Therefore, the frequency response of the inverse is

$$G(j\omega) = \frac{1}{H(j\omega)} = \frac{-\omega^2 + 6j\omega + 9}{-\omega^2 + 3j\omega + 2}.$$

The differential equation describing the inverse system is

$$\frac{d^2y(t)}{dt} + 3\frac{dy(t)}{dt} + 2y(t) = \frac{d^2x(t)}{dt} + 6\frac{dx(t)}{dt} + 9x(t).$$

Using partial fraction expansion followed by application of the inverse Fourier transform, we find the impulse responses to be

$$h(t) = \delta(t) - 3e^{-3t}u(t) + 2te^{-3t}u(t)$$

and

$$g(t) = \delta(t) - e^{-2t}u(t) + 4e^{-t}u(t).$$

4.52. (a) Since the step response is $s(t) = (1 - e^{-t/2})u(t)$, the impulse response has to be

$$h(t) = \frac{1}{2}e^{-t/2}u(t).$$

The frequency response of the system is

$$H(j\omega) = \frac{1/2}{\frac{1}{2} + j\omega}.$$

We now desire to build an inverse for the above system. Therefore, the frequency response of the inverse system has to be

$$G(j\omega) = \frac{1}{H(j\omega)} = 2\left[\frac{1}{2} + j\omega\right].$$

Taking the inverse Fourier transform we obtain

$$g(t) = \delta(t) + 2u_1(t).$$

(b) When $\sin(\omega t)$ passes through the inverse system, the output will be

$$y(t) = \sin(\omega t) + 2\omega\cos(\omega t).$$

We see that the output is directly proportional to ω. Therefore, as ω increases, the contribution to the output due to the noise also increases.

(c) In this case we require that $|H(j\omega)| \le \frac{1}{4}$ when $\omega = 6$. Since

$$|H(j\omega)|^2 = \frac{1}{a^2 + \omega^2},$$

we require that

$$\frac{1}{a^2 + 36} \le \frac{1}{16}.$$

Therefore, $a \le \frac{6}{\sqrt{15}}$.

4.53. (a) From the given definition we obtain

$$\begin{aligned} X(j\omega_1, j\omega_2) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(t_1, t_2)e^{-j(\omega_1 t_1 + \omega_2 t_2)}dt_1 dt_2 \\ &= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} x(t_1, t_2)e^{-j\omega_1 t_1}dt_1\right]e^{-j\omega_2 t_2}dt_2 \\ &= \int_{-\infty}^{\infty} X(\omega_1, t_2)e^{-j\omega_2 t_2}dt_2 \end{aligned}$$

(b) From the result of part (a) we may write

$$x(t_1, t_2) = \mathcal{FT}_{\omega_1}^{-1}\{\mathcal{FT}_{\omega_2}^{-1}\{X(j\omega_1, j\omega_2)\}\} = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X(j\omega_1, j\omega_2)e^{j(\omega_1 t_1 + \omega_2 t_2)}d\omega_1 d\omega_2$$

(c) (i) $X(j\omega_1, \omega_2) = \frac{e^{-(1+j\omega_1)}}{(1+j\omega_1)}\frac{e^{2(2-j\omega_2)}}{(2-j\omega_2)}$

(ii) $X(j\omega_1, \omega_2) = \frac{[1-e^{-(1+j\omega_1)}][1-e^{-(1-j\omega_2)}]}{(1+j\omega_1)(1-j\omega_2)} + \frac{[1-e^{-(1+j\omega_1)}][1-e^{-(1+j\omega_2)}]}{(1+j\omega_1)(1+j\omega_2)}$

(iii) $X(j\omega_1, \omega_2) = \frac{2-e^{-(1+j\omega_1)}-e^{-(1+j\omega_2)}-[1-e^{-(1+j\omega_1)}][1-e^{-(1+j\omega_2)}]}{(1+j\omega_1)(1+j\omega_2)} +$
$\frac{1-e^{-(1+j\omega_1)}}{(1+j\omega_1)(1-j\omega_2)} + \frac{1-e^{-(1+j\omega_2)}}{(1-j\omega_1)(1+j\omega_2)}$

(iv) $X(\omega_1, \omega_2) = -\frac{1}{j\omega_2}\left[\frac{e^{-j\omega_2}(1-e^{j(\omega_1+\omega_2)})+e^{j\omega_2}(1-e^{-j(\omega_1+\omega_2)})}{-j(\omega_1+\omega_2)}\right.$
$\left. + \frac{e^{j\omega_2}(1-e^{j(\omega_1-\omega_2)})+e^{-j\omega_2}(e^{-j(\omega_1-\omega_2)}-1)}{-j(\omega_1-\omega_2)}\right]$

(v) As shown in the Figure S4.53, this signal has six different regions in the (t_1, t_2) plane.

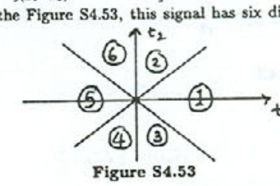

Figure S4.53

The signal $x(t_1, t_2)$ is given by

$$x(t_1, t_2) = \begin{cases} e^{-2t_1}, & \text{in region 1} \\ e^{-2t_2}, & \text{in region 2} \\ e^{2t_2}, & \text{in region 3} \\ e^{2t_2}, & \text{in region 4} \\ e^{2t_1}, & \text{in region 5} \\ e^{-2t_2}, & \text{in region 6} \end{cases}$$

Therefore,

$$\begin{aligned} X(\omega_1, \omega_2) &= \frac{1}{(2 + j\omega_1 + j\omega_2)(2 + j\omega_1 - j\omega_2)} + \frac{1}{(2 + j\omega_2)(2 + j\omega_1 + j\omega_2)} \\ &+ \frac{1}{(2 - j\omega_2)(2 + j\omega_1 - j\omega_2)} + \frac{1}{(2 - j\omega_2)(2 - j\omega_1 - j\omega_2)} \\ &- \frac{2}{(2 - j\omega_1 - j\omega_2)(2 - j\omega_1 + j\omega_2)} + \frac{1}{(j\omega_2)(2 - j\omega_1 - j\omega_2)} \end{aligned}$$

(d) $x(t_1, t_2) = e^{-4(t_1+2t_2)}u(t_1 + 2t_2)$

(e) (i) $e^{-j\omega_1 T_1}e^{-j\omega_2 T_2}X(j\omega_1, j\omega_2)$

(ii) $\frac{1}{|ab|}X(j\frac{\omega_1}{a}, j\frac{\omega_2}{b})$

(iii) $X(j\omega_1, j\omega_2)H(j\omega_1, j\omega_2)$

Chapter 5 Answers

5.1. (a) Let $x[n] = (1/2)^{n-1}u[n-1]$. Using the Fourier transform analysis equation (5.9), the Fourier transform $X(e^{j\omega})$ of this signal is

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \\ &= \sum_{n=1}^{\infty}(1/2)^{n-1}e^{-j\omega n} \\ &= \sum_{n=0}^{\infty}(1/2)^{n}e^{-j\omega(n+1)} \\ &= e^{-j\omega}\frac{1}{(1 - (1/2)e^{-j\omega})} \end{aligned}$$

(b) Let $x[n] = (1/2)^{|n-1|}$. Using the Fourier transform analysis equation (5.9), the Fourier transform $X(e^{j\omega})$ of this signal is

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \\ &= \sum_{n=-\infty}^{0} (1/2)^{-(n-1)}e^{-j\omega n} + \sum_{n=1}^{\infty}(1/2)^{n-1}e^{-j\omega n} \end{aligned}$$

The second summation in the right-hand side of the above equation is exactly the same as the result of part (a). Now,

$$\sum_{n=-\infty}^{0} (1/2)^{-(n-1)}e^{-j\omega n} = \sum_{n=0}^{\infty}(1/2)^{(n+1)}e^{j\omega n} = \left(\frac{1}{2}\right)\frac{1}{1 - (1/2)e^{j\omega}}.$$

Therefore,

$$X(e^{j\omega}) = \left(\frac{1}{2}\right)\frac{1}{1 - (1/2)e^{j\omega}} + e^{-j\omega}\frac{1}{(1 - (1/2)e^{-j\omega})} = \frac{0.75e^{-j\omega}}{1.25 - \cos\omega}.$$

5.2. (a) Let $x[n] = \delta[n-1] + \delta[n+1]$. Using the Fourier transform analysis equation (5.9), the Fourier transform $X(e^{j\omega})$ of this signal is

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \\ &= e^{-j\omega} + e^{j\omega} = 2\cos\omega \end{aligned}$$

(b) Let $x[n] = \delta[n+2] - \delta[n-2]$. Using the Fourier transform analysis equation (5.9), the Fourier transform $X(e^{j\omega})$ of this signal is

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \\ &= e^{2j\omega} - e^{-2j\omega} = 2j\sin(2\omega) \end{aligned}$$

5.3. We note from Section 5.2 that a periodic signal $x[n]$ with Fourier series representation

$$x[n] = \sum_{k=<N>} a_k e^{jk(2\pi/N)n}$$

has a Fourier transform

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta\left(\omega - \frac{2\pi k}{N}\right).$$

(a) Consider the signal $x_1[n] = \sin(\frac{\pi}{3}n + \frac{\pi}{4})$. We note that the fundamental period of the signal $x_1[n]$ is $N = 6$. The signal may be written as

$$x_1[n] = (1/2j)e^{j(\frac{\pi}{3}n+\frac{\pi}{4})} - (1/2j)e^{-j(\frac{\pi}{3}n+\frac{\pi}{4})} = (1/2j)e^{j\frac{\pi}{4}}e^{j\frac{2\pi}{6}n} - (1/2j)e^{-j\frac{\pi}{4}}e^{-j\frac{2\pi}{6}n}.$$

From this, we obtain the non-zero Fourier series coefficients a_k of $x_1[n]$ in the range $-2 \le k \le 3$ as

$$a_1 = (1/2j)e^{j\frac{\pi}{4}}, \qquad a_{-1} = -(1/2j)e^{-j\frac{\pi}{4}}.$$

Therefore, in the range $-\pi \le \omega \le \pi$, we obtain

$$\begin{aligned} X(e^{j\omega}) &= 2\pi a_1\delta(\omega - \frac{2\pi}{6}) + 2\pi a_{-1}\delta(\omega + \frac{2\pi}{6}) \\ &= (\pi/j)\{e^{j\pi/4}\delta(\omega - 2\pi/6) - e^{-j\pi/4}\delta(\omega + 2\pi/6)\} \end{aligned}$$

(b) Consider the signal $x_2[n] = 2 + \cos(\frac{\pi}{6}n + \frac{\pi}{8})$. We note that the fundamental period of the signal $x_1[n]$ is $N = 12$. The signal may be written as

$$x_1[n] = 2 + (1/2)e^{j(\frac{\pi}{6}n+\frac{\pi}{8})} + (1/2)e^{-j(\frac{\pi}{6}n+\frac{\pi}{8})} = 2 + (1/2)e^{j\frac{\pi}{8}}e^{j\frac{2\pi}{12}n} + (1/2)e^{-j\frac{\pi}{8}}e^{-j\frac{2\pi}{12}n}.$$

From this, we obtain the non-zero Fourier series coefficients a_k of $x_2[n]$ in the range $-5 \le k \le 6$ as

$$a_0 = 2, \qquad a_1 = (1/2)e^{j\frac{\pi}{8}}, \qquad a_{-1} = (1/2)e^{-j\frac{\pi}{8}}.$$

Therefore, in the range $-\pi \le \omega \le \pi$, we obtain

$$\begin{aligned} X(e^{j\omega}) &= 2\pi a_0\delta(\omega) + 2\pi a_1\delta(\omega - \frac{2\pi}{12}) + 2\pi a_{-1}\delta(\omega + \frac{2\pi}{12}) \\ &= 4\pi\delta(\omega) + \pi\{e^{j\pi/8}\delta(\omega - \pi/6) + e^{-j\pi/8}\delta(\omega + \pi/6)\} \end{aligned}$$

5.4. (a) Using the Fourier transform synthesis equation (5.8),

$$\begin{aligned} x_1[n] &= (1/2\pi)\int_{-\pi}^{\pi} X_1(e^{j\omega})e^{j\omega n}d\omega \\ &= (1/2\pi)\int_{-\pi}^{\pi} [2\pi\delta(\omega) + \pi\delta(\omega - \pi/2) + \pi\delta(\omega + \pi/2)]e^{j\omega n}d\omega \\ &= e^{j0} + (1/2)e^{j(\pi/2)n} + (1/2)e^{-j(\pi/2)n} \\ &= 1 + \cos(\pi n/2) \end{aligned}$$

(b) Using the Fourier transform synthesis equation (5.8),

$$\begin{aligned} x_2[n] &= (1/2\pi)\int_{-\pi}^{\pi} X_2(e^{j\omega})e^{j\omega n}d\omega \\ &= -(1/2\pi)\int_{-\pi}^{0} 2je^{j\omega n}d\omega + (1/2\pi)\int_{0}^{\pi} 2je^{j\omega n}d\omega \\ &= (j/\pi)\left[-\frac{1 - e^{-jn\pi}}{jn} + \frac{e^{jn\pi} - 1}{jn}\right] \\ &= -(4/(n\pi))\sin^2(n\pi/2) \end{aligned}$$

5.5. From the given information,

$$\begin{aligned} x[n] &= (1/2\pi)\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega \\ &= (1/2\pi)\int_{-\pi}^{\pi} |X(e^{j\omega})|e^{j\sphericalangle\{X(e^{j\omega})\}}e^{j\omega n}d\omega \\ &= (1/2\pi)\int_{-\pi/4}^{\pi/4} e^{-\frac{3}{2}\omega}e^{j\omega n}d\omega \\ &= \frac{\sin(\frac{\pi}{4}(n - 3/2))}{\pi(n - 3/2)} \end{aligned}$$

The signal $x[n]$ is zero when $\frac{\pi}{4}(n - 3/2)$ is a nonzero integer multiple of π or when $|n| \to \infty$. The value of $\frac{\pi}{4}(n - 3/2)$ can never be such that it is a nonzero integer multiple of π. Therefore, $x[n] = 0$ only for $n = \pm\infty$.

