) = \frac{3}{2}\left[e^{-4t} + e^{-2t}\right]u(t).$$

(c) We have

$$\frac{Y(j\omega)}{X(j\omega)} = \frac{(9+3j\omega)}{8+6j\omega-\omega^2}.$$

Cross-multiplying and taking the inverse Fourier transform, we obtain

$$\frac{d^2y(t)}{dt^2} + 6\frac{dy(t)}{dt} + 8y(t) = 3\frac{dx(t)}{dt} + 9x(t).$$

4.37. (a) Note that

$$x(t) = x_1(t) * x_1(t),$$

where

$$x_1(t) = \begin{cases} 1, & |\omega| < \frac{1}{2} \\ 0, & \text{otherwise} \end{cases}$$

Also, the Fourier transform $X_1(j\omega)$ of $x_1(t)$ is

$$X_1(j\omega) = 2\frac{\sin(\omega/2)}{\omega}.$$

Using the convolution property we have

$$X(j\omega) = X_1(j\omega)X_1(j\omega) = \left[2\frac{\sin(\omega/2)}{\omega}\right]^2.$$

(b) The signal $\tilde{x}(t)$ is as shown in Figure S4.37

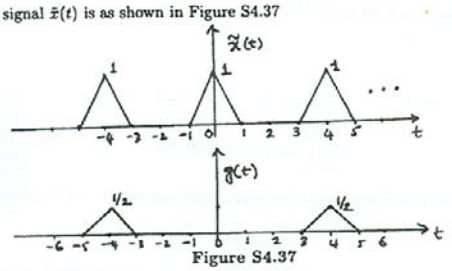

Figure S4.37

(c) One possible choice of $g(t)$ is as shown in Figure S4.37.

(d) Note that

$$\tilde{X}(j\omega) = X(j\omega)\frac{\pi}{2}\sum_{k=-\infty}^{\infty}\delta(j(\omega - k\frac{\pi}{2})) = G(j\omega)\frac{\pi}{2}\sum_{k=-\infty}^{\infty}\delta(j(\omega - k\frac{\pi}{2})).$$

This may also be written as

$$\tilde{X}(j\omega) = \frac{\pi}{2}\sum_{k=-\infty}^{\infty}X(j\pi k/2)\delta(j(\omega - k\frac{\pi}{2})) = \frac{\pi}{2}\sum_{k=-\infty}^{\infty}G(j\pi k/2)\delta(j(\omega - k\frac{\pi}{2})).$$

Clearly, this is possible only if

$$G(j\pi k/2) = X(j\pi k/2).$$

4.38. (a) Applying a frequency shift to the analysis equation, we have

$$X(j(\omega-\omega_0)) = \int_{-\infty}^{\infty}x(t)e^{-j(\omega-\omega_0)t}dt = \int_{-\infty}^{\infty}x(t)e^{j\omega_0 t}e^{-j\omega t}dt = \mathcal{FT}\{x(t)e^{j\omega_0 t}\}.$$

(b) We have

$$w(t) = e^{j\omega_0 t} \overset{FT}{\longleftrightarrow} W(j\omega) = 2\pi\delta(\omega-\omega_0).$$

Also,

$$\begin{aligned} x(t)w(t) \overset{FT}{\longleftrightarrow} & \frac{1}{2\pi}\left[X(j\omega) * W(j\omega)\right] \\ = & X(j\omega) * \delta(\omega-\omega_0) \\ = & X(j(\omega-\omega_0)) \end{aligned}$$

4.39. (a) From the Fourier transform analysis equation, we have

$$G(j\omega) = \int_{-\infty}^{\infty}g(t)e^{-j\omega t}dt = \int_{-\infty}^{\infty}X(jt)e^{-j\omega t}dt. \qquad \text{(S4.39-1)}$$

Also from the Fourier transform synthesis equation, we have

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}X(j\omega)e^{j\omega t}d\omega.$$

Switching the variables t and ω, we have

$$x(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty}X(jt)e^{j\omega t}dt.$$

We may also write this equation as

$$2\pi x(-\omega) = \int_{-\infty}^{\infty}X(jt)e^{-j\omega t}dt.$$

Substituting this equation in eq. (S4.39-1), we obtain

$$G(j\omega) = 2\pi x(-\omega).$$

(b) If in part (a) we have $x(t) = \delta(t+B)$, then we would have $g(t) = X(jt) = e^{jBt}$ and $G(j\omega) = 2\pi x(-\omega) = 2\pi\delta(-\omega+B) = 2\pi\delta(\omega-B)$.