5.6. Throughout this problem, we assume that

$$x[n] \stackrel{FT}{\longleftrightarrow} X_1(e^{j\omega}).$$

(a) Using the time reversal property (Sec. 5.3.6), we have

$$x[-n] \stackrel{FT}{\longleftrightarrow} X(e^{-j\omega})$$

Using the time shifting property (Sec. 5.3.3) on this, we have

$$x[-n+1] \stackrel{FT}{\longleftrightarrow} e^{-j\omega n}X(e^{-j\omega}) \quad \text{and} \quad x[-n-1] \stackrel{FT}{\longleftrightarrow} e^{j\omega n}X(e^{-j\omega})$$

Therefore,

$$\begin{aligned} x_1[n] = x[-n+1] + x[-n-1] &\stackrel{FT}{\longleftrightarrow} e^{-j\omega n}X(e^{-j\omega}) + e^{j\omega n}X(e^{-j\omega}) \\ &\stackrel{FT}{\longleftrightarrow} 2X(e^{-j\omega})\cos\omega \end{aligned}$$

(b) Using the time reversal property (Sec. 5.3.6), we have

$$x[-n] \stackrel{FT}{\longleftrightarrow} X(e^{-j\omega})$$

Using the conjugation property on this, we have

$$x^*[-n] \stackrel{FT}{\longleftrightarrow} X^*(e^{j\omega})$$

Therefore,

$$\begin{aligned} x_2[n] = (1/2)(x^*[-n] + x[n]) &\stackrel{FT}{\longleftrightarrow} (1/2)(X(e^{j\omega}) + X^*(e^{j\omega}) \\ &\stackrel{FT}{\longleftrightarrow} \mathcal{R}e\{X(e^{j\omega})\} \end{aligned}$$

(c) Using the differentiation in frequency property (Sec. 5.3.8), we have

$$nx[n] \stackrel{FT}{\longleftrightarrow} j\frac{dX(e^{j\omega})}{d\omega}$$

Using the same property a second time,

$$n^2x[n] \stackrel{FT}{\longleftrightarrow} -\frac{d^2X(e^{j\omega})}{d\omega^2}$$

Therefore,

$$x_3[n] = n^2x[n] - 2nx[n] + 1 \stackrel{FT}{\longleftrightarrow} -\frac{d^2X(e^{j\omega})}{d\omega^2} - 2j\frac{dX(e^{j\omega})}{d\omega} + X(e^{j\omega})$$

5.7. (a) Consider the signal $y_1[n]$ with Fourier transform

$$Y_1(e^{j\omega}) = \sum_{k=1}^{10} \sin(k\omega).$$

We see that $Y_1(e^{j\omega})$ is real and odd. From Table 5.1, we know that the Fourier transform of a real and odd signal is purely imaginary and odd. Therefore, we may say that the Fourier transform of a purely imaginary and odd signal is real and odd. Using this observation, we conclude that $y_1[n]$ is purely imaginary and odd.

Note now that

$$X_1(e^{j\omega}) = e^{-j\omega}Y_1(e^{j\omega}).$$

Therefore, $x_1[n] = y_1[n-1]$. Therefore, $x_1[n]$ is also **purely imaginary**. But $x_1[n]$ is **neither even nor odd**.

(b) We note that $X_2(e^{j\omega})$ is purely imaginary and odd. Therefore, $x_2[n]$ has to be **real and odd**.

(c) Consider a signal $y_3[n]$ whose magnitude of the Fourier transform is $|Y_3(e^{j\omega})| = A(\omega)$, and whose phase of the Fourier transform is $\sphericalangle\{Y_3(e^{j\omega})\} = -(3/2)\omega$. Since $|Y_3(e^{j\omega})| = |Y_3(e^{-j\omega})|$ and $\sphericalangle\{Y_3(e^{j\omega})\} = -\sphericalangle\{Y_3(e^{-j\omega})\}$, we may conclude that the signal $y_3[n]$ is real (See Table 5.1, Property 5.3.4).

Now, consider the signal $x_3[n]$ with Fourier transform $X_3(e^{j\omega}) = Y_3(e^{j\omega})e^{j\pi} = -Y_3(j\omega)$. Using the result from the previous paragraph and the linearity property of the Fourier transform, we may conclude that $x_3[n]$ has to **real**. Since the Fourier transform $X_3(e^{j\omega})$ is neither purely imaginary nor purely real, the signal $x_3[n]$ is **neither even nor odd**.

5.8. Consider the signal

$$x_1[n] = \begin{cases} 1, & |n| \le 1 \\ 0, & |n| > 1 \end{cases}$$

From Table 5.2, we know that

$$x_1[n] \stackrel{FT}{\longleftrightarrow} X_1(e^{j\omega}) = \frac{\sin(3\omega/2)}{\sin(\omega/2)}$$

Using the accumulation property (Table 5.1, Property 5.3.5), we have

$$\sum_{k=-\infty}^{n} x_1[k] \stackrel{FT}{\longleftrightarrow} \frac{1}{1 - e^{-j\omega}}X_1(e^{j\omega}) + \pi X_1(e^{j0})\sum_{k=-\infty}^{\infty}\delta(\omega - 2\pi k).$$

Therefore, in the range $-\pi < \omega \le \pi$,

$$\sum_{k=-\infty}^{n} x_1[k] \stackrel{FT}{\longleftrightarrow} \frac{1}{1 - e^{-j\omega}}X_1(e^{j\omega}) + 3\pi\delta(\omega).$$

Also, in the range $-\pi < \omega \le \pi$,

$$1 \stackrel{FT}{\longleftrightarrow} 2\pi\delta(\omega)$$

Therefore, in the range $-\pi < \omega \le \pi$,

$$x[n] = 1 + \sum_{k=-\infty}^{n} x_1[k] \stackrel{FT}{\longleftrightarrow} \frac{1}{1 - e^{-j\omega}}X_1(e^{j\omega}) + 5\pi\delta(\omega).$$

The signal $x[n]$ has the desired Fourier transform. We may express $x[n]$ mathematically as

$$x[n] = 1 + \sum_{k=-\infty}^{n} x_1[k] = \begin{cases} 1, & n \le -2 \\ n + 3, & -1 \le n \le 1 \\ 4, & n \ge 2 \end{cases}$$

5.9. From Property 5.3.4 in Table 5.1, we know that for a real signal $x[n]$,

$$Od\{x[n]\} \overset{\mathcal{FT}}{\longleftrightarrow} j\mathcal{I}m\{X(e^{j\omega})\}$$

From the given information,

$$\begin{aligned} j\mathcal{I}m\{X(e^{j\omega})\} &= j\sin\omega - j\sin 2\omega \\ &= (1/2)(e^{j\omega} - e^{-j\omega} - e^{2j\omega} + e^{-2j\omega}) \end{aligned}$$

Therefore,

$$Od\{x[n]\} = \mathcal{IFT}\{j\mathcal{I}m\{X(e^{j\omega})\}\} = (1/2)(\delta[n+1] - \delta[n-1] - \delta[n+2] + \delta[n-2])$$

We also know that

$$Od\{x[n]\} = \frac{x[n] - x[-n]}{2}$$

and that $x[n] = 0$ for $n > 0$. Therefore,

$$x[n] = 2Od\{x[n]\} = \delta[n+1] - \delta[n+2], \qquad \text{for } n < 0.$$

Now we only have to find $x[0]$. Using Parseval's relation, we have

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} |X(e^{j\omega})|^2 d\omega = \sum_{n=-\infty}^{\infty} |x[n]|^2.$$

From the given information, we can write

$$3 = (x[0])^2 + \sum_{n=-\infty}^{-1} |x[n]|^2 = (x[0])^2 + 2$$

This gives $x[0] = \pm 1$. But since we are given that $x[0] > 0$, we conclude that $x[0] = 1$. Therefore,

$$x[n] = \delta[n] + \delta[n+1] - \delta[n+2].$$

5.10. From Table 5.2, we know that

$$\left(\frac{1}{2}\right)^n u[n] \overset{\mathcal{FT}}{\longleftrightarrow} \frac{1}{1 - \frac{1}{2}e^{-j\omega}}$$

Using Property 5.3.8 in Table 5.1,

$$x[n] = n\left(\frac{1}{2}\right)^n u[n] \overset{\mathcal{FT}}{\longleftrightarrow} X(e^{j\omega}) = j\frac{d}{d\omega}\left\{\frac{1}{1 - \frac{1}{2}e^{-j\omega}}\right\} = \frac{\frac{1}{2}e^{-j\omega}}{(1 - \frac{1}{2}e^{-j\omega})^2}$$

Therefore,

$$\sum_{n=0}^{\infty} n\left(\frac{1}{2}\right)^n = \sum_{n=-\infty}^{\infty} x[n] = X(e^{j0}) = 2$$

5.11. We know from the time expansion property (Table 5.1, Property 5.3.7) that

$$g[n] = x_{(2)}[n] \overset{\mathcal{FT}}{\longleftrightarrow} G(e^{j\omega}) = X(e^{j2\omega}).$$

Therefore, $G(e^{j\omega})$ is obtained by compressing $X(e^{j\omega})$ by a factor of 2. Since we know that $X(e^{j\omega})$ is periodic with a period of 2π, we may conclude that $G(e^{j\omega})$ has a period which is $(1/2)2\pi = \pi$. Therefore,

$$G(e^{j\omega}) = G(e^{j(\omega - \pi)}) \qquad \text{and } \alpha = \pi.$$

5.12. Consider the signal

$$x_1[n] = \left(\frac{\sin\frac{\pi}{4}n}{\pi n}\right).$$

From Table 5.2, we obtain the Fourier transform of $x_1[n]$ to be

$$X_1(e^{j\omega}) = \begin{cases} 1, & 0 \le |\omega| \le \frac{\pi}{4} \\ 0, & \frac{\pi}{4} < |\omega| < \pi \end{cases}.$$

The plot of $X_1(e^{j\omega})$ is as shown in the Figure S5.12. Now consider the signal $x_2[n] = (x_1[n])^2$. Using the multiplication property (Table 5.1, Property 5.5), we obtain the Fourier tranform of $x_2[n]$ to be

$$X_2(e^{j\omega}) = (1/2\pi)[X_1(e^{j\omega}) * X_1(e^{j\omega})].$$

This is plotted in the Figure S5.12.

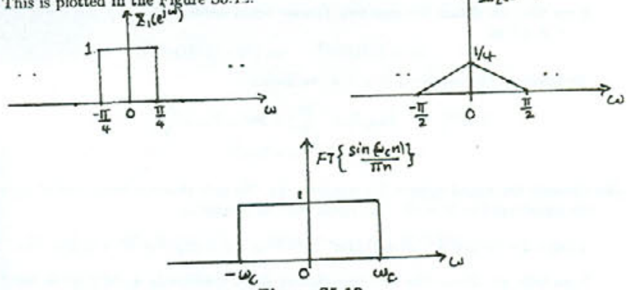

Figure S5.12

From Figure S5.12 it is clear that $X_2(e^{j\omega})$ is zero for $|\omega| > \pi/2$. By using the convolution property (Table 5.1, Property 5.4), we note that

$$Y(e^{j\omega}) = X_2(e^{j\omega})\mathcal{FT}\left\{\frac{\sin(\omega_c n)}{\pi n}\right\}.$$

The plot of $\mathcal{FT}\left\{\frac{\sin(\omega_c n)}{\pi n}\right\}$ is shown in Figure S5.12. It is clear that if $Y(e^{j\omega}) = X_2(e^{j\omega})$, then $(\pi/2) \le \omega_c \le \pi$.

5.13. When two LTI systems are connected in parallel, the impulse response of the overall system is the sum of the impulse responses of the individual systems. Therefore,

$$h[n] = h_1[n] + h_2[n].$$

Using the linearity property (Table 5.1, Property 5.3.2),

$$H(e^{j\omega}) = H_1(e^{j\omega}) + H_2(e^{j\omega})$$

Given that $h_1[n] = (1/2)^n u[n]$, we obtain

$$H_1(e^{j\omega}) = \frac{1}{1 - \frac{1}{2}e^{-j\omega}}$$

Therefore,

$$H_2(e^{j\omega}) = \frac{-12 + 5^{-j\omega}}{12 - 7e^{-j\omega} + e^{-2j\omega}} - \frac{1}{1 - \frac{1}{2}e^{-j\omega}} = \frac{-2}{1 - \frac{1}{4}e^{-j\omega}}.$$

Taking the inverse Fourier transform,

$$h_2[n] = -2\left(\frac{1}{4}\right)^n u[n].$$

5.14. From the given information, we have the Fourier transform $G(e^{j\omega})$ of $g[n]$ to be

$$G(e^{j\omega}) = g[0] + g[1]e^{-j\omega}.$$

Also, when the input to the system is $x[n] = (1/4)^n u[n]$, the output is $g[n]$. Therefore

$$H(e^{j\omega}) = \frac{G(e^{j\omega})}{X(e^{j\omega})}.$$

From Table 5.2, we obtain

$$X(e^{j\omega}) = \frac{1}{1 - \frac{1}{4}e^{-j\omega}}.$$

Therefore,

$$H(e^{j\omega}) = \{g[0] + g[1]e^{-j\omega}\}\{1 - \frac{1}{4}e^{-j\omega}\} = g[0] + \{g[1] - \frac{1}{4}g[0]\}e^{-j\omega} - g[1]e^{-2j\omega}$$

Clearly, $h[n]$ is a three point sequence.

We have

$$H(e^{j\omega}) = h[0] + h[1]e^{-j\omega} + h[2]e^{-2j\omega}$$

and

$$\begin{aligned} H(e^{j(\omega-\pi)}) &= h[0] + h[1]e^{-j(\omega-\pi)} + h[2]e^{-2j(\omega-\pi)} \\ &= h[0] - h[1]e^{-j\omega} + h[2]e^{-2j\omega} \end{aligned}$$

We see that $H(e^{j\omega}) = H(e^{j(\omega-\pi)})$ only if $h[1] = 0$.

We also have

$$\begin{aligned} H(e^{j\pi/2}) &= h[0] + h[1]e^{-j\pi/2} + h[2]e^{-2j\pi/2} \\ &= h[0] - h[2] \end{aligned}$$

Since we are also given that $H(e^{j\pi/2}) = 1$, we have

$$h[0] - h[2] = 1. \qquad \text{(S5.14-1)}$$

Now note that

$$\begin{aligned} g[n] &= h[n] * \{(1/4)^n u[n]\} \\ &= \sum_{k=0}^{2} h[k](1/4)^{n-k} u[n-k] \end{aligned}$$

Evaluating this equation at $n = 2$, we have

$$g[2] = 0 = \frac{1}{16}h[0] + \frac{1}{4}h[1] + h[2]$$

Since $h[1] = 0$,

$$\frac{1}{16}h[0] + h[2] = 0. \qquad \text{(S5.14-2)}$$

Solving equations (S5.14-1) and (S5.14-2), we obtain

$$h[0] = \frac{16}{17}, \quad \text{and} \quad h[2] = -\frac{1}{17}.$$

Therefore,

$$h[n] = \frac{16}{17}\delta[n] - \frac{1}{17}\delta[n-2].$$

5.15. Consider $x[n] = \sin(\omega_c n)/(\pi n)$. The Fourier transform $X(e^{j\omega})$ of $x[n]$ is as shown in Figure S5.15. We note that the given signal $y[n] = x[n]x[n]$. Therefore, the Fourier transform $Y(e^{j\omega})$ of $y[n]$ is

$$Y(e^{j\omega}) = \frac{1}{2\pi}\int_{<2\pi>} X(e^{j\theta})X(e^{j(\omega-\theta)})d\theta.$$

Employing the approach used in Example 5.15, we can convert the above periodic convolution into an aperiodic signal by defining

$$\hat{X}(e^{j\omega}) = \begin{cases} X(e^{j\omega}), & -\pi < \omega \le \pi \\ 0, & \text{otherwise} \end{cases}$$

Then we may write

$$Y(e^{j\omega}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{X}(e^{j\theta})X(e^{j(\omega-\theta)})d\theta.$$

This is the aperiodic convolution of the rectangular pulse $\hat{X}(e^{j\omega})$ shown in Figure S5.15 with the periodic square wave $X(e^{j\omega})$. The result of this convolution is as shown in the Figure S5.15.

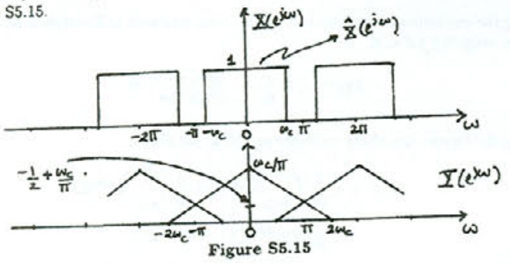

Figure S5.15

From the figure, it is clear that we require $-1+(2\omega_c/\pi)$ to be 1/2. Therefore, $\omega_c = 3\pi/4$.

5.16. We may write

$$X(e^{j\omega}) = \frac{1}{2\pi}\left\{\frac{1}{1-\frac{1}{4}e^{-j\omega}} * \left[2\pi\sum_{k=0}^{3}\delta(\omega - \frac{\pi k}{2})\right]\right\}$$

where $*$ denotes aperiodic convolution. We may also rewrite this as a periodic convolution

$$X(e^{j\omega}) = \frac{1}{2\pi}\int_0^{2\pi} G(e^{j\theta})Q(e^{j(\omega-\theta)})d\theta$$

where

$$G(e^{j\omega}) = \frac{1}{1-\frac{1}{4}e^{-j\omega}}$$

and

$$Q(e^{j\omega}) = 2\pi\sum_{k=0}^{3}\delta(\omega - \frac{\pi k}{2}) \quad \text{for } 0 \le \omega < 2\pi.$$

(a) Taking the inverse Fourier transform of $G(e^{j\omega})$ (see Table 5.2), we get $g[n] = (1/4)^n u[n]$. Therefore, $\alpha = \frac{1}{4}$.

(b) Taking the inverse Fourier transform of $Q(e^{j\omega})$ (see Table 5.2), we get

$$q[n] = 1 + \frac{1}{2}e^{j(\pi/2)n} + \frac{1}{4}e^{j\pi n} + \frac{1}{8}e^{j(3\pi/2)n}.$$

This signal is periodic with a fundamental period of $N = 4$.

(c) We can easily show that $X(e^{j\omega})$ is not conjugate symmetric. Therefore, $x[n]$ is not real.

5.17. Using the duality property, we have

$$(-1)^n \stackrel{FS}{\longleftrightarrow} a_k \Rightarrow a_n \stackrel{FS}{\longleftrightarrow} \frac{1}{N}(-1)^{-k} = \frac{1}{2}(-1)^k.$$

5.18. Knowing that

$$\left(\frac{1}{2}\right)^{|n|} \stackrel{FT}{\longleftrightarrow} \frac{1-\frac{1}{4}}{1-\cos\omega+\frac{1}{4}} = \frac{3}{5-4\cos\omega},$$

we may use the Fourier transform analysis equation to write

$$\frac{3}{5-4\cos\omega} = \sum_{n=-\infty}^{\infty}\left(\frac{1}{2}\right)^{|n|}e^{-j\omega n}$$

Putting $\omega = -2\pi t$ in this equation, and replacing the variable n by the variable k

$$\frac{1}{5-4\cos(2\pi t)} = \sum_{k=-\infty}^{\infty}\frac{1}{3}\left(\frac{1}{2}\right)^{|k|}e^{j2\pi kt}.$$

By comparing this with the continuous-time Fourier series synthesis equation, it is immediately apparent that $a_k = \frac{1}{3}\left(\frac{1}{2}\right)^{|k|}$ are the Fourier series coefficients of the signal $1/(5-4\cos(2\pi t))$.

5.19. (a) Taking the Fourier transform of both sides of the difference equation, we have

$$Y(e^{j\omega})\left[1-\frac{1}{6}e^{-j\omega}-\frac{1}{6}e^{-2j\omega}\right] = X(e^{j\omega}).$$

Therefore,

$$H(ej\omega) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1}{1-\frac{1}{6}e^{-j\omega}-\frac{1}{6}e^{-2j\omega}} = \frac{1}{(1-\frac{1}{2}e^{-j\omega})(1+\frac{1}{3}e^{-j\omega})}.$$

(b) Using Partial fraction expansion,

$$H(ej\omega) = \frac{3/5}{1-\frac{1}{2}e^{-j\omega}} + \frac{2/5}{1+\frac{1}{3}e^{-j\omega}}.$$

Using Table 5.2, and taking the inverse Fourier trasform, we obtain

$$h[n] = \frac{3}{5}\left(\frac{1}{2}\right)^n u[n] + \frac{2}{5}\left(-\frac{1}{3}\right)^n u[n].$$

5.20. (a) Since the LTI system is causal and stable, a single input-output pair is sufficient to determine the frequency response of the system. In this case, the input is $x[n] = (4/5)^n u[n]$ and the output is $y[n] = n(4/5)^n u[n]$. The frequency response is given by

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})}$$

where $X(e^{j\omega})$ and $Y(e^{j\omega})$ are the Fourier transforms of $x[n]$ and $y[n]$ respectively. Using Table 5.2, we have

$$x[n] = \left(\frac{4}{5}\right)^n u[n] \stackrel{FT}{\longleftrightarrow} X(e^{j\omega}) = \frac{1}{1-\frac{4}{5}e^{-j\omega}}.$$

Using the differentiation in frequency property (Table 5.1, Property 5.3.8), we have

$$y[n] = n\left(\frac{4}{5}\right)^n u[n] \stackrel{FT}{\longleftrightarrow} Y(e^{j\omega}) = j\frac{dX(e^{j\omega})}{d\omega} = \frac{(4/5)e^{-j\omega}}{(1-\frac{4}{5}e^{-j\omega})^2}.$$

Therefore,

$$H(e^{j\omega}) = \frac{(4/5)e^{-j\omega}}{1-\frac{4}{5}e^{-j\omega}}.$$

(b) Since $H(ej\omega) = Y(e^{j\omega})/X(e^{j\omega})$, we may write

$$Y(e^{j\omega})\left[1-\frac{4}{5}e^{-j\omega}\right] = X(e^{j\omega})\left[(4/5)e^{-j\omega}\right].$$

Taking the inverse Fourier tranform of both sides

$$y[n] - \frac{4}{5}y[n-1] = \frac{4}{5}x[n].$$

5.21. (a) The given signal is

$$x[n] = u[n-2] - u[n-6] = \delta[n-2] + \delta[n-3] + \delta[n-4] + \delta[n-5].$$

Using the Fourier transform analysis eq. (5.9), we obtain

$$X(e^{j\omega}) = e^{-2j\omega} + e^{-3j\omega} + e^{-4j\omega} + e^{-5j\omega}.$$

(b) Using the Fourier transform analysis eq. (5.9), we obtain

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=-\infty}^{-1}(\tfrac{1}{2})^{-n}e^{-j\omega n} \\ &= \sum_{n=1}^{\infty}(\tfrac{1}{2}e^{j\omega})^n \\ &= \frac{e^{j\omega}}{2}\frac{1}{(1-\frac{1}{2}e^{j\omega})} \end{aligned}$$

(c) Using the Fourier transform analysis eq. (5.9), we obtain

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=-\infty}^{-2}(\tfrac{1}{3})^{-n}e^{-j\omega n} \\ &= \sum_{n=2}^{\infty}(\tfrac{1}{3}e^{j\omega})^n \\ &= \frac{e^{2j\omega}}{9}\frac{1}{(1-\frac{1}{3}e^{j\omega})} \end{aligned}$$

(d) Using the Fourier transform analysis eq. (5.9), we obtain

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=-\infty}^{0} 2^n\sin(\pi n/4)e^{-j\omega n} \\ &= -\sum_{n=0}^{\infty}2^{-n}\sin(\pi n/4)e^{j\omega n} \\ &= -\frac{1}{2j}\sum_{n=0}^{\infty}[(1/2)^n e^{j\pi n/4}e^{j\omega n} - (1/2)^n e^{-j\pi n/4}e^{j\omega n}] \\ &= -\frac{1}{2j}\left[\frac{1}{1-(1/2)e^{j\pi/4}e^{j\omega}} - \frac{1}{1-(1/2)e^{-j\pi/4}e^{j\omega}}\right] \end{aligned}$$

(e) Using the Fourier transform analysis eq. (5.9), we obtain

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty}(1/2)^{|n|}\cos[\pi(n-1)/8]e^{-j\omega n} \\ &= \frac{1}{2}\left[\frac{e^{-j\pi/8}}{1-(1/2)e^{j\pi/8}e^{-j\omega}} + \frac{e^{j\pi/8}}{1-(1/2)e^{-j\pi/8}e^{-j\omega}}\right] \\ &+ \frac{1}{4}\left[\frac{e^{j\pi/4}e^{j\omega}}{1-(1/2)e^{j\pi/8}e^{j\omega}} + \frac{e^{-j\pi/4}e^{j\omega}}{1-(1/2)e^{-j\pi/8}e^{j\omega}}\right] \end{aligned}$$

(f) The given signal is

$$x[n] = -3\delta[n+3] - 2\delta[n+2] - \delta[n+1] + \delta[n-1] + 2\delta[n-2] + 3\delta[n-3].$$

Using the Fourier transform analysis eq. (5.9), we obtain

$$X(e^{j\omega}) = -3e^{3j\omega} - 2e^{2j\omega} - e^{j\omega} + e^{-j\omega} + 2e^{-2j\omega} + 3e^{-3j\omega}.$$

(g) The given signal is

$$x[n] = \sin(\pi n/2) + \cos(n) = \frac{1}{2j}[e^{j\pi n/2} - e^{-j\pi n/2}] + \frac{1}{2}[e^{jn} + e^{-jn}].$$

Therefore,

$$X(e^{j\omega}) = \frac{\pi}{j}[\delta(\omega-\pi/2) - \delta(\omega+\pi/2)] + \pi[\delta(\omega-1) + \delta(\omega+1)], \quad \text{in } 0 \le |\omega| < \pi.$$

(h) The given signal is

$$\begin{aligned} x[n] &= \sin(5\pi n/3) + \cos(7\pi n/3) \\ &= -\sin(\pi n/3) + \cos(\pi n/3) \\ &= -\frac{1}{2j}[e^{j\pi n/3} - e^{-j\pi n/3}] + \frac{1}{2}[e^{j\pi n/3} + e^{-j\pi n/3}]. \end{aligned}$$

Therefore,

$$X(e^{j\omega}) = -\frac{\pi}{j}[\delta(\omega - \pi/3) - \delta(\omega + \pi/3)] + \pi[\delta(\omega - \pi/3) + \delta(\omega + \pi/3)], \quad \text{in } 0 \le |\omega| < \pi.$$

(i) $x[n]$ is periodic with period 6. The Fourier series coefficients of $x[n]$ are given by

$$\begin{aligned} a_k &= \frac{1}{6}\sum_{n=0}^{5} x[n] e^{-j(2\pi/6)kn} \\ &= \frac{1}{6}\sum_{n=0}^{4} e^{-j(2\pi/6)kn} \\ &= \frac{1}{6}\left[\frac{1 - e^{-j5\pi k/3}}{1 - e^{-j(2\pi/6)k}}\right] \end{aligned}$$

Therefore, from the results of Section 5.2

$$X(e^{j\omega}) = \sum_{l=-\infty}^{\infty} 2\pi\left(\frac{1}{6}\right)\left[\frac{1 - e^{-j5\pi k/3}}{1 - e^{-j(2\pi/6)k}}\right]\delta(\omega - \frac{2\pi}{6} - 2\pi l).$$

(j) Using the Fourier transform analysis eq. (5.9) we obtain

$$\left(\frac{1}{3}\right)^{|n|} \stackrel{FT}{\longleftrightarrow} \frac{4}{5 - 3\cos\omega}.$$

Using the differentiation in frequency property of the Fourier transform,

$$n\left(\frac{1}{3}\right)^{|n|} \stackrel{FT}{\longleftrightarrow} -j\frac{12\sin\omega}{(5 - 3\cos\omega)^2}.$$

Therefore,

$$x[n] = n\left(\frac{1}{3}\right)^{|n|} - \left(\frac{1}{3}\right)^{|n|} \stackrel{FT}{\longleftrightarrow} \frac{4}{5 - 3\cos\omega} - j\frac{12\sin\omega}{(5 - 3\cos\omega)^2}.$$

(k) We have

$$x_1[n] = \frac{\sin(\pi n/5)}{\pi n} \stackrel{FT}{\longleftrightarrow} X_1(e^{j\omega}) = \begin{cases} 1, & |\omega| < \frac{\pi}{5} \\ 0, & \frac{\pi}{5} \le |\omega| < \pi \end{cases}.$$

Also,

$$x_2[n] = \cos(7\pi n/2) = \cos(\pi n/2) \stackrel{FT}{\longleftrightarrow} X_2(e^{j\omega}) = \pi\{\delta(\omega - \pi/2) + \delta(\omega + \pi/2)\},$$

in the range $0 \le |\omega| < \pi$. Therefore, if $x[n] = x_1[n]x_2[n]$, then

$$X(e^{j\omega}) = \text{Periodic convolution of } X_1(e^{j\omega}) \text{ and } X_2(e^{j\omega}).$$

Using the mechanics of periodic convolution demosntrated in Example 5.15, we obtain in the range $0 \le |\omega| < \pi$,

$$X(e^{j\omega}) = \begin{cases} 1, & \frac{3\pi}{10} < |\omega| < \frac{7\pi}{10} \\ 0, & \text{otherwise} \end{cases}.$$

5.22. (a) Using the Fourier transform synthesis eq. (5.8), we obtain

$$\begin{aligned} x[n] &= \frac{1}{2\pi}\int_{-3\pi/4}^{-\pi/4} e^{j\omega n} d\omega + \frac{1}{2\pi}\int_{\pi/4}^{3\pi/4} e^{j\omega n} d\omega \\ &= \frac{1}{\pi n}[\sin(3\pi n/4) - \sin(\pi n/4)] \end{aligned}$$

(b) Comparing the given Fourier transform with the analysis eq. (5.8), we obtain

$$x[n] = \delta[n] + 3\delta[n-1] + 2\delta[n-2] - 4\delta[n-3] + \delta[n-10].$$

(c) Using the Fourier transform synthesis eq. (5.8), we obtain

$$\begin{aligned} x[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-j\omega/2} e^{j\omega n} d\omega \\ &= \frac{(-1)^{n+1}}{\pi(n - \frac{1}{2})} \end{aligned}$$

(d) The given Fourier transform is

$$\begin{aligned} X(e^{j\omega}) &= \cos^2\omega + \sin^2(3\omega) \\ &= \frac{1 + \cos(2\omega)}{2} + \frac{1 - \cos(3\omega)}{2} \\ &= 1 + \frac{1}{4}e^{2j\omega} + \frac{1}{4}e^{-2j\omega} + -\frac{1}{4}e^{3j\omega} - \frac{1}{4}e^{-3j\omega} \end{aligned}$$

Comparing the given Fourier transform with the analysis eq. (5.8), we obtain

$$x[n] = \delta[n] + \frac{1}{4}\delta[n-2] + \frac{1}{4}\delta[n+2] - \frac{1}{4}\delta[n-3] - \frac{1}{4}\delta[n+3].$$