4.40. When $n = 1$, $x_1(t) = e^{-at}u(t)$ and $X_1(j\omega) = 1/(a+j\omega)$.
When $n = 2$, $x_2(t) = te^{-at}u(t)$ and $X_2(j\omega) = 1/(a+j\omega)^2$.

Now, let us assume that the given statement is true when $n = m$, that is,

$$x_m(t) = \frac{t^{m-1}}{(m-1)!}e^{-at}u(t) \overset{FT}{\longleftrightarrow} X_m(j\omega) = \frac{1}{(1+j\omega)^m}.$$

For $n = m+1$ we may use the differentiation in frequency property to write,

$$x_{m+1}(t) = \frac{t}{m}x_m(t) \overset{FT}{\longleftrightarrow} X_{m+1}(j\omega) = \frac{1}{m}j\frac{dX_m(j\omega)}{d\omega} = \frac{1}{(1+j\omega)^{m+1}}.$$

This shows that if we assume that the given statement is true for $n = m$, then it is true for $n = m+1$. Since we also shown that the given statement is true for $n = 2$, we may argue that it is true for $n = 2+1 = 3$, $n = 3+1 = 4$, and so on. Therefore, the given statement is true for any n.

4.41. (a) We have

$$\begin{aligned} g(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{2\pi}\left[X(j\omega) * Y(j\omega)\right]e^{j\omega t}d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{2\pi}\left[\int_{-\infty}^{\infty}X(j\theta)Y(j(\omega-\theta))d\theta\right]e^{j\omega t}d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}X(j\theta)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}Y(j(\omega-\theta))e^{j\omega t}d\omega\right]d\theta \end{aligned}$$

(b) Using the frequency shift property of the Fourier transform we have

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}Y(j(\omega-\theta))e^{j\omega t}d\omega = e^{j\theta t}y(t).$$

(c) Combining the results of parts (a) and (b),

$$\begin{aligned} g(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}X(j\theta)e^{j\theta t}y(t)d\theta \\ &= y(t)\frac{1}{2\pi}\int_{-\infty}^{\infty}X(j\theta)e^{j\theta t}d\theta \\ &= y(t)x(t). \end{aligned}$$

4.42. $x(t)$ is a periodic signal with Fourier series coefficients a_k. The fundamental frequency of $x(t)$ is $\omega_f = 100$ rad/sec. From Section 4.2 we know that the Fourier transform $X(j\omega)$ of $x(t)$ is

$$X(j\omega) = \sum_{k=-\infty}^{\infty}2\pi a_k\delta(\omega-100k).$$

(a) Since

$$y_1(t) = x(t)\cos(\omega_0 t) \overset{FT}{\longleftrightarrow} Y_1(j\omega) = \frac{1}{2}\{X(j(\omega-\omega_0)) + X(j(\omega+\omega_0))\}$$

we have

$$\begin{aligned} Y_1(j\omega) &= \pi\sum_{k=-\infty}^{\infty}\left[a_k\delta(\omega-100k-\omega_0) + a_k\delta(\omega-100k+\omega_0)\right] \\ &= \pi\sum_{k=-\infty}^{\infty}\left[a_{-k}\delta(\omega+100k-\omega_0) + a_k\delta(\omega-100k+\omega_0)\right] \qquad \text{(S4.42-1)} \end{aligned}$$

If $\omega_0 = 500$, then the term in the above summation with $k = 5$ becomes

$$\pi a_{-5}\delta(\omega) + \pi a_5\delta(\omega).$$

Since $x(t)$ is real, $a_k = a^*_{-k}$. Therefore, the above expression becomes $2\pi\mathcal{R}e\{a_5\}\delta(\omega)$, which is an impulse at $\omega = 0$. Note that the inverse Fourier transform of $2\pi\mathcal{R}e\{a_5\}\delta(\omega)$ is $g_1(t) = \mathcal{R}e\{a_5\}$. Therefore, we now need to find a $H(j\omega)$ such that

$$Y_1(j\omega)H(j\omega) = G_1(j\omega) = 2\pi\mathcal{R}e\{a_5\}\delta(\omega).$$

We may easily obtain such a $H(j\omega)$ by noting that the other terms (other than that for $k = 5$) in the summation of eq. (S4.42-1) result in impulses at $\omega = 100m$, $m \neq 0$. Therefore, we my choose any $H(j\omega)$ which is zero for $\omega = 100m$, where $m = \pm 1, \pm 2, \cdots$.