(e) This is the Fourier transform of a periodic signal with fundamental frequency $\pi/2$. Therefore, its fundamental period is 4. Also, the Fourier series coefficients of this signal are $a_k = (-1)^k$. Therefore, the signal is given by

$$x[n] = \sum_{k=0}^{3}(-1)^k e^{jk(\pi/2)n} = 1 - e^{j\pi n/2} + e^{j\pi n} - e^{j3\pi n/2}.$$

(f) The given Fourier transform may be written as

$$\begin{aligned} X(e^{j\omega}) &= e^{-j\omega}\sum_{n=0}^{\infty}(1/5)^n e^{-j\omega n} - (1/5)\sum_{n=0}^{\infty}(1/5)^n e^{-j\omega n} \\ &= 5\sum_{n=1}^{\infty}(1/5)^n e^{-j\omega n} - (1/5)\sum_{n=0}^{\infty}(1/5)^n e^{-j\omega n} \end{aligned}$$

Comparing each of the two terms in the right-hand side of the above equation with the Fourier transform analysis eq. (5.9) we obtain

$$x[n] = \left(\frac{1}{5}\right)^{n-1} u[n-1] - \left(\frac{1}{5}\right)^{n+1} u[n].$$

(g) The given Fourier transform may be written as

$$X(e^{j\omega}) = \frac{2/9}{1 - \frac{1}{2}e^{-j\omega}} + \frac{7/9}{1 + \frac{1}{4}e^{-j\omega}}.$$

Therefore,

$$x[n] = \frac{2}{9}\left(\frac{1}{2}\right)^n u[n] + \frac{7}{9}\left(-\frac{1}{4}\right)^n u[n].$$

(h) The given Fourier transform may be written as

$$X(e^{j\omega}) = 1 + \frac{1}{3}e^{-j\omega} + \frac{1}{3^2}e^{-j2\omega} + \frac{1}{3^3}e^{-j3\omega} + \frac{1}{3^4}e^{-j4\omega} + \frac{1}{3^5}e^{-j5\omega}.$$

Comparing the given Fourier transform with the analysis eq. (5.8), we obtain

$$x[n] = \delta[n] + \frac{1}{3}\delta[n-1] + \frac{1}{9}\delta[n-2] + \frac{1}{27}\delta[n-3] + \frac{1}{81}\delta[n-4] + \frac{1}{243}\delta[n-5].$$

5.23. (a) We have from eq. (5.9)

$$X(e^{j0}) = \sum_{n=-\infty}^{\infty} x[n] = 6.$$

(b) Note that $y[n] = x[n+2]$ is an even signal. Therefore, $Y(e^{j\omega})$ is real and even. This implies that $\sphericalangle Y(e^{j\omega}) = 0$. Furthermore, from the time shifting property of the Fourier transform we have $Y(e^{j\omega}) = e^{j2\omega}X(e^{j\omega})$. Therefore, $\sphericalangle X(e^{j\omega}) = e^{-j2\omega}$.

(c) We have from eq. (5.8)

$$2\pi x[0] = \int_{-\pi}^{\pi} X(e^{j\omega}) d\omega.$$

Therefore,

$$\int_{-\pi}^{\pi} X(e^{j\omega}) d\omega = 4\pi.$$

(d) We have from eq. (5.9)

$$X(e^{j\pi}) = \sum_{n=-\infty}^{\infty} x[n](-1)^n = 2.$$

(e) From Table 5.1, we have

$$\mathcal{E}v\{x[n]\} \stackrel{FT}{\longleftrightarrow} \mathcal{R}e\{X(e^{j\omega})\}.$$

Therefore, the desired signal is $\mathcal{E}v\{x[n]\} = (x[n] + x[-n])/2$. This is as shown in Figure S5.23.

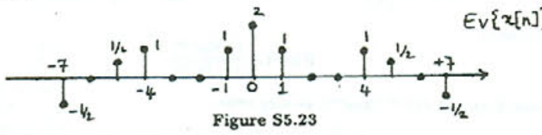

Figure S5.23

(f) (i) From Parseval's theorem we have

$$\int_{-\infty}^{\infty}|X(e^{j\omega})|^2 = 2\pi\sum_{n=-\infty}^{\infty}|x[n]|^2 = 28\pi.$$

(ii) Using the differentiation in frequency property of the Fourier transform we obtain

$$nx[n] \stackrel{FT}{\longleftrightarrow} j\frac{dX(e^{j\omega})}{d\omega}.$$

Again using Parseval's theorem, we obtain

$$\int_{-\infty}^{\infty}\left|\frac{dX(e^{j\omega})}{d\omega}\right|^2 = 2\pi\sum_{n=-\infty}^{\infty}|n|^2|x[n]|^2 = 316\pi.$$

5.24. (1) For $\mathcal{R}e\{X(e^{j\omega})\}$ to be zero, the signal must be real and odd. Only signals (b) and (i) are real and odd.

(2) For $\mathcal{I}m\{X(e^{j\omega})\}$ to be zero, the signal must be real and even. Only signals (d) and (h) are real and even.

(3) Assume $Y(e^{j\omega}) = e^{j\alpha\omega}X(e^{j\omega})$. Using the time shifting property of the Fourier transform we have $y[n] = x[n+\alpha]$. If $Y(e^{j\omega})$ is real, then $y[n]$ is real and even (assuming that $x[n]$ is real). Therefore, $x[n]$ has to be symmetric about α. This is true only for signals (a), (b), (d), (e), (f), and (h).

(4) Since $\int_{-\pi}^{\pi} X(e^{j\omega})d\omega = 2\pi x[0]$, the given condition is satisfied only if $x[0] = 0$. This is true for signals (b), (e), (f), (h), and (i).

(5) $X(e^{j\omega})$ is always periodic with period 2π. Therefore, all signals satisfy this condition.

(6) Since $X(e^{j0}) = \sum_{n=-\infty}^{\infty} x[n]$, the given condition is satisfied only if the samples of the signal add up to zero. This is true for signals (b), (g), and (i).

5.25. If the inverse Fourier transform of $X(e^{j\omega})$ is $x[n]$, then

$$x_e[n] = \mathcal{E}v\{x[n]\} = \frac{x[n]+x[-n]}{2} \overset{FT}{\longleftrightarrow} A(\omega)$$

and

$$x_o[n] = \mathcal{O}d\{x[n]\} = \frac{x[n]-x[-n]}{2} \overset{FT}{\longleftrightarrow} jB(\omega)$$

Therefore, the inverse Fourier transform of $B(\omega)$ is $-jx_o[n]$. Also, the inverse Fourier transform of $A(\omega)e^{j\omega}$ is $x_e[n+1]$. Therefore, the time function corresponding to the inverse Fourier transform of $B(\omega) + A(\omega)e^{j\omega}$ will be $x_e[n+1] - jx_o[n]$. This is as shown in the Figure S5.25.

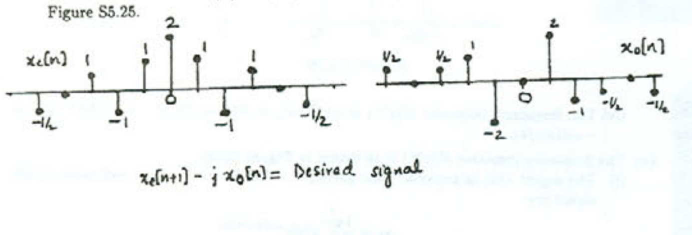

Figure S5.25

5.26. (a) We may express $X_2(e^{j\omega})$ as

$$X_2(e^{j\omega}) = \mathcal{R}e\{X_1(e^{j\omega})\} + \mathcal{R}e\{X_1(e^{j(\omega-2\pi/3)})\} + \mathcal{R}e\{X_1(e^{j(\omega+2\pi/3)})\}.$$

Therefore,

$$x_2[n] = \mathcal{E}v\{x_1[n]\}\left[1 + e^{j2\pi/3} + e^{-j2\pi/3}\right].$$

(b) We may express $X_3(e^{j\omega})$ as

$$X_3(e^{j\omega}) = \mathcal{I}m\{X_1(e^{j(\omega-\pi)})\} + \mathcal{I}m\{X_1(e^{j(\omega+\pi)})\}.$$

Therefore,

$$x_3[n] = \mathcal{O}d\{x_1[n]\}\left[e^{j\pi n} + e^{-j\pi n}\right] = 2(-1)^n\mathcal{O}d\{x_1[n]\}.$$

(c) We may express α as

$$\alpha = \frac{j\left.\frac{dX_1(e^{j\omega})}{d\omega}\right|_{\omega=0}}{X_1(e^{j\omega})|_{\omega=0}} = \frac{j(-6j/\pi)}{1} = \frac{6}{\pi}.$$

(d) Using the fact that $H(e^{j\omega})$ is the frequency response of an ideal lowpass filter with cutoff frequency $\pi/6$, we may draw $X_4(e^{j\omega})$ as shown in Figure S5.26.

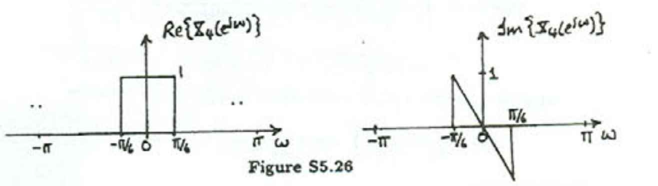

Figure S5.26

5.27. (a) $W(e^{j\omega})$ will be the periodic convolution of $X(e^{j\omega})$ with $P(e^{j\omega})$. The Fourier transforms are sketched in Figure S5.27.

(b) The Fourier transform of $Y(e^{j\omega})$ of $y[n]$ is $Y(e^{j\omega}) = P(e^{j\omega})H(e^{j\omega})$. The LTI system with unit sample response $h[n]$ is an ideal lowpass filter with cutoff frequency $\pi/2$. Therefore, $Y(e^{j\omega})$ for each choice of $p[n]$ are as shown in Figure S5.27. Therefore, $y[n]$ in each case is:

(i) $y[n] = 0$

(ii) $y[n] = \frac{\sin(\pi n/2)}{2\pi n} - \frac{1-\cos(\pi n/2)}{\pi^2 n^2}$

(iii) $y[n] = \frac{\sin(\pi n/2)}{\pi^2 n^2} - \frac{\cos(\pi n/2)}{2\pi n}$

(iv) $y[n] = 2\left[\frac{\sin(\pi n/4)}{\pi n}\right]^2$

(v) $y[n] = \frac{1}{4}\left[\frac{\sin(\pi n/2)}{\pi n}\right]$

5.28. Let

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})G(e^{j(\omega-\theta)})d\theta = 1 + e^{-j\omega} = Y(e^{j\omega}).$$

Taking the inverse Fourier transform of the above equation, we obtain

$$g[n]x[n] = \delta[n] + \delta[n-1] = y[n].$$

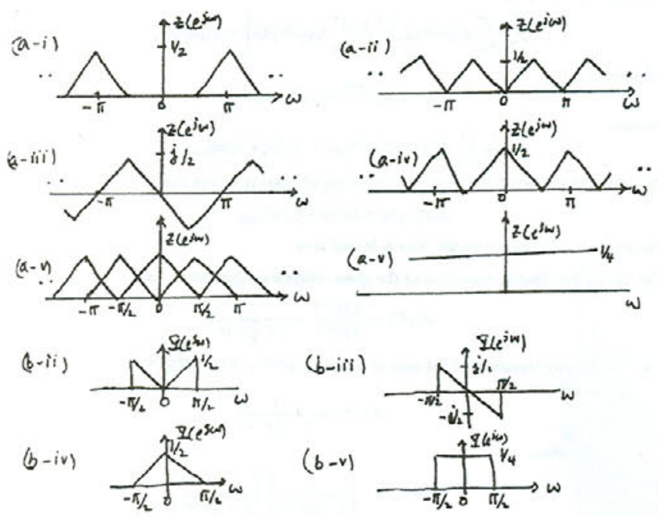

Figure S5.27

(a) If $x[n] = (-1)^n$,

$$g[n] = \delta[n] - \delta[n-1].$$

(b) If $x[n] = (1/2)^n u[n]$, $g[n]$ has to be chosen such that

$$g[n] = \begin{cases} 1, & n = 0 \\ 2, & n = 1 \\ 0, & n > 1 \\ \text{any value}, & \text{otherwise} \end{cases}$$

Therefore, there are many possible choices for $g[n]$.

5.29. (a) Let the output of the system be $y[n]$. We know that

$$Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega}).$$

In this part of the problem

$$H(e^{j\omega}) = \frac{1}{1-\frac{1}{2}e^{-j\omega}}.$$

(i) We have

$$X(e^{j\omega}) = \frac{1}{1-\frac{3}{4}e^{-j\omega}}.$$

Therefore,

$$Y(e^{j\omega}) = \left[\frac{1}{1-\frac{3}{4}e^{-j\omega}}\right]\left[\frac{1}{1-\frac{1}{2}e^{-j\omega}}\right] = \frac{-2}{1-\frac{1}{2}e^{-j\omega}} + \frac{3}{1-\frac{3}{4}e^{-j\omega}}$$

Taking the inverse Fourier transform, we obtain

$$y[n] = 3\left(\frac{3}{4}\right)^n u[n] - 2\left(\frac{1}{2}\right)^n u[n].$$

(ii) We have

$$X(e^{j\omega}) = \frac{1}{(1-\frac{1}{4}e^{-j\omega})^2}.$$

Therefore,

$$Y(e^{j\omega}) = \left[\frac{1}{(1-\frac{1}{4}e^{-j\omega})^2}\right]\left[\frac{1}{1-\frac{1}{2}e^{-j\omega}}\right] = \frac{4}{1-\frac{1}{2}e^{-j\omega}} - \frac{2}{1-\frac{1}{4}e^{-j\omega}} - \frac{3}{(1-\frac{1}{4}e^{-j\omega})^2}$$

Taking the inverse Fourier transform, we obtain

$$y[n] = 4\left(\frac{1}{2}\right)^n u[n] - 2\left(\frac{1}{4}\right)^n u[n] - 3(n+1)\left(\frac{1}{4}\right)^n u[n].$$

(iii) We have

$$X(e^{j\omega}) = 2\pi\sum_{k=-\infty}^{\infty}\delta(\omega-(2k+1)\pi).$$

Therefore,

$$Y(e^{j\omega}) = \left[2\pi\sum_{k=-\infty}^{\infty}\delta(\omega-(2k+1)\pi)\right]\left[\frac{1}{1-\frac{1}{2}e^{-j\omega}}\right] = \frac{4\pi}{3}\sum_{k=-\infty}^{\infty}\delta(\omega-(2k+1)\pi)$$

Taking the inverse Fourier transform, we obtain

$$x[n] = \frac{2}{3}(-1)^n.$$

(b) Given

$$h[n]=\frac{1}{2}\left(\frac{1}{2}e^{j\pi/2}\right)^n u[n]+\frac{1}{2}\left(\frac{1}{2}e^{-j\pi/2}\right)^n u[n],$$

we obtain

$$H(e^{j\omega})=\frac{1/2}{1-\frac{1}{2}e^{j\pi/2}e^{-j\omega}}+\frac{1/2}{1-\frac{1}{2}e^{-j\pi/2}e^{-j\omega}}.$$

(i) We have

$$X(e^{j\omega})=\frac{1}{1-\frac{1}{2}e^{-j\omega}}.$$

Therefore,

$$\begin{aligned}Y(e^{j\omega}) &= \left[\frac{1/2}{1-\frac{1}{2}e^{j\pi/2}e^{-j\omega}}+\frac{1/2}{1-\frac{1}{2}e^{-j\pi/2}e^{-j\omega}}\right]\left[\frac{1}{1-\frac{1}{2}e^{-j\omega}}\right]\\ &= \frac{A}{1-(1/2)e^{j\pi/2}e^{-j\omega}}+\frac{B}{1-(1/2)e^{-j\omega}}+\frac{C}{1-(1/2)e^{-j\pi/2}e^{-j\omega}}\end{aligned}$$

where $A=-j/[2(1-j)]$, $B=1/2$, and $C=1/[2(1+j)]$. Therefore,

$$y[n]=\frac{-j}{2(1-j)}\left(\frac{j}{2}\right)^n u[n]+\frac{1}{2(1+j)}\left(-\frac{j}{2}\right)^n u[n]+\frac{1}{2}\left(\frac{1}{2}\right)^n u[n].$$

(ii) In this case,

$$y[n]=\frac{\cos(\pi n/2)}{3}\left[4-(\tfrac{1}{2})^n\right]u[n].$$

(c) Here,

$$\begin{aligned}Y(e^{j\omega}) &= X(e^{j\omega})H(e^{j\omega})=-3e^{-2j\omega}-e^{j\omega}+1-2e^{-j2\omega}\\ &\quad +6e^{-j\omega}+2e^{-j2\omega}-2e^{-j3\omega}+4e^{-j5\omega}\\ &\quad +3e^{j5\omega}+e^{j4\omega}-e^{j3\omega}+2e^{j\omega}\end{aligned}$$

Therefore,

$$\begin{aligned}y[n] &= 3\delta[n+5]+\delta[n+4]-\delta[n+3]-3\delta[n+2]\\ &\quad +\delta[n+1]+\delta[n]+6\delta[n-1]-2\delta[n-3]+4\delta[n-5].\end{aligned}$$

5.30. (a) The frequency response of the system is as shown in Figure S5.30.