Similarly since

$$y_2(t) = x(t)\sin(\omega_0 t) \overset{FT}{\longleftrightarrow} Y_2(j\omega) = \frac{1}{2j}\{X(j(\omega - \omega_0)) - X(j(\omega + \omega_0))\},$$

we have

$$\begin{aligned} Y_2(j\omega) &= \frac{\pi}{j}\sum_{k=-\infty}^{\infty}[a_k\delta(\omega - 100k - \omega_0) - a_k\delta(\omega - 100k + \omega_0)] \\ &= \frac{\pi}{j}\sum_{k=-\infty}^{\infty}[a_{-k}\delta(\omega + 100k - \omega_0) - a_k\delta(\omega - 100k + \omega_0)] \quad \text{(S4.42-2)} \end{aligned}$$

If $\omega_0 = 500$, then the term in the above summation with $k = 5$ becomes

$$\frac{\pi}{j}a_{-5}\delta(\omega) - \frac{\pi}{j}a_5\delta(\omega).$$

Since $x(t)$ is real, $a_k = a^*_{-k}$. Therefore, the above expression becomes $2\pi\mathcal{I}m\{a_5\}\delta(\omega)$, which is an impulse at $\omega = 0$. Note that the inverse Fourier transform of $2\pi\mathcal{I}m\{a_5\}\delta(\omega)$ is $g_2(t) = \mathcal{I}m\{a_5\}$. Therefore, we now need to find a $H(j\omega)$ such that

$$Y_2(j\omega)H(j\omega) = G_2(j\omega) = 2\pi\mathcal{R}e\{a_5\}\delta(\omega).$$

We may easily obtain such a $H(j\omega)$ by noting that the other terms (other than that for $k = 5$) in the summation of eq. (S4.42-2) result in impulses at $\omega = 100m$, $m \neq 0$. Therefore, we my choose any $H(j\omega)$ which is zero for $\omega = 100m$, where $m = \pm 1, \pm 2, \cdots$.

(b) An example of a valid $H(j\omega)$ would be the frequency response of an ideal lowpass filter with passband gain of unity and cutoff frequency of 50 rad/sec. In this case,

$$h(t) = \frac{\sin(50t)}{\pi t}.$$

4.43. Since

$$y_1(t) = \cos^2 t = \frac{1 + \cos(2t)}{2},$$

we obtain

$$Y_1(j\omega) = \pi\delta(\omega) + \frac{\pi}{2}\delta(\omega - 2) + \frac{\pi}{2}\delta(\omega + 2).$$

Therefore,

$$y_2(t) = x(t)y_1(t) = x(t)\cos^2(t) \overset{FT}{\longleftrightarrow} Y_2(j\omega) = \frac{1}{2\pi}\{X(j\omega) * Y_1(j\omega)\}.$$

This gives

$$Y_2(j\omega) = \frac{1}{2}X(j\omega) + \frac{1}{4}X(j(\omega - 2)) + \frac{1}{4}X(j(\omega + 2)).$$

$X(j\omega)$ and $Y_2(j\omega)$ are as shown in Figure S4.43.

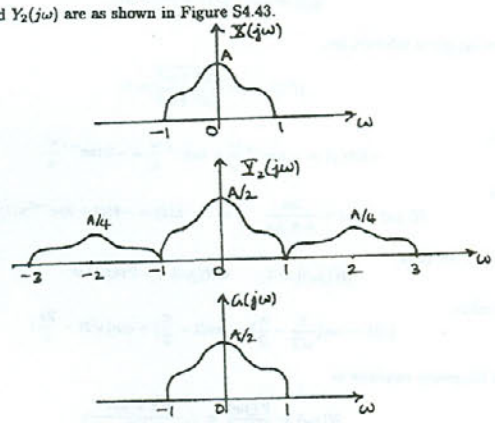

Figure S4.43

Now,

$$y_3(t) = \frac{\sin t}{\pi t} \overset{FT}{\longleftrightarrow} Y_3(j\omega) = \begin{cases} 1, & |\omega| < 1 \\ 0, & \text{otherwise} \end{cases}.$$

Also,

$$g(t) = y_2(t) * y_3(t) \overset{FT}{\longleftrightarrow} G(j\omega) = Y_2(j\omega)Y_3(j\omega).$$

From Figure S4.43 it is clear that

$$G(j\omega) = \frac{1}{2}X(j\omega).$$

Therefore, an LTI system with impulse response $h(t) = \frac{1}{2}\delta(t)$ may be used to obtain $g(t)$ from $x(t)$.

4.44. (a) Taking the Fourier transform of both sides of the given differential equation, we have

$$Y(j\omega)[10 + j\omega] = X(j\omega)[Z(j\omega) - 1].$$

Since, $Z(j\omega) = \frac{1}{1+j\omega} + 3$, we obtain from the above equation

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{3 + 2j\omega}{(1 + j\omega)(10 + j\omega)}.$$

(b) Finding the partial fraction expansion of $H(j\omega)$ and then taking its inverse Fourier transform we obtain

$$h(t) = \frac{1}{9}e^{-t}u(t) + \frac{17}{9}e^{-10t}u(t).$$

4.45. We have

$$y(t) = x(t) * h(t) \quad \Rightarrow \quad Y(j\omega) = X(j\omega)H(j\omega).$$

From Parseval's relation the total energy in $y(t)$ is

$$\begin{aligned} E &= \int_{-\infty}^{\infty}|y(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}|Y(j\omega)|^2 d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}|X(j\omega)|^2|H(j\omega)|^2 d\omega \\ &= \frac{1}{2\pi}\int_{-\omega_0-\Delta/2}^{-\omega_0+\Delta/2}|X(j\omega)|^2 d\omega + \frac{1}{2\pi}\int_{\omega_0-\Delta/2}^{\omega_0+\Delta/2}|X(j\omega)|^2 d\omega \\ &\approx \frac{1}{2\pi}|X(-j\omega_0)|^2\Delta + \frac{1}{2\pi}|X(j\omega_0)|^2\Delta \end{aligned}$$

For real $x(t)$, $|X(-j\omega_0)|^2 = |X(j\omega_0)|^2$. Therefore,

$$E = \frac{1}{\pi}|X(j\omega_0)|^2\Delta.$$

4.46. Let $g_1(t)$ be the response of $H_1(j\omega)$ to $x(t)\cos\omega_c t$. Let $g_2(t)$ be the response of $H_2(j\omega)$ to $x(t)\sin\omega_c t$. Then, with reference to Figure 4.30,

$$y(t) = x(t)e^{j\omega_c t} = x(t)\cos\omega_c t + jx(t)\sin\omega_c t,$$

and

$$w(t) = g_1(t) + jg_2(t).$$

Also,

$$f(t) = e^{-j\omega_c t}w(t) = [\cos\omega_c t - j\sin\omega_c t][g_1(t) + jg_2(t)].$$

Therefore,

$$\mathcal{R}e\{f(t)\} = g_1(t)\cos\omega_c t + g_2(t)\sin\omega_c t.$$

This is exactly what Figure P4.46 implements.

4.47. (a) We have

$$h_e(t) = \frac{h(t) + h(-t)}{2}.$$

Since $h(t)$ is causal, the non-zero portions of $h(t)$ and $h(-t)$ overlap *only* at $t = 0$. Therefore,

$$h(t) = \begin{cases} 0, & t < 0 \\ h_e(t), & t = 0 \\ 2h_e(t), & t > 0 \end{cases}. \quad \text{(S4.47-1)}$$

Also, from Table 4.1 we have

$$h_e(t) \overset{FT}{\longleftrightarrow} \mathcal{R}e\{H(j\omega)\}.$$

Given $\mathcal{R}e\{H(j\omega)$, we can obtain $h_e(t)$. From $h_e(t)$, we can recover $h(t)$ (and consequently $H(j\omega)$) by using eq. (S4.47-1). Therefore, $H(j\omega)$ is completely specified by $\mathcal{R}e\{H(j\omega)$.

(b) If

$$\mathcal{R}e\{H(j\omega)\} = \cos t = \frac{1}{2}e^{j\omega t} + \frac{1}{2}e^{-j\omega t}$$

then,

$$h_e(t) = \frac{1}{2}\delta(t + 1) + \frac{1}{2}\delta(t - 1).$$

Therefore from eq. (S4.47-1),

$$h(t) = \delta(t - 1).$$

(c) We have

$$h_o(t) = \frac{h(t) + h(-t)}{2}.$$

Since $h(t)$ is causal, the non-zero portions of $h(t)$ and $h(-t)$ overlap *only* at $t = 0$ and $h_o(t)$ will be zero at $t = 0$. Therefore,

$$h(t) = \begin{cases} 0, & t < 0 \\ \text{unknown}, & t = 0 \\ 2h_o(t), & t > 0 \end{cases}. \quad \text{(S4.47-2)}$$

Also, from Table 4.1 we have

$$h_o(t) \overset{FT}{\longleftrightarrow} \mathcal{I}m\{H(j\omega)\}.$$

Given $\mathcal{I}m\{H(j\omega)$, we can obtain $h_o(t)$. From $h_o(t)$, we can recover $h(t)$ except for $t = 0$ by using eq. (S4.47-1). If there are no singularities in $h(t)$ at $t = 0$, then $H(j\omega)$ can be recovered from $h(t)$ even if $h(0)$ is unknown. Therefore $H(j\omega)$ is completely specified by $\mathcal{I}m\{H(j\omega)$ in this case.