(b) The Fourier transform $X(e^{j\omega})$ of $x[n]$ is as shown in Figure S5.30.

(i) The frequency response $H(e^{j\omega})$ is as shown in Figure S5.30. Therefore, $y[n]=\sin(\pi n/8)$.

(ii) The frequency response $H(e^{j\omega})$ is as shown in Figure S5.30. Therefore, $y[n]=2\sin(\pi n/8)-2\cos(\pi n/4)$.

(iii) The frequency response $H(e^{j\omega})$ is as shown in Figure S5.30. Therefore, $y[n]=\frac{1}{6}\sin(\pi n/8)-\frac{1}{4}\cos(\pi n/4)$.

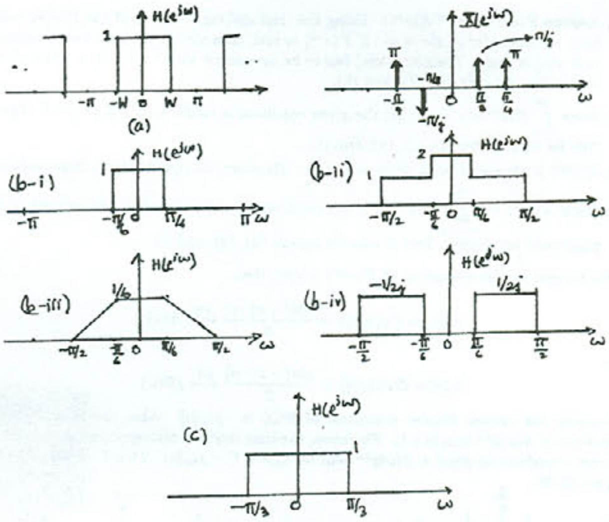

Figure S5.30

(iv) The frequency response $H(e^{j\omega})$ is as shown in Figure S5.30. Therefore, $y[n]=-\sin(\pi n/4)$.

(c) The frequency response $H(e^{j\omega})$ is as shown in Figure S5.30.

(i) The signal $x[n]$ is periodic with period 8. The Fourier series coefficients of the signal are

$$a_k=\frac{1}{8}\sum_{n=0}^{7}x[n]e^{-j(2\pi/8)kn}.$$

The Fourier transform of this signal is

$$X(e^{j\omega})=\sum_{k=-\infty}^{\infty}2\pi a_k\delta(\omega-2\pi k/8).$$

The Fourier transform $Y(e^{j\omega})$ of the output is $Y(e^{j\omega})=X(e^{j\omega})H(e^{j\omega})$. Therefore,

$$Y(e^{j\omega})=2\pi[a_0\delta(\omega)+a_1\delta(\omega-\pi/4)+a_{-1}\delta(\omega+\pi/4)]$$

in the range $0\le|\omega|\le\pi$. Therefore,

$$y[n]=a_0+a_1e^{j\pi n/4}+a_{-1}e^{-j\pi/4}=\frac{5}{8}+[(1/4)+(1/2)(1/\sqrt{2})]\cos(\pi n/4).$$

(ii) The signal $x[n]$ is periodic with period 8. The Fourier series coefficients of the signal are

$$a_k=\frac{1}{8}\sum_{n=0}^{7}x[n]e^{-j(2\pi/8)kn}.$$

The Fourier transform of this signal is

$$X(e^{j\omega})=\sum_{k=-\infty}^{\infty}2\pi a_k\delta(\omega-2\pi k/8).$$

The Fourier transform $Y(e^{j\omega})$ of the output is $Y(e^{j\omega})=X(e^{j\omega})H(e^{j\omega})$. Therefore,

$$Y(e^{j\omega})=2\pi[a_0\delta(\omega)+a_1\delta(\omega-\pi/4)+a_{-1}\delta(\omega+\pi/4)]$$

in the range $0\le|\omega|\le\pi$. Therefore,

$$y[n]=a_0+a_1e^{j\pi n/4}+a_{-1}e^{-j\pi/4}=\frac{1}{8}+\frac{1}{4}\cos(\pi n/4).$$

(iii) Again in this case, the Fourier transform $X(e^{j\omega})$ of the signal $x[n]$ is of the form shown in part (i). Therefore,

$$y[n]=a_0+a_1e^{j\pi n/4}+a_{-1}e^{-j\pi/4}=\frac{1}{8}+[(1/4)-(1/2)(1/\sqrt{2})]\cos(\pi n/4).$$

(iv) In this case, the output is

$$y[n]=h[n]*x[n]=\frac{\sin[\pi/3(n-1)]}{\pi(n-1)}+\frac{\sin[\pi/3(n+1)]}{\pi(n+1)}.$$

5.31. (a) From the given information, it is clear that when the input to the system is a complex exponential of frequency ω_0, the output is a complex exponential of the same frequency but scaled by the $|\omega_0|$. Therefore, the frequency response of the system is

$$H(e^{j\omega})=|\omega|,\qquad \text{for } 0\le|\omega|\le\pi.$$

(b) Taking the inverse Fourier transform of the frequency response, we obtain

$$\begin{aligned}h[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi}H(e^{j\omega})e^{j\omega n}d\omega\\ &= \frac{1}{2\pi}\int_{-\pi}^{0}-\omega e^{j\omega n}d\omega+\frac{1}{2\pi}\int_{0}^{\pi}\omega e^{j\omega n}d\omega\\ &= \frac{1}{\pi}\int_{0}^{\pi}\omega\cos(\omega n)d\omega\\ &= \frac{1}{\pi}\left[\frac{\cos(n\pi)-1}{n^2}\right]\end{aligned}$$

5.32. From the synthesis equation (5.8) we have

$$\left[\frac{1}{2\pi}\int_{-\pi}^{\pi}H_1(e^{j\omega})d\omega\right]\left[\frac{1}{2\pi}\int_{-\pi}^{\pi}H_2(e^{j\omega})d\omega\right]=h_1[0]h_2[0].$$

Also, since

$$h_1[n]*h_2[n]\overset{FT}{\longleftrightarrow}H_1(e^{j\omega})H_2(e^{j\omega}),$$

we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}H_1(e^{j\omega})H_2(e^{j\omega})d\omega=[h_1[n]*h_2[n]]_{n=0}.$$

Therefore, the question here amounts to asking whether it is true that

$$h_1[0]h_2[0]=[h_1[n]*h_2[n]]_{n=0}.$$

Since $h_1[n]$ and $h_2[n]$ are causal, this is indeed true.

5.33. (a) Taking the Fourier transform of the given difference equation we have

$$H(e^{j\omega})=\frac{Y(e^{j\omega})}{X(e^{j\omega})}=\frac{1}{1+\frac{1}{2}e^{-j\omega}}.$$

(b) The Fourier transform of the output will be $Y(e^{j\omega})=X(e^{j\omega})H(e^{j\omega})$.

(i) In this case

$$X(e^{j\omega})=\frac{1}{1-\frac{1}{2}e^{-j\omega}}.$$

Therefore,

$$\begin{aligned}Y(e^{j\omega}) &= \left[\frac{1}{1-\frac{1}{2}e^{-j\omega}}\right]\left[\frac{1}{1+\frac{1}{2}e^{-j\omega}}\right]\\ &= \frac{1/2}{1-\frac{1}{2}e^{-j\omega}}+\frac{1/2}{1+\frac{1}{2}e^{-j\omega}}\end{aligned}$$

Taking the inverse Fourier transform, we obtain

$$y[n]=\frac{1}{2}\left(\frac{1}{2}\right)^n u[n]+\frac{1}{2}\left(-\frac{1}{2}\right)^n u[n].$$

(ii) In this case

$$X(e^{j\omega})=\frac{1}{1+\frac{1}{2}e^{-j\omega}}.$$

Therefore,

$$Y(e^{j\omega})=\left[\frac{1}{1-\frac{1}{2}e^{-j\omega}}\right]^2.$$

Taking the inverse Fourier transform, we obtain

$$y[n]=(n+1)\left(-\frac{1}{2}\right)^n u[n].$$

(iii) In this case

$$X(e^{j\omega}) = 1 + \frac{1}{2}e^{-j\omega}.$$

Therefore,

$$Y(e^{j\omega}) = 1.$$

Taking the inverse Fourier transform, we obtain

$$y[n] = \delta[n].$$

(iv) In this case

$$X(e^{j\omega}) = 1 - \frac{1}{2}e^{-j\omega}.$$

Therefore,

$$\begin{aligned} Y(e^{j\omega}) &= \left[1 - \frac{1}{2}e^{-j\omega}\right]\left[\frac{1}{1+\frac{1}{2}e^{-j\omega}}\right] \\ &= -1 + \frac{2}{1+\frac{1}{2}e^{-j\omega}} \end{aligned}$$

Taking the inverse Fourier transform, we obtain

$$y[n] = -\delta[n] + 2\left(-\frac{1}{2}\right)^n u[n].$$

(c) (i) We have

$$\begin{aligned} Y(e^{j\omega}) &= \left[\frac{1-\frac{1}{4}e^{-j\omega}}{1+\frac{1}{2}e^{-j\omega}}\right]\left[\frac{1}{1+\frac{1}{2}e^{-j\omega}}\right] \\ &= \frac{1}{(1+\frac{1}{2}e^{-j\omega})^2} - \frac{\frac{1}{4}e^{-j\omega}}{(1+\frac{1}{2}e^{-j\omega})^2} \end{aligned}$$

Taking the inverse Fourier transform, we obtain

$$y[n] = (n+1)\left(-\frac{1}{2}\right)^n u[n] - \frac{1}{4}n\left(-\frac{1}{2}\right)^{n-1} u[n-1].$$

(ii) We have

$$\begin{aligned} Y(e^{j\omega}) &= \left[\frac{1+\frac{1}{2}e^{-j\omega}}{1-\frac{1}{4}e^{-j\omega}}\right]\left[\frac{1}{1+\frac{1}{2}e^{-j\omega}}\right] \\ &= \frac{1}{1-\frac{1}{4}e^{-j\omega}} \end{aligned}$$

Taking the inverse Fourier transform, we obtain

$$y[n] = \left(\frac{1}{4}\right)^n u[n].$$

(iii) We have

$$\begin{aligned} Y(e^{j\omega}) &= \left[\frac{1}{(1+\frac{1}{2}e^{-j\omega})(1-\frac{1}{4}e^{-j\omega})}\right]\left[\frac{1}{1+\frac{1}{2}e^{-j\omega}}\right] \\ &= \frac{2/3}{(1+\frac{1}{2}e^{-j\omega})^2} + \frac{2/9}{1+\frac{1}{2}e^{-j\omega}} + \frac{1/9}{1-\frac{1}{4}e^{-j\omega}} \end{aligned}$$

Taking the inverse Fourier transform, we obtain

$$y[n] = \frac{2}{3}(n+1)\left(-\frac{1}{2}\right)^n u[n] + \frac{2}{9}\left(-\frac{1}{2}\right)^n u[n] + \frac{1}{9}\left(\frac{1}{4}\right)^n u[n].$$

(iv) We have

$$\begin{aligned} Y(e^{j\omega}) &= \left[1 + 2e^{-3j\omega}\right]\left[\frac{1}{1+\frac{1}{2}e^{-j\omega}}\right] \\ &= \frac{1}{1+\frac{1}{2}e^{-j\omega}} + \frac{2e^{-3j\omega}}{1+\frac{1}{2}e^{-j\omega}} \end{aligned}$$

Taking the inverse Fourier transform, we obtain

$$y[n] = \left(-\frac{1}{2}\right)^n u[n] + 2\left(-\frac{1}{2}\right)^{n-3} u[n-3].$$

5.34. (a) Since the two systems are cascaded, the frequency response of the overall system is

$$\begin{aligned} H(e^{j\omega}) &= H_1(e^{j\omega})H_2(e^{j\omega}) \\ &= \frac{2 - e^{-j\omega}}{1 + \frac{1}{8}e^{-j3\omega}} \end{aligned}$$

Therefore, the Fourier transforms of the input and output of the overall system are related by

$$\frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{2 - e^{-j\omega}}{1 + \frac{1}{8}e^{-j3\omega}}.$$

Cross-multiplying and taking the inverse Fourier transform, we get

$$y[n] + \frac{1}{8}y[n-3] = 2x[n] - x[n-1].$$

(b) We may rewrite the overall frequency response as

$$H(e^{j\omega}) = \frac{4/3}{1+\frac{1}{2}e^{-j\omega}} + \frac{(1+j\sqrt{3})/3}{1-\frac{1}{2}e^{j120}e^{-j\omega}} + \frac{(1-j\sqrt{3})/3}{1-\frac{1}{2}e^{-j120}e^{-j\omega}}.$$

Taking the inverse Fourier transform we get

$$h[n] = \frac{4}{3}\left(-\frac{1}{2}\right)^n u[n] + \frac{1+j\sqrt{3}}{3}\left(\frac{1}{2}e^{j120}\right)^n u[n] + \frac{1-j\sqrt{3}}{3}\left(\frac{1}{2}e^{-j120}\right)^n u[n].$$

5.35. (a) Taking the Fourier transform of both sides of the given difference equation we obtain

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{b + e^{-j\omega}}{1 - ae^{-j\omega}}.$$

In order for $|H(e^{j\omega})|$ to be one, we must ensure that

$$\begin{aligned} |b + e^{-j\omega}| &= |1 - ae^{-j\omega}| \\ 1 + b^2 + 2b\cos\omega &= 1 + a^2 - 2a\cos\omega \end{aligned}$$

This is possible only if $b = -a$.