4.48. (a) Using the multiplication property we have

$$h(t) = h(t)u(t) \overset{FT}{\longleftrightarrow} H(j\omega) = \frac{1}{2\pi}\left\{H(j\omega) * \left[\frac{1}{j\omega} + \pi\delta(\omega)\right]\right\}.$$

The right-hand side may be written as

$$H(j\omega) = \frac{1}{2}H(j\omega) + \frac{1}{2\pi j}\left[H(j\omega) * \frac{1}{\omega}\right].$$

That is,

$$H(j\omega) = \frac{1}{\pi j}\int_{-\infty}^{\infty}\frac{H(j\eta)}{\omega - \eta}d\eta.$$

Breaking up $H(j\omega)$ into real and imaginary parts,

$$H_R(j\omega) + jH_I(j\omega) = \frac{1}{\pi j}\int_{-\infty}^{\infty}\frac{H_R(j\eta) + jH_I(j\eta)}{\omega - \eta}d\eta = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{H_I(j\eta) - jH_R(j\eta)}{\omega - \eta}d\eta.$$

Comparing real and imaginary parts on both sides, we obtain

$$H_R(j\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{H_I(j\eta)}{\omega - \eta}d\eta \quad \text{and} \quad H_I(j\omega) = -\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{H_R(j\eta)}{\omega - \eta}d\eta.$$

(b) From eq. (P4.48-3), we may write

$$y(t) = x(t) * \frac{1}{\pi t} \quad \Rightarrow \quad Y(j\omega) = X(j\omega)\mathcal{FT}\{1/(\pi t)\} \qquad \text{(S4.48-1)}$$

Also, from Table 4.2

$$u(t) \overset{FT}{\longleftrightarrow} \frac{1}{j\omega} + \pi\delta(\omega).$$

Therefore,

$$2u(t) - 1 \overset{FT}{\longleftrightarrow} 2\frac{1}{j\omega}.$$

Using the duality property, we have

$$\frac{2}{jt} \overset{FT}{\longleftrightarrow} 2\pi[2u(-\omega) - 1]$$

or

$$\frac{1}{\pi t} \overset{FT}{\longleftrightarrow} j[2u(-\omega) - 1].$$

Therefore, from eq.(S4.48-1), we have

$$Y(j\omega) = X(j\omega)H(j\omega)$$

where

$$H(j\omega) = j[2u(-\omega) - 1] = \begin{cases} -j, & \omega > 0 \\ j, & \omega > 0 \end{cases}.$$

(c) Let $y(t)$ be the Hilbert transform of $x(t) = \cos(3t)$. Then,

$$Y(j\omega) = X(j\omega)H(j\omega) = \pi[\delta(\omega - 3) + \delta(\omega + 3)]H(j\omega) = -j\pi\delta(\omega - 3) + j\pi\delta(\omega + 3)$$

Therefore,

$$y(t) = \sin(3t).$$

4.49. (a) (i) Since $H(j\omega)$ is real and even, $h(t)$ is also real and even.
(ii)

$$|h(t)| = \left|\frac{1}{2\pi}\int_{-\infty}^{\infty}H(j\omega)e^{j\omega t}d\omega\right| \le \frac{1}{2\pi}\int_{-\infty}^{\infty}|H(j\omega)|e^{j\omega t}d\omega.$$

Since $H(j\omega)$ is real and positive,

$$|h(t)| \le \frac{1}{2\pi}\int_{-\infty}^{\infty}H(j\omega)e^{j\omega t}d\omega = h(0).$$

Therefore,

$$\max[|h(t)|] = h[0].$$

(b) The bandwidth of this system is $2W$.

(c) We have

$$B_w H(j0) = \text{Area under } H(j\omega).$$

Therefore,

$$B_w = \frac{1}{H(j0)}\int_{-\infty}^{\infty}H(j\omega)d\omega.$$

(d) We have

$$t_r = \frac{s(\infty)}{h(0)} = \frac{\int_{-\infty}^{\infty}h(t)dt}{\frac{1}{2\pi}\int_{-\infty}^{\infty}H(j\omega)d\omega} = \frac{H(j0)}{\frac{1}{2\pi}\int_{-\infty}^{\infty}H(j\omega)d\omega} = \frac{2\pi}{B_w}.$$

(e) Therefore,

$$B_w t_r = B_w\frac{2\pi}{B_w} = 2\pi.$$

4.50. (a) We know from problems 1.45 and 2.67 that

$$\phi_{xy}(t) = \phi_{yx}(-t).$$

Therefore,

$$\Phi_{xy}(j\omega) = \Phi_{yx}(-j\omega).$$

Since $\phi_{yx}(t)$ is real,

$$\Phi_{xy}(j\omega) = \Phi_{yx}^*(j\omega).$$

(b) We may write

$$\phi_{xy}(t) = \int_{-\infty}^{\infty}x(t+\tau)y(\tau)d\tau = x(t) * y(-t).$$

Therefore,

$$\Phi_{xy}(j\omega) = X(j\omega)Y(-j\omega).$$

Since $y(t)$ is real, we may write this as

$$\Phi_{xy}(j\omega) = X(j\omega)Y^*(j\omega).$$

(c) Using the results of part (b) with $y(t) = x(t)$,

$$\Phi_{xx}(j\omega) = X(j\omega)X^*(j\omega) = |X(j\omega)|^2 \ge 0.$$

(d) From part (b) we have

$$\begin{aligned}\Phi_{xy}(j\omega) &= X(j\omega)Y^*(j\omega) \\ &= X(j\omega)[H(j\omega)X(j\omega)]^* \\ &= \Phi_{xx}(j\omega)H^*(j\omega)\end{aligned}$$

Also,

$$\begin{aligned}\Phi_{yy}(j\omega) &= Y(j\omega)Y^*(j\omega) \\ &= [H(j\omega)X(j\omega)][H(j\omega)X(j\omega)]^* \\ &= \Phi_{xx}(j\omega)|H(j\omega)|^2\end{aligned}$$

(e) From the given information, we have

$$X(j\omega) = \frac{e^{-j\omega} - 1}{\omega^2} - j\frac{e^{-j\omega}}{\omega}$$

and

$$H(j\omega) = \frac{1}{a + j\omega}.$$

Therefore,

$$\Phi_{xx}(j\omega) = |X(j\omega)|^2 = \frac{2 - 2\cos\omega}{\omega^4} - \frac{2\sin\omega}{\omega^2} + \frac{1}{\omega^2},$$

$$\Phi_{xy}(j\omega) = \Phi_{xx}(j\omega)H^*(j\omega) = \left[\frac{2 - 2\cos\omega}{\omega^4} - \frac{2\sin\omega}{\omega^2} + \frac{1}{\omega^2}\right]\left[\frac{1}{a - j\omega}\right],$$

and

$$\Phi_{yy}(j\omega) = \Phi_{xx}(j\omega)|H(j\omega)|^2 = \left[\frac{2 - 2\cos\omega}{\omega^4} - \frac{2\sin\omega}{\omega^2} + \frac{1}{\omega^2}\right]\left[\frac{1}{a^2 + \omega^2}\right].$$

(f) We require that

$$|H(j\omega)|^2 = \frac{\omega^2 + 100}{\omega^2 + 25}.$$

The possible causal and stable choices for $H(j\omega)$ are

$$H_1(j\omega) = \frac{10 + j\omega}{5 + j\omega} \quad \text{and} \quad H_2(j\omega) = \frac{10 - j\omega}{5 + j\omega}.$$

The corresponding impulse responses are

$$h_1(t) = \delta(t) + 5e^{-5t}u(t) \quad \text{and} \quad h_2(t) = -\delta(t) + 15e^{-5t}u(t).$$

Only the system with impulse response $h_1(t)$ has a causal and stable inverse.

4.51. (a) $H(j\omega) = 1/G(j\omega)$.

(b) (i) If we denote the output by $y(t)$, then we have

$$Y(j0) = \frac{1}{2}.$$

Since $H(j0) = 0$, it is impossible for us to have $Y(j0) = X(j0)H(j0)$. Therefore, we cannot find an $x(t)$ which produces an output which looks like Figure P4.50.

(ii) This system is not invertible because $1/H(j\omega)$ is not defined for all ω.