(b) The plot is as shown Figure S5.35.

(c) The plot is as shown Figure S5.35.

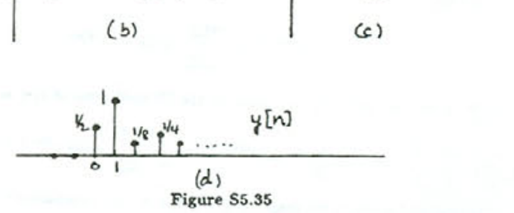

Figure S5.35

(d) When $a = -\frac{1}{2}$,

$$H(e^{j\omega}) = \frac{\frac{1}{2} + e^{-j\omega}}{1 + \frac{1}{2}e^{-j\omega}}.$$

Also,

$$X(e^{j\omega}) = \frac{1}{1 - \frac{1}{2}e^{-j\omega}}.$$

Therefore,

$$\begin{aligned} Y(e^{j\omega}) &= \frac{\frac{1}{2} + e^{-j\omega}}{(1+\frac{1}{2}e^{-j\omega})(1-\frac{1}{2}e^{-j\omega})} \\ &= \frac{5/4}{1-\frac{1}{2}e^{-j\omega}} - \frac{3/4}{1+\frac{1}{2}e^{-j\omega}} \end{aligned}$$

Taking the inverse Fourier transform we obtain

$$y[n] = \frac{5}{4}\left(\frac{1}{2}\right)^n u[n] - \frac{3}{4}\left(-\frac{1}{2}\right)^n u[n].$$

This is as sketched in Figure S5.35.

5.36. (a) The frequency responses are related by the following expression:

$$G(e^{j\omega}) = \frac{1}{H(e^{j\omega})}.$$

(b) (i) Here, $H(e^{j\omega}) = 1 - \frac{1}{4}e^{-j\omega}$. Therefore, $G(e^{j\omega}) = 1/(1 - \frac{1}{4}e^{-j\omega})$ and $g[n] = (\frac{1}{4})^n u[n]$. Since

$$G(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1}{1 - \frac{1}{4}e^{-j\omega}},$$

the difference equation relating the input $x[n]$ and output $y[n]$ is

$$y[n] - \frac{1}{4}y[n-1] = x[n].$$

(ii) Here, $H(e^{j\omega}) = 1/(1 + \frac{1}{2}e^{-j\omega})$. Therefore, $G(e^{j\omega}) = 1 + \frac{1}{2}e^{-j\omega}$ and $g[n] = \delta[n] + \frac{1}{2}\delta[n-1]$. Since

$$G(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = 1 + \frac{1}{2}e^{-j\omega},$$

the difference equation relating the input $x[n]$ and output $y[n]$ is

$$y[n] = x[n] + \frac{1}{2}x[n-1].$$

(iii) Here, $H(e^{j\omega}) = (1 - \frac{1}{4}e^{-j\omega})/(1 + \frac{1}{2}e^{-j\omega})$. Therefore, $G(e^{j\omega}) = (1 + \frac{1}{2}e^{-j\omega})/(1 - \frac{1}{4}e^{-j\omega})$ and $g[n] = (\frac{1}{4})^n u[n] + \frac{1}{2}(\frac{1}{4})^{n-1}u[n-1]$. Since

$$G(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1 + \frac{1}{2}e^{-j\omega}}{1 - \frac{1}{4}e^{-j\omega}},$$

the difference equation relating the input $x[n]$ and output $y[n]$ is

$$y[n] - \frac{1}{4}y[n-1] = x[n] + \frac{1}{2}x[n-1].$$

(iv) Here, $H(e^{j\omega}) = (1 - \frac{1}{4}e^{-j\omega} - \frac{1}{8}e^{-2j\omega})/(1 + \frac{5}{4}e^{-j\omega} - \frac{1}{8}e^{-2j\omega})$. Therefore, $G(e^{j\omega}) = (1 + \frac{5}{4}e^{-j\omega} - \frac{1}{8}e^{-2j\omega})/(1 - \frac{1}{4}e^{-j\omega} - \frac{1}{8}e^{-2j\omega})$. Therefore,

$$G(e^{j\omega}) = 1 + \frac{2}{1 - (1/2)e^{-j\omega}} - \frac{2}{1 + (1/4)e^{-j\omega}}$$

and

$$g[n] = \delta[n] + 2\left(\frac{1}{2}\right)^n u[n] - 2\left(-\frac{1}{4}\right)^n u[n].$$

Since

$$G(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{(1 + \frac{5}{4}e^{-j\omega} - \frac{1}{8}e^{-2j\omega})}{(1 - \frac{1}{4}e^{-j\omega} - \frac{1}{8}e^{-2j\omega})},$$

the difference equation relating the input $x[n]$ and output $y[n]$ is

$$y[n] - \frac{1}{4}y[n-1] - \frac{1}{8}y[n-1] = x[n] + \frac{5}{4}x[n-1] - \frac{1}{8}x[n-2].$$

(v) Here, $H(e^{j\omega}) = (1-\frac{1}{2}e^{-j\omega})/(1+\frac{5}{4}e^{-j\omega}-\frac{1}{8}e^{-2j\omega})$. Therefore, $G(e^{j\omega}) = (1+\frac{5}{4}e^{-j\omega}-\frac{1}{8}e^{-2j\omega})/(1-\frac{1}{2}e^{-j\omega})$ Since

$$G(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{(1+\frac{5}{4}e^{-j\omega}-\frac{1}{8}e^{-2j\omega})}{(1-\frac{1}{2}e^{-j\omega})},$$

the difference equation relating the input $x[n]$ and output $y[n]$ is

$$y[n] - \frac{1}{2}y[n-1] = x[n] + \frac{5}{4}x[n-1] - \frac{1}{8}x[n-2].$$

(vi) Here, $H(e^{j\omega}) = 1/(1+\frac{5}{4}e^{-j\omega}-\frac{1}{8}e^{-2j\omega})$. Therefore, $G(e^{j\omega}) = (1+\frac{5}{4}e^{-j\omega}-\frac{1}{8}e^{-2j\omega})$ Since

$$G(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = (1+\frac{5}{4}e^{-j\omega}-\frac{1}{8}e^{-2j\omega})$$

we have

$$g[n] = \delta[n] + \frac{5}{4}\delta[n-1] - \frac{1}{8}\delta[n-2]$$

and the difference equation relating the input $x[n]$ and output $y[n]$ is

$$y[n] = x[n] + \frac{5}{4}x[n-1] - \frac{1}{8}x[n-2].$$

(c) The frequency response of the given system is

$$H(e^{j\omega}) = \frac{e^{-j\omega} - \frac{1}{2}e^{-2j\omega}}{1+e^{-j\omega}+\frac{1}{4}e^{-2j\omega}}.$$

The frequency response of the inverse system is

$$G(e^{j\omega}) = \frac{1}{H(e^{j\omega})} = \frac{e^{j\omega}+1+\frac{1}{4}e^{-j\omega}}{1-\frac{1}{2}e^{-j\omega}}.$$

Therefore,

$$g[n] = \left(\frac{1}{2}\right)^{n+1}u[n+1] + \left(\frac{1}{2}\right)^{n}u[n] + \frac{1}{4}\left(\frac{1}{2}\right)^{n-1}u[n-1].$$

Clearly, $g[n]$ is not a causal impulse response.

If we delay this impulse response by 1 sample, then it becomes causal. Furthermore, the output of the inverse system will then be $x[n-1]$. The impulse response of this causal system is

$$g_1[n] = g[n-1] = \left(\frac{1}{2}\right)^{n}u[n] + \left(\frac{1}{2}\right)^{n-1}u[n-1] + \frac{1}{4}\left(\frac{1}{2}\right)^{n-2}u[n-2].$$

5.37. Given that

$$x[n] \overset{FT}{\longleftrightarrow} X(e^{j\omega}).$$

(i) Since

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n},$$

we may write

$$X^*(e^{-j\omega}) = \sum_{n=-\infty}^{\infty} x^*[n]e^{-j\omega n}.$$

Comparing with the analysis eq. (5.9), we conclude that

$$x^*[n] \overset{FT}{\longleftrightarrow} X^*(e^{-j\omega}).$$

Therefore,

$$\mathcal{R}e\{x[n]\} = \frac{x[n]+x^*[n]}{2} \overset{FT}{\longleftrightarrow} \frac{X(e^{j\omega})+X^*(e^{-j\omega})}{2}.$$

(ii) Since

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n},$$

we may write

$$X(e^{-j\omega}) = \sum_{n=-\infty}^{\infty} x[-n]e^{-j\omega n}.$$

Therefore,

$$x[-n] \overset{FT}{\longleftrightarrow} X(e^{-j\omega}).$$

From the previous part we know that

$$x^*[n] \overset{FT}{\longleftrightarrow} X^*(e^{-j\omega}).$$

Therefore, putting these two statements together we get

$$x^*[-n] \overset{FT}{\longleftrightarrow} X^*(e^{j\omega}).$$

(iii) From our previous results we know that

$$\mathcal{E}v\{x[n]\} = \frac{x[n]+x[-n]}{2} \overset{FT}{\longleftrightarrow} \frac{X(e^{j\omega})+X(e^{-j\omega})}{2}.$$

5.38. From the synthesis equation (5.8) we obtain

$$\begin{aligned} x[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega \\ &= \frac{1}{2\pi}\int_{0}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega + \frac{1}{2\pi}\int_{0}^{\pi} X(e^{-j\omega})e^{-j\omega n}d\omega \end{aligned}$$

Since $x[n]$ is real, $X(e^{-j\omega}) = X^*(e^{j\omega})$. Therefore,

$$\begin{aligned} x[n] &= \frac{1}{2\pi}\int_{0}^{\pi} \mathcal{R}e\{X(e^{j\omega})\}\{e^{\omega n}+e^{-j\omega n}\}d\omega + \frac{j}{2\pi}\int_{0}^{\pi} \mathcal{I}m\{X(e^{j\omega})\}\{e^{\omega n}-e^{-j\omega n}\}d\omega \\ &= \frac{1}{\pi}\int_{0}^{\pi} \mathcal{R}e\{X(e^{j\omega})\}2\cos(\omega n)d\omega - \frac{}{\pi}\int_{0}^{\pi} \mathcal{I}m\{X(e^{j\omega})\}\sin(\omega n)d\omega \end{aligned}$$

Therefore,

$$B(\omega) = \frac{1}{\pi}\mathcal{R}e\{X(e^{j\omega})\}\cos(\omega n), \quad \text{and} \quad -\frac{1}{\pi}\mathcal{I}m\{X(e^{j\omega})\}\sin(\omega n).$$

5.39. Let $y[n] = x[n] * h[n]$. Then

$$\begin{aligned} Y(e^{j\omega}) &= \sum_{n=-\infty}^{\infty}\{x[n]*h[n]\}e^{-j\omega n} \\ &= \sum_{n=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} x[k]h[n-k]e^{-j\omega n} \\ &= \sum_{k=-\infty}^{\infty} x[k]\sum_{n=-\infty}^{\infty} h[n-k]e^{-j\omega n} \\ &= \sum_{k=-\infty}^{\infty} x[k]e^{-j\omega k}H(e^{j\omega}) \\ &= H(e^{j\omega})\sum_{k=-\infty}^{\infty} x[k]e^{-j\omega k} \\ &= H(e^{j\omega})X(e^{j\omega}) \end{aligned}$$

5.40. Let $y[n] = x[n] * h[n]$. Then using the convolution sum

$$y[0] = \sum_{k=-\infty}^{\infty} x[k]h[-k] \qquad \text{(S5.40-1)}$$

Using the convolution property of the Fourier transform,

$$y[0] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})H(e^{j\omega})d\omega \qquad \text{(S5.40-2)}$$

Now let $h[n] = x^*[-n]$. Then $H(e^{j\omega}) = X^*(e^{j\omega})$. Substituting in the right-hand sides of equations (S5.40-1) and (S5.40-2) and equating them,

$$\sum_{k=-\infty}^{\infty} x[k]x^*[k] = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(e^{j\omega})X^*(e^{j\omega})d\omega.$$

Therefore,

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega.$$

Now let $h[n] = z^*[-n]$. Then $H(e^{j\omega}) = Z^*(e^{j\omega})$. Substituting in the right-hand sides of equations (S5.40-1) and (S5.40-2) and equating them,

$$\sum_{k=-\infty}^{\infty} x[k]z^*[k] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})Z^*(e^{j\omega})d\omega.$$

5.41. (a) The Fourier transform $X(e^{j\omega})$ of the signal $x[n]$ is

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} = \sum_{n_0}^{n_0+N-1} x[n]e^{-j\omega n}.$$

Therefore,

$$X(e^{j2\pi k/N}) = \sum_{n=n_0}^{n_0+N-1} x[n]e^{-j(2\pi/N)kn}. \qquad \text{(S5.41-1)}$$

Now, we may write the expression for the FS coefficients of $\tilde{x}[n]$ as

$$a_k = \frac{1}{N}\sum_{<N>} \tilde{x}[n]e^{-j(2\pi/N)kn} = \frac{1}{N}\sum_{n=n_0}^{n_0+N-1} x[n]e^{-j(2\pi/N)kn}.$$

(Because $x[n] = \tilde{x}[n]$ in the range $n_0 \le n \le n_0+N-1$). Comparing the above equation with eq. (S5.41-1), we get

$$a_k = \frac{1}{N}X(e^{j2\pi k/N}).$$

(b) (i) From the given information,

$$\begin{aligned} X(e^{j\omega}) &= 1 + e^{-j\omega} + e^{-2j\omega} + e^{-3j\omega} \\ &= e^{-j(3/2)\omega}\{e^{j(3/2)\omega} + e^{-j(3/2)\omega}\} + e^{-j(3/2)\omega}\{e^{j(1/2)\omega} + e^{-j(1/2)\omega}\} \\ &= 2e^{-j(3/2)\omega}\{\cos(3\omega/2) + \cos(\omega/2)\} \end{aligned}$$

(ii) From part (a),

$$a_k = \frac{1}{N}X(e^{j2\pi k/N}) = \frac{1}{N}2e^{-j(3/2)2\pi k/N}\{\cos(6\pi k/(2N)) + \cos(\pi k/N)\}.$$

5.42. (a) $P(e^{j\omega}) = 2\pi\delta(\omega - \omega_0)$ for $|\omega| < \pi$. This is as shown in Figure S5.42.