(c) We have

$$H(j\omega) = \sum_{k=0}^{\infty}e^{-kT}e^{-j\omega kT} = \frac{1}{1 - e^{-(1+j\omega)T}}.$$

We now need to find a $G(j\omega)$ such that

$$H(j\omega)G(j\omega) = 1.$$

Thus $G(j\omega)$ is the inverse system of $H(j\omega)$, and is given by

$$G(j\omega) = 1 - e^{-(1+j\omega)T}.$$

(d) Since $H(j\omega) = 2 + j\omega$,

$$G(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{1}{2 + j\omega}.$$

Cross-multiplying and taking the inverse Fourier transform, we obtain

$$\frac{dy(t)}{dt} + 2y(t) = x(t).$$

(e) We have

$$H(j\omega) = \frac{-\omega^2 + 3j\omega + 2}{-\omega^2 + 6j\omega + 9}.$$

Therefore, the frequency response of the inverse is

$$G(j\omega) = \frac{1}{H(j\omega)} = \frac{-\omega^2 + 6j\omega + 9}{-\omega^2 + 3j\omega + 2}.$$

The differential equation describing the inverse system is

$$\frac{d^2y(t)}{dt} + 3\frac{dy(t)}{dt} + 2y(t) = \frac{d^2x(t)}{dt} + 6\frac{dx(t)}{dt} + 9x(t).$$

Using partial fraction expansion followed by application of the inverse Fourier transform, we find the impulse responses to be

$$h(t) = \delta(t) - 3e^{-3t}u(t) + 2te^{-3t}u(t)$$

and

$$g(t) = \delta(t) - e^{-2t}u(t) + 4e^{-t}u(t).$$

4.52. (a) Since the step response is $s(t) = (1 - e^{-t/2})u(t)$, the impulse response has to be

$$h(t) = \frac{1}{2}e^{-t/2}u(t).$$

The frequency response of the system is

$$H(j\omega) = \frac{1/2}{\frac{1}{2} + j\omega}.$$

We now desire to build an inverse for the above system. Therefore, the frequency response of the inverse system has to be

$$G(j\omega) = \frac{1}{H(j\omega)} = 2\left[\frac{1}{2} + j\omega\right].$$

Taking the inverse Fourier transform we obtain

$$g(t) = \delta(t) + 2tu_1(t).$$

(b) When $\sin(\omega t)$ passes through the inverse system, the output will be

$$y(t) = \sin(\omega t) + 2\omega\cos(\omega t).$$

We see that the output is directly proportional to ω. Therefore, as ω increases, the contribution to the output due to the noise also increases.

(c) In this case we require that $|H(j\omega)| \le \frac{1}{4}$ when $\omega = 6$. Since

$$|H(j\omega)|^2 = \frac{1}{a^2 + \omega^2},$$

we require that

$$\frac{1}{a^2 + 36} \le \frac{1}{16}.$$

Therefore, $a \le \frac{6}{\sqrt{15}}$.

4.53. (a) From the given definition we obtain

$$\begin{aligned} X(j\omega_1, j\omega_2) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(t_1, t_2)e^{-j(\omega_1 t_1 + \omega_2 t_2)}dt_1 dt_2 \\ &= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} x(t_1, t_2)e^{-j\omega_1 t_1}dt_1\right]e^{-j\omega_2 t_2}dt_2 \\ &= \int_{-\infty}^{\infty} X(\omega_1, t_2)e^{-j\omega_2 t_2}dt_2 \end{aligned}$$

(b) From the result of part (a) we may write

$$x(t_1, t_2) = \mathcal{FT}_{\omega_1}^{-1}\{\mathcal{FT}_{\omega_2}^{-1}\{X(j\omega_1, j\omega_2)\}\} = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X(j\omega_1, j\omega_2)e^{j(\omega_1 t_1 + \omega_2 t_2)}d\omega_1 d\omega_2$$

(c) (i) $X(j\omega_1, \omega_2) = \frac{e^{-(1+j\omega_1)}}{(1+j\omega_1)}\frac{e^{2(2-j\omega_2)}}{(2-j\omega_2)}$

(ii) $X(j\omega_1, \omega_2) = \frac{[1-e^{-(1+j\omega_1)}][1-e^{-(1-j\omega_2)}]}{(1+j\omega_1)(1-j\omega_2)} + \frac{[1-e^{-(1+j\omega_1)}][1-e^{-(1+j\omega_2)}]}{(1+j\omega_1)(1+j\omega_2)}$

(iii) $X(j\omega_1, \omega_2) = \frac{2-e^{-(1+j\omega_1)}-e^{-(1+j\omega_2)}-[1-e^{-(1+j\omega_1)}][1-e^{-(1+j\omega_2)}]}{(1+j\omega_1)(1+j\omega_2)} + \frac{1-e^{-(1+j\omega_1)}}{(1+j\omega_1)(1-j\omega_2)} + \frac{1-e^{-(1+j\omega_2)}}{(1-j\omega_1)(1+j\omega_2)}$

(iv) $X(\omega_1, \omega_2) = -\frac{1}{j\omega_2}\left[\frac{e^{-j\omega_2}(1-e^{j(\omega_1+\omega_2)})+e^{j\omega_2}(1-e^{-j(\omega_1+\omega_2)})}{-j(\omega_1+\omega_2)} + \frac{e^{j\omega_2}(1-e^{j(\omega_1-\omega_2)})+e^{-j\omega_2}(e^{-j(\omega_1-\omega_2)}-1)}{-j(\omega_1-\omega_2)}\right]$

(v) As shown in the Figure S4.53, this signal has six different regions in the (t_1, t_2) plane.

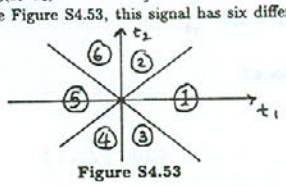

Figure S4.53

The signal $x(t_1, t_2)$ is given by

$$x(t_1, t_2) = \begin{cases} e^{-2t_1}, & \text{in region 1} \\ e^{-2t_2}, & \text{in region 2} \\ e^{2t_2}, & \text{in region 3} \\ e^{2t_2}, & \text{in region 4} \\ e^{2t_1}, & \text{in region 5} \\ e^{-2t_2}, & \text{in region 6} \end{cases}$$

Therefore,

$$\begin{aligned} X(\omega_1, \omega_2) &= \frac{1}{(2 + j\omega_1 + j\omega_2)(2 + j\omega_1 - j\omega_2)} + \frac{1}{(2 + j\omega_2)(2 + j\omega_1 + j\omega_2)} \\ &+ \frac{1}{(2 - j\omega_2)(2 + j\omega_1 - j\omega_2)} + \frac{1}{(2 - j\omega_2)(2 - j\omega_1 - j\omega_2)} \\ &- \frac{2}{(2 - j\omega_1 - j\omega_2)(2 - j\omega_1 + j\omega_2)} + \frac{1}{(j\omega_2)(2 - j\omega_1 - j\omega_2)} \end{aligned}$$

(d) $x(t_1, t_2) = e^{-4(t_1+2t_2)}u(t_1 + 2t_2)$

(e) (i) $e^{-j\omega_1 T_1}e^{-j\omega_2 T_2}X(j\omega_1, j\omega_2)$

(ii) $\frac{1}{|ab|}X(j\frac{\omega_1}{a}, j\frac{\omega_2}{b})$

(iii) $X(j\omega_1, j\omega_2)H(j\omega_1, j\omega_2)$

Chapter 5 Answers

5.1. (a) Let $x[n] = (1/2)^{n-1}u[n-1]$. Using the Fourier transform analysis equation (5.9), the Fourier transform $X(e^{j\omega})$ of this signal is

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \\ &= \sum_{n=1}^{\infty}(1/2)^{n-1}e^{-j\omega n} \\ &= \sum_{n=0}^{\infty}(1/2)^{n}e^{-j\omega(n+1)} \\ &= e^{-j\omega}\frac{1}{(1 - (1/2)e^{-j\omega})} \end{aligned}$$

(b) Let $x[n] = (1/2)^{|n-1|}$. Using the Fourier transform analysis equation (5.9), the Fourier transform $X(e^{j\omega})$ of this signal is

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \\ &= \sum_{n=-\infty}^{0}(1/2)^{-(n-1)}e^{-j\omega n} + \sum_{n=1}^{\infty}(1/2)^{n-1}e^{-j\omega n} \end{aligned}$$

The second summation in the right-hand side of the above equation is exactly the same as the result of part (a). Now,

$$\sum_{n=-\infty}^{0}(1/2)^{-(n-1)}e^{-j\omega n} = \sum_{n=0}^{\infty}(1/2)^{(n+1)}e^{j\omega n} = \left(\frac{1}{2}\right)\frac{1}{1 - (1/2)e^{j\omega}}.$$

Therefore,

$$X(e^{j\omega}) = \left(\frac{1}{2}\right)\frac{1}{1 - (1/2)e^{j\omega}} + e^{-j\omega}\frac{1}{(1 - (1/2)e^{-j\omega})} = \frac{0.75e^{-j\omega}}{1.25 - \cos\omega}.$$

5.2. (a) Let $x[n] = \delta[n-1] + \delta[n+1]$. Using the Fourier transform analysis equation (5.9), the Fourier transform $X(e^{j\omega})$ of this signal is

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \\ &= e^{-j\omega} + e^{j\omega} = 2\cos\omega \end{aligned}$$