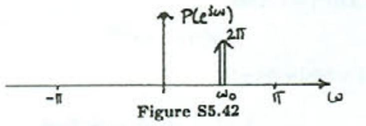

Figure S5.42

(b) From the multiplication property of the Foutier transform we have

$$\begin{aligned} G(e^{j\omega}) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})P(e^{j(\omega-\theta)})d\theta \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})2\pi\delta(\omega-\theta-\omega_0)d\theta \\ &= X(e^{j(\omega-\omega_0)}) \end{aligned}$$

5.43. (a) Using the frequency shift and linearity properties,

$$V(e^{j\omega}) = \frac{X(e^{j(\omega-\pi)}) + X(e^{j\omega})}{2}.$$

(b) Let $y[n] = v[2n]$. Then

$$Y(e^{j\omega}) = \sum_{n=-\infty}^{\infty} v[2n]e^{-j\omega n}.$$

Since the odd-indexed samples of $v[n]$ are zero, we may put $m = 2n$ in the above equation to get

$$Y(e^{j\omega}) = \sum_{m=-\infty}^{\infty} v[m]e^{-j\omega m/2} = V(e^{j\omega/2}).$$

(Note that the substitution of n by $2m$ is valid *only* if the odd-indexed samples in the summation are zero.)

(c) $x[2n]$ is a new sequence which consists of only the even indexed samples of $x[n]$. $v[n]$ is a sequence whose even-indexed samples are equal to $x[n]$. The odd-indexed samples of $v[n]$ are zero. $v[2n]$ is a new sequence which consists of only the even indexed samples of $v[n]$. This implies that $v[2n]$ is a sequence which consists of only the even indexed samples of $x[n]$. This idea is illustrated in Figure S5.43.

From part (a),

$$G(e^{j\omega}) = \frac{X(e^{j(\omega/2-\pi)}) + X(e^{j\omega/2})}{2}.$$

5.44. (a) The signal $x_1[n]$ is as shown in Figure S5.44.

(i) Taking the inverse Fourier transform, the signal $x_2[n]$ is

$$x_2[n] = x_1[n+1].$$

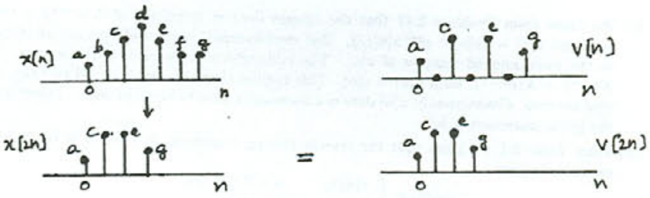

Figure S5.43

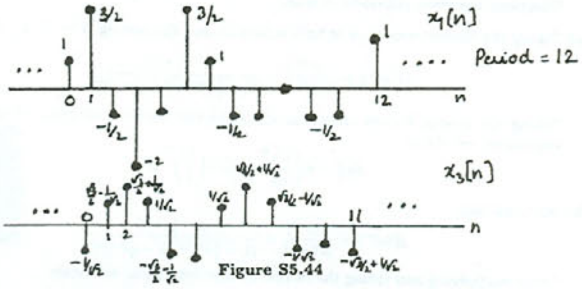

Figure S5.44

(ii) Taking the inverse Fourier transform, the signal $x_3[n]$ is

$$x_2[n] = x_1[n-3/2] = \sin(\pi n/3) + \sin(\pi n/2)\cos(3\pi/4) - \cos(\pi n/2)\sin(3\pi/4).$$

This is as shown in Figure S5.44.

(b) From part (a),

$$x_2[n] = x_1[n+1] = w(nT+T).$$

Also,

$$x_3[n] = x_1[n-3/2] = w(nT - 3T/2).$$

Therefore, $\alpha = -1$ and $\beta = 3/2$.

5.45. From the Fourier transform analysis equation

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}.$$

(a) Comparing the equation for $x_1(t)$ with the above equation, we obtain

$$x_1(t) = X(e^{-j(2\pi/10)t}).$$

Therefore $x_1(t)$ is as shown in Figure S5.45.

(b) Comparing the equation for $x_2(t)$ with the equation for $X(e^{j\omega})$, we obtain

$$x_2(t) = X(e^{j(2\pi/10)t}) = x_1(-t).$$

Therefore $x_2(t)$ is as shown in Figure S5.45.

(c) We know that $\mathcal{O}d\{x[n]\} = (x[n] - x[-n])/2$. Therefore,

$$\frac{X(e^{j\omega}) - X(e^{-j\omega})}{2} = \sum_{n=-\infty}^{\infty} \mathcal{O}d\{x[n]\}e^{-j\omega n}.$$

Comparing this with the given equation for $x_3(t)$, we obtain

$$x_3(t) = \frac{X(e^{-j(2\pi/8)t}) - X(e^{j(2\pi/8)t})}{2}.$$

Therefore $x_3(t)$ is as shown in Figure S5.45.

(d) We know that $\mathcal{R}e\{x[n]\} = (x[n] + x^*[n])/2$. Therefore,

$$\frac{X(e^{j\omega}) - X^*(e^{-j\omega})}{2} = \sum_{n=-\infty}^{\infty} \mathcal{R}e\{x[n]\}e^{-j\omega n}.$$

Comparing this with the given equation for $x_4(t)$, we obtain

$$x_4(t) = \frac{X(e^{-j(2\pi/6)t}) + X^*(e^{j(2\pi/6)t})}{2}.$$

Therefore $x_4(t)$ is as shown in the Figure S5.45.

5.46. (a) Let $x[n] = \alpha^n u[n]$. Then $X(e^{j\omega}) = \frac{1}{1-\alpha e^{-j\omega}}$. Using the differentiation in frequency property,

$$n\alpha^n u[n] \stackrel{FT}{\longleftrightarrow} j\frac{dX(e^{j\omega})}{d\omega} = \frac{\alpha e^{-j\omega}}{(1-\alpha e^{-j\omega})^2}.$$

Therefore,

$$(n+1)\alpha^n u[n] \stackrel{FT}{\longleftrightarrow} j\frac{dX(e^{j\omega})}{d\omega} + X(e^{j\omega}) = \frac{1}{(1-\alpha e^{-j\omega})^2}.$$

(b) From part (a), it is clear that the result is true for $r = 1$ and $r = 2$. Let us assume that it is also true for $k = r - 1$. We will now attempt to prove that the result is true for $k = r$. We have

$$x_{r-1}[n] = \frac{(n+r-2)!}{n!(r-2)!}\alpha^n u[n] \stackrel{FT}{\longleftrightarrow} X_{r-1}(e^{j\omega}) = \frac{1}{(1-\alpha e^{-j\omega})^{r-1}}.$$

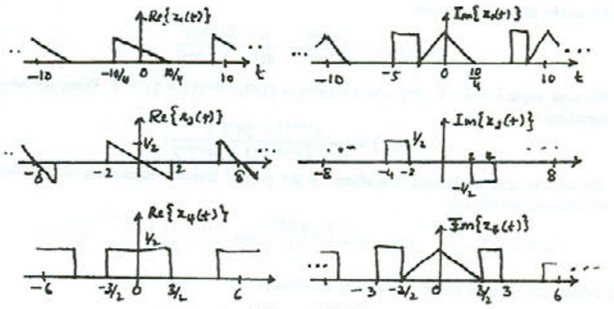

Figure S5.45

From the differentiation in frequency property,

$$nx_{r-1}[n] \stackrel{FT}{\longleftrightarrow} \frac{\alpha(r-1)e^{-j\omega}}{(1-\alpha e^{-j\omega})^{r-1}}.$$

Therefore,

$$\frac{(n+1)x_{r-1}[n+1]}{\alpha(r-1)} \stackrel{FT}{\longleftrightarrow} \frac{1}{(1-\alpha e^{-j\omega})^r}.$$

The left hand side of the above expression is

$$\frac{(n+1)x_{r-1}[n+1]}{\alpha(r-1)} = \frac{(n+r-1)!}{n!(r-1)!}\alpha^n u[n] = x_r[n].$$

Therefore, we have shown that the result is valid for r if it is valid for $r-1$. Since, we know that the result is valid for $r = 2$, we may conclude that it is valid for $r = 3$, $r = 4$, and so on.

5.47. (a) If $X(e^{j\omega}) = X(e^{j(\omega-1)})$ then $X(e^{j\omega})$ is periodic with a period of 1. But we already know that $X(e^{j\omega})$ is periodic with a period of 2π. This is only possible if $X(e^{j\omega})$ is a constant for all ω. This implies that $x[n]$ is of the form $k\delta[n]$ where k is a constant. Therefore, the given statemet is true.

(b) If $X(e^{j\omega}) = X(e^{j(\omega-\pi)})$ then $X(e^{j\omega})$ is periodic with a period of π. We also know that $X(e^{j\omega})$ is periodic with a period of 2π. Both these conditions can be satisfied even if $X(e^{j\omega})$ has some arbitrary shape in the region $0 \le |\omega| \le \pi/2$. Therefore, $X(e^{j\omega})$ need not necessarily be a constant. Consequently, $x[n]$ need not be just an impulse. Therefore, the given statement is false.

(c) We know from Problem 5.43 that the inverse Fourier transform of $X(e^{j\omega/2})$ is the sequence $v[n] = (x[n] + e^{j\pi n}x[n])/2$. The even-indexed samples of $v[n]$ are identical to the even-indexed samples of $x[n]$. The odd-indexed samples of $v[n]$ are zero. If $X(e^{j\omega}) = X(e^{j\omega/2})$, then $x[n] = v[n]$. This implies that the even-indexed samples of $x[n]$ are zero. Consequently, $x[n]$ does not necessarily have to be an impulse. Therefore, the given statement is false.

(d) From Table 5.1 we know that the inverse Fourier transform of $X(e^{j2\omega})$ is the time-expanded signal

$$x_{(2)}[n] = \begin{cases} x[n/2], & n = 0, \pm 2, \pm 4, \cdots \\ 0, & \text{otherwise} \end{cases}$$

If $X(e^{j\omega}) = X(e^{j2\omega})$, then $x[n] = x_{(2)}[n]$. This is possible only if $x[n]$ is an impulse. Therefore, the given statement is true.

5.48. (a) Taking the Fourier transform of both equations and eliminating $W(e^{j\omega})$, we obtain

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{3 - \frac{1}{2}e^{-j\omega}}{(1 - \frac{1}{2}e^{-j\omega})(1 - \frac{1}{4}e^{-j\omega})}.$$

Taking the inverse Fourier transform of the partial fraction expansion of the above expression, we obtain

$$h[n] = 4\left(\frac{1}{2}\right)^n u[n] - \left(\frac{1}{4}\right)^n u[n].$$

(b) We know that

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{3 - \frac{1}{2}e^{-j\omega}}{(1 - \frac{1}{2}e^{-j\omega})(1 - \frac{1}{4}e^{-j\omega})}.$$

Cross-multiplying and taking the inverse Fourier transform, we obtain

$$y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = 3x[n] - \frac{1}{2}x[n-1].$$

5.49. (a) (i) Consider the signal $x[n] = ax_1[n] + bx_2[n]$, where a and b are constants. Then, $X(e^{j\omega}) = aX_1(e^{j\omega}) + bX_2(e^{j\omega})$. Also let the responses of the system to $x_1[n]$ and $x_2[n]$ be $y_1[n]$ and $y_2[n]$, respectively. Substituting for $X(e^{j\omega})$ in the equation given in the problem and simplifying we obtain $Y(e^{j\omega}) = aY_1(e^{j\omega}) + bY_2(e^{j\omega})$. Therefore, the system is linear

(ii) Consider the signal $x_1[n] = x[n-1]$. Then, $X_1(e^{j\omega}) = e^{-j\omega}X(e^{j\omega})$. Let the response of the system to this signal be $y_1[n]$. From the given equation,

$$\begin{aligned} Y_1(e^{j\omega}) &= 2X_1(e^{j\omega}) + e^{-j\omega}X_1(e^{j\omega}) - \frac{dX_1(e^{j\omega})}{d\omega} \\ &= e^{-j\omega}\left[2X(e^{j\omega}) + e^{-j\omega}X(e^{j\omega}) - \frac{dX(e^{j\omega})}{d\omega}\right] + je^{-j\omega}X(e^{j\omega}) \\ &\neq e^{-j\omega}Y(e^{j\omega}) \end{aligned}$$

Therefore, the system is not time invariant.

(iii) If $x[n] = \delta[n]$, $X(e^{j\omega}) = 1$. Then,

$$Y(e^{j\omega}) = 2 + e^{-j\omega}.$$

Therefore, $y[n] = 2\delta[n] + \delta[n-1]$.

(b) We may write

$$Y(e^{j\omega}) = \frac{1}{2\pi}\int_{\omega-\pi/4}^{\omega+\pi/4} X(e^{j\theta})H(e^{j(\omega-\theta)})d\theta,$$

where $H(e^{j\omega})$ is as shown in the Figure S5.49.

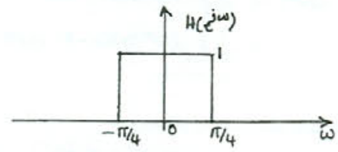

Figure S5.49

Using the multiplication property of the Fourier transform and Table 5.2, we obtain

$$y[n] = 2x[n]\frac{\sin(\pi n/4)}{n}.$$

5.50. (a) (i) From the given information,

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1 - \frac{1}{2}e^{-j\omega}}{(1 - \frac{1}{3}e^{-j\omega})(1 - \frac{1}{4}e^{-j\omega})}.$$

Taking the inverse Fourier transform, we obtain

$$h[n] = 3\left(\frac{1}{4}\right)^n u[n] - 2\left(\frac{1}{3}\right)^n u[n].$$

(ii) From part (a), we know that

$$\frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1 - \frac{1}{2}e^{-j\omega}}{(1 - \frac{1}{3}e^{-j\omega})(1 - \frac{1}{4}e^{-j\omega})}.$$

Cross-multiplying and taking the inverse Fourier transform

$$y[n] - \frac{7}{12}y[n-1] + \frac{1}{12}y[n-2] = x[n] - \frac{1}{2}x[n-1].$$

(b) From the given information,

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{(1 - \frac{1}{2}e^{-j\omega})^2}{2(1 - \frac{1}{4}e^{-j\omega})^2}.$$

We now want to find $X(e^{j\omega})$ when $Y(e^{j\omega}) = (1/2)e^{-j\omega}/(1 + \frac{1}{2}e^{-j\omega})$. From the above equation we obtain

$$X(e^{j\omega}) = \frac{e^{-j\omega}(1 - \frac{1}{4}e^{-j\omega})^2}{(1 - \frac{1}{2}e^{-j\omega})^2(1 + \frac{1}{2}e^{-j\omega})}.$$

Taking the inverse Fourier transform of the partial fraction expansion of the above expression, we obtain

$$x[n] = \frac{3}{8}\left(-\frac{1}{2}\right)^{n-1} u[n-1] + \frac{3}{8}\left(\frac{1}{2}\right)^{n-1} u[n-1] + \frac{1}{8}n\left(\frac{1}{2}\right)^{n-1} u[n-1].$$

5.51. (a) Taking the Fourier transform of $h[n]$ we obtain

$$H(e^{j\omega}) = Y(e^{j\omega})/X(e^{j\omega}) = \frac{\frac{3}{2} - \frac{1}{2}e^{-j\omega}}{1 - \frac{3}{4}e^{-j\omega} + \frac{1}{8}e^{-j2\omega}}.$$

Cross-multiplying and taking the inverse Fourier transform we obtain

$$y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = \frac{3}{2}x[n] - \frac{1}{2}x[n-1].$$

(b) (i) Let us name the intemediate output $w[n]$ (See Figure S5.51).

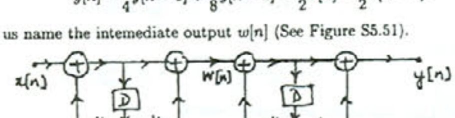

Figure S5.51

We may then write the following difference equations:

$$y[n] + \frac{1}{2}y[n-1] = \frac{1}{4}w[n] + w[n-1]$$

and

$$w[n] - \frac{1}{3}w[n-1] = x[n] - \frac{1}{2}x[n-1].$$

Taking the Fourier transform of both these equations and eliminating $W(e^{j\omega})$, we obtain

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{\frac{1}{4} + \frac{7}{8}e^{-j\omega} - \frac{1}{2}e^{-2j\omega}}{1 - \frac{1}{4}e^{-2j\omega}}.$$

Cross-multiplying and taking the inverse Fourier transform we obtain

$$y[n] - \frac{1}{4}y[n-2] = \frac{1}{4}x[n] + \frac{7}{8}x[n-1] - \frac{1}{2}x[n-2].$$

(ii) From (i)

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{\frac{1}{4} + \frac{7}{8}e^{-j\omega} - \frac{1}{2}e^{2j\omega}}{1 - \frac{1}{4}e^{-2j\omega}}.$$

(iii) Taking the inverse Fourier transform of the partial fraction expansion of $H(e^{j\omega})$, we obtain

$$h[n] = 2\delta[n] - \frac{21}{16}\left(-\frac{1}{2}\right)^n u[n] + \frac{7}{16}\left(\frac{1}{2}\right)^n u[n].$$

5.52. (a) Since $h[n]$ is causal, the nonzero sample values of $h[n]$ and $h[-n]$ overlap only at $n = 0$. Therefore,

$$\mathcal{E}v\{h[n]\} = \frac{h[n] + h[-n]}{2} = \begin{cases} h[n]/2, & n > 0 \\ h[0], & n = 0 \\ h[-n]/2, & n < 0 \end{cases}.$$

In other words,

$$h[n] = \begin{cases} 2\mathcal{E}v\{h[n]\}, & n > 0 \\ \mathcal{E}v\{h[0]\}, & n = 0 \\ 0, & n < 0 \end{cases} \qquad \text{(S5.52-1)}$$

Now note that if

$$h[n] \xleftrightarrow{FT} H(e^{j\omega})$$

then

$$\mathcal{E}v\{h[n]\} = \frac{h[n] + h[-n]}{2} \xleftrightarrow{FT} \mathcal{R}e\{H(e^{j\omega})\}.$$

Clearly, we can recover $\mathcal{E}v\{h[n]\}$ from $\mathcal{R}e\{H(e^{j\omega})\}$. From $\mathcal{E}v\{h[n]\}$ we can use eq.(S5.52-1) to recover $h[n]$. Obviously, from $h[n]$ we can once again obtain $H(e^{j\omega})$. Therefore, the system is completely specified by $\mathcal{R}e\{H(e^{j\omega})\}$.

(b) Taking the inverse Fourier transform of $\mathcal{R}e\{H(e^{j\omega})\}$, we obtain

$$\mathcal{E}v\{h[n]\} = \delta[n] + \frac{\alpha}{2}\delta[n-2] + \frac{\alpha}{2}\delta[n+2].$$

Therefore,

$$h[n] = \delta[n] + \alpha\delta[n-2],$$

and

$$H(e^{j\omega}) = 1 + \alpha e^{-j2\omega}.$$

(c) Since $h[n]$ is causal, the nonzero sample values of $h[n]$ and $h[-n]$ overlap only at $n = 0$. Therefore,

$$\mathcal{O}d\{h[n]\} = \frac{h[n] - h[-n]}{2} = \begin{cases} h[n]/2, & n > 0 \\ 0, & n = 0 \\ -h[-n]/2, & n < 0 \end{cases}.$$

In other words,

$$h[n] = \begin{cases} 2Od\{h[n]\}, & n > 0 \\ \text{some value}, & n = 0 \\ 0, & n < 0 \end{cases} \qquad \text{(S5.52-2)}$$

Now note that if

$$h[n] \overset{FT}{\longleftrightarrow} H(e^{j\omega})$$

then

$$Od\{h[n]\} = \frac{h[n] - h[-n]}{2} \overset{FT}{\longleftrightarrow} j\mathcal{I}m\{H(e^{j\omega})\}.$$

Clearly, we can recover $Od\{h[n]\}$ from $\mathcal{I}m\{H(e^{j\omega})\}$. From $Od\{h[n]\}$ we can use eq.(S5.52-2) to recover $h[n]$ (provided $h[0]$ is given). Obviously, from $h[n]$ we can once again obtain $H(e^{j\omega})$. Therefore, the system is completely specified by $\mathcal{I}m\{H(e^{j\omega})\}$ and $h[0]$.

(d) Let $\mathcal{I}m\{H(e^{j\omega})\} = \sin\omega$. Then,

$$Od\{x[n]\} = \frac{1}{2}\delta[n-1] - \frac{1}{2}\delta[n+1].$$

Therefore,

$$h[n] = h[0]\delta[n] + \delta[n-1].$$

We may choose two different values for $h[0]$ (say 1 and 2) to obtain two different systems whose frequncy responses have imaginary parts equal to $\sin\omega$.

5.53. (a) The analysis equation of the Fourier transform is

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}.$$

Comparing with eq. (P5.53-2), we have

$$\tilde{X}[k] = \frac{1}{N}X(e^{j(2\pi k/N)}).$$

(b) From the figures we obtain

$$X_1(e^{j\omega}) = 1 - e^{-j\omega} + 2e^{-3j\omega}$$

and

$$X_2(e^{j\omega}) = -e^{2j\omega} - e^{j\omega} - 1 + e^{-2j\omega} + e^{-3j\omega} + 2e^{-j4\omega} - e^{-j5\omega} + 2e^{-j7\omega}.$$

Now,

$$X_1(e^{j(2\pi k/4)}) = 1 - e^{-j\pi k/2} + 2e^{-3j\pi k/2}$$

and

$$X_2(e^{j(2\pi k/4)}) = 1 - e^{-j\pi k/2} + 2e^{-3j\pi k/2} = X_1(e^{j(2\pi k/4)}).$$

5.54. (a) From eq. (P5.54-1) it is clear that to compute $\tilde{X}[k]$ for one particular value of k, we need to perform N complex multiplications. Therefore, in order to compute $\tilde{X}[k]$ for N different values of k, we need to perform $N.N = N^2$ complex multiplications.

(b) (i) Since $f[n] = x[2n]$, we have $f[0] = x[0]$, $f[1] = x[2], \cdots, f[(N/2)-1] = x[N-2]$. Since $x[n]$ is nonzero only in the range $0 \le n \le N-1$, $f[n]$ is nonzero only in the range $0 \le n \le (N/2)-1$.

Similarly, since $g[n] = x[2n+1]$, we have $g[0] = x[1]$, $g[1] = x[3], \cdots, g[(N/2)-1] = x[N]$. Since $x[n]$ is nonzero only in the range $0 \le n \le N-1$, $g[n]$ is nonzero only in the range $0 \le n \le (N/2)-1$.

(ii) We may rewrite eq. (5.54-1) as

$$\tilde{X}[k] = \frac{1}{N}\sum_{n=0}^{(N/2)-1} x[2n]W_N^{2nk} + W_N^k\frac{1}{N}\sum_{n=0}^{(N/2)-1} x[2n+1]W_N^{2nk}.$$

Since $W_N^{2nk} = W_{N/2}^{nk}$, we may rewrite the above equation as

$$\begin{aligned}\tilde{X}[k] &= \tfrac{1}{N}\sum_{n=0}^{(N/2)-1} f[n]W_{N/2}^{nk} + W_N^k\tfrac{1}{N}\sum_{n=0}^{(N/2)-1} g[n]W_{N/2}^{nk} \\ &= \tfrac{1}{2}\tilde{F}[k] + \tfrac{1}{2}W_N^k\tilde{G}[k]\end{aligned} \qquad \text{(S5.54-1)}$$

(iii) We have

$$\tilde{F}[k+N/2] = \frac{2}{N}\sum_{n=0}^{(N/2)-1} f[n]W_{N/2}^{kn}W_{N/2}^{nN/2} = \tilde{F}[k].$$

Similarly,

$$\tilde{G}[k+N/2] = \tilde{G}[k].$$

(iv) Since $\tilde{F}[k]$ is a $N/2$ point DFT, we may use an approach similar to the one in part (a) to show that we need $N^2/4$ complex multplications to compute it. Similarly we may show that the computation of $\tilde{F}[k]$ requires $N^2/4$ multiplications. From eq. (S5.54-1), it is clear that we need $N^2/2 + N$ complex multiplcations to compute $\tilde{X}[k]$.

(c) By decomposing $g[n]$ and $f[n]$ into their odd and even indexed samples, we can bring down the number of computations to $N^2/4 + N/2$. Repeating this decomposition $\log_2 N$ times, we make the required computation $N\log_2 N$. We tabulate below the computations required by the direct method and the FFT method for values of N.

N	Direct method	FFT method
32	1024	160
256	65536	2048
1024	1048576	10240
4096	16777216	49152

5.55. (a) (i) From Table 5.2, we have

$$X(e^{j\omega}) = 2\pi\sum_{k=-\infty}^{\infty}\delta(\omega - 2\pi k).$$

(ii) When $M = 1$, $P(e^{j\omega}) = e^{j\omega} + 1 + e^{-j\omega} = 1 + 2\cos\omega$.

(iii) When $M = 10$, we may use Table 5.2 to find that

$$P(e^{j\omega}) = \frac{\sin(21\omega/2)}{\omega/2}.$$

(b) The plots are as shown in Figure S5.55.

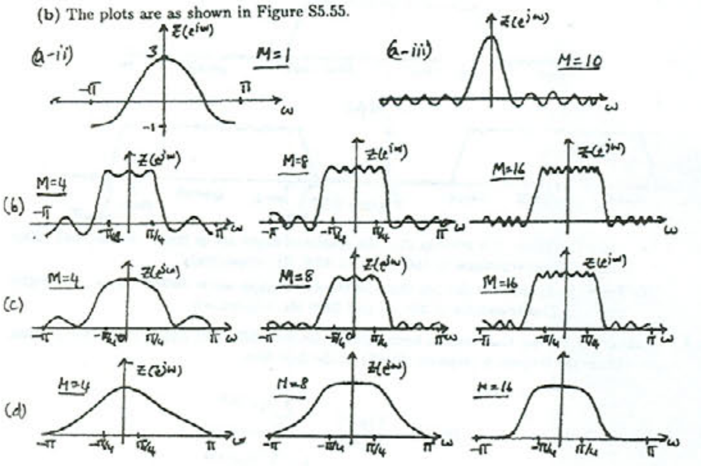

Figure S5.55

(c) We have $W(e^{j\omega}) = \frac{\sin^2[(M+1)\omega/2]}{\sin^2(\omega/2)}$. The plots are as shown in Figure S5.55

(d) The plots are as shown Figure S5.55.

5.56. (a) We have

$$\begin{aligned}X(e^{j\omega_1}, e^{j\omega_2}) &= \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} x[m,n]e^{-j(\omega_1 m + \omega_2 n)} \\ &= \sum_{n=-\infty}^{\infty}\left[\sum_{m=-\infty}^{\infty} x[m,n]e^{-j\omega_1 m}\right]e^{-j\omega_2 n} \\ &= \sum_{n=-\infty}^{\infty} X(e^{j\omega_1}, n)e^{-j\omega_2 n}\end{aligned}$$

Therefore, we may write

$$X(e^{j\omega_1}, n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega_1}, e^{j\omega_2})e^{j\omega_2 n}d\omega_2.$$

From this we obtain

$$x[m,n] = \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} X(e^{j\omega_1}, e^{j\omega_2})e^{j\omega_1 m}e^{j\omega_2 n}d\omega_1 d\omega_2.$$

(b) We may easily show that

$$X(e^{j\omega_1}, e^{j\omega_2}) = A(e^{j\omega})B(e^{j\omega}).$$

(c) We use the result of the previous part in many of the problems of this part.

(i) $X(e^{j\omega_1}, e^{j\omega_2}) = e^{-j\omega_1}e^{4j\omega_2}$.

(ii) $X(e^{j\omega_1}, e^{j\omega_2}) = \left[\frac{e^{-j2\omega_2}}{(1-\frac{1}{2}e^{-j\omega_2})}\right]\left[\frac{1}{(1-\frac{1}{3}e^{-j\omega_1})}\right]$

(iii) $X(e^{j\omega_1}, e^{j\omega_2}) = \left[\frac{1}{(1-\frac{1}{2}e^{-j\omega_2})}\right]\left[\pi\sum_{k=-\infty}^{\infty}\delta(\omega_1 - \frac{2\pi}{3} - 2\pi k) + \pi\sum_{k=-\infty}^{\infty}\delta(\omega_1 + \frac{2\pi}{3} - 2\pi k)\right]$.

(iv) Here $x[n,m] = \{u[m+1] - u[m-2]\}\{u[n+4] - u[n-5]\}$. Therefore,

$$X(e^{j\omega_1}, e^{j\omega_2}) = \left[\frac{\sin(7\omega_2/2)}{\sin(\omega_2/2)}\right]\left[\frac{\sin(3\omega_1/2)}{\sin(\omega_1/2)}\right].$$

(v) From the definition of the 2D Fourier transform we obtain

$$X(e^{j\omega_1}, e^{j\omega_2}) = \frac{e^{j(\omega_1+3\omega_2)}}{1 - e^{-j\omega_1}}\left[\frac{1 - e^{-j7(\omega_1+\omega_2)}}{1 - e^{-j(\omega_1+\omega_2)}} - e^{-j\omega_1}\left(\frac{1 - e^{-j7(3\omega_1+\omega_2)}}{1 - e^{-j(3\omega_1+\omega_2)}}\right)\right].$$

(vi) From the definition of the 2D Fourier transform we obtain

$$\begin{aligned}X(e^{j\omega_1}, e^{j\omega_2}) &= \frac{\pi}{j}\sum_{l=-\infty}^{\infty}\sum_{r=-\infty}^{\infty}\left[\delta(\omega_1 - \tfrac{2\pi}{5} + 2\pi l)\delta(\omega_2 - \tfrac{\pi}{3} + 2\pi r) - \right. \\ &\left.\delta(\omega_1 + \tfrac{2\pi}{5} + 2\pi l)\delta(\omega_2 + \tfrac{\pi}{3} + 2\pi r)\right].\end{aligned}$$

(d) (i) $X(e^{j(\omega_1 - W_1)}, e^{j(\omega_2 - W_2)})$

(ii) $X(e^{2\omega_1}, e^{3\omega_2})$

(iii) $\frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} X(e^{j\zeta}, e^{j\theta})H(e^{j(\omega_1-\zeta)}, e^{j(\omega_1-\theta)})d\zeta d\theta$