Chapter 6 Answers

6.1. The signal $x(t)$ may be broken up into a sum of the two complex exponentials $x_1(t) = (1/2)e^{j\omega_0 t+\phi_0}$ and $x_2(t) = (1/2)e^{-j\omega_0 t-\phi_0}$. Since complex exponentials are Eigen functions of LTI systems, we know that when $x_1(t)$ passes through the LTI system, the output is

$$\begin{aligned} y_1(t) &= x_1(t)H(j\omega_0) = x_1(t)|H(j\omega_0)|e^{j\sphericalangle H(j\omega_0)} \\ &= (1/2)|H(j\omega_0)|e^{j(\omega_0 t+\phi_0+\sphericalangle H(j\omega_0))} \end{aligned}$$

Similarly, when the input is $x_2(t)$, the output is

$$y_2(t) = (1/2)|H(-j\omega_0)|e^{-j(\omega_0 t+\phi_0-\sphericalangle H(-j\omega_0))}.$$

But since $h[n]$ is given to be real, $|H(j\omega_0)| = |H(-j\omega_0)|$ and $\sphericalangle H(j\omega_0) = -\sphericalangle H(-j\omega_0)$. Therefore,

$$y_2(t) = (1/2)|H(j\omega_0)|e^{-j(\omega_0 t+\phi_0+\sphericalangle H(j\omega_0))}.$$

Using linearity we may argue that when the input to the LTI system is $x(t) = x_1(t) + x_2(t)$, the output will be $y(t) = y_1(t) = y_2(t)$. Therefore,

$$y(t) = |H(j\omega_0)|\cos(\omega_0 t + \phi_0 + \sphericalangle H(j\omega_0)) = |H(j\omega_0)|\cos\left(\omega_0(t - \frac{-\sphericalangle H(j\omega_0)}{\omega_0}) + \phi_0\right)$$

(a) From $y(t)$, we have $A = |H(j\omega_0)|$.

(b) From $y(t)$, we have $t_0 = \frac{-\sphericalangle H(j\omega_0)}{\omega_0}$.

6.2. The signal $x[n]$ may be broken up into a sum of the two complex exponentials $x_1[n] = (1/2j)e^{j\omega_0 n+\phi_0}$ and $x_2[n] = (-1/2j)e^{-j\omega_0 n-\phi_0}$. Since complex exponentials are Eigen functions of LTI systems, we know that when $x_1[n]$ passes through the LTI system, the output is

$$\begin{aligned} y_1[n] &= x_1[n]H(e^{j\omega_0}) = x_1[n]|H(e^{j\omega_0})|e^{j\sphericalangle H(e^{j\omega_0})} \\ &= (1/2j)|H(e^{j\omega_0})|e^{j(\omega_0 n+\phi_0+\sphericalangle H(e^{j\omega_0}))} \end{aligned}$$

Similarly, when the input is $x_2[n]$, the output is

$$y_2[n] = (-1/2j)|H(e^{-j\omega_0})|e^{-j(\omega_0 n+\phi_0-\sphericalangle H(e^{-j\omega_0}))}.$$

But since $h(t)$ is given to be real, $|H(e^{j\omega_0})| = |H(e^{-j\omega_0})|$ and $\sphericalangle H(e^{j\omega_0}) = -\sphericalangle H(e^{-j\omega_0})$. Therefore,

$$y_2[n] = (-1/2j)|H(e^{j\omega_0})|e^{-j(\omega_0 t+\phi_0+\sphericalangle H(e^{j\omega_0}))}.$$

Using linearity we may argue that when the input to the LTI system is $x[n] = x_1[n] + x_2[n]$, the output will be $y[n] = y_1[n] + y_2[n]$. Therefore,

$$y[n] = |H(e^{j\omega_0})|\sin(\omega_0 n + \phi_0 + \sphericalangle H(e^{j\omega_0})) = |H(e^{j\omega_0})|\sin\left(\omega_0(n - \frac{-\sphericalangle H(e^{j\omega_0})}{\omega_0}) + \phi_0\right)$$

Now note that if we require that $y[n] = |H(e^{j\omega_0})|x[n-n_0]$, then $n_0 = -\sphericalangle H(e^{j\omega_0})/\omega_0$ has to be an integer. Therefore, $\sphericalangle H(e^{j\omega_0}) = -n_0\omega_0$. Now also, note that if we add an integer multiple of 2π to this $\sphericalangle H(e^{j\omega_0})$, it does not make any difference. Therefore, we require in general that $\sphericalangle H(e^{j\omega_0}) = -n_0(\omega_0 + 2k\pi)$.

6.3. (a) We have

$$|H(j\omega)| = \frac{|1-j\omega|}{|1+j\omega|} = \frac{\sqrt{1+\omega^2}}{\sqrt{1+\omega^2}} = 1.$$

Therefore, $A = 1$.

(b) We have

$$\sphericalangle H(j\omega) = \tan^{-1}(-\omega) - \tan^{-1}(\omega) = 2\tan^{-1}(\omega).$$

Therefore, the group delay is

$$\tau(\omega) = -\frac{d}{d\omega}\sphericalangle H(j\omega) = \frac{2}{1+\omega^2}.$$

Clearly, $\tau(\omega) > 0$ for $\omega > 0$. Therefore, statement 2 is true.

6.4. (a) The signal $\cos(\pi n/2)$ can be broken up into a sum of two complex exponentials $x_1[n] = (1/2)e^{j\pi n/2}$ and $x_2[n] = (1/2)e^{-j\pi n/2}$. From the given information, we know that $x_1[n]$ passes through the given LTI system, it experiences a delay of 2 samples. Since the system has a real impulse response, it has an even group delay function. Therefore, the complex exponential $x_2[n]$ with frequency $-\omega_0$ also experiences a group delay of 2 samples. The output $y[n]$ of the LTI system when the input is $x[n] = x_1[n] + x_2[n]$ is therefore

$$y[n] = 2x_1[n-2] + 2x_2[n-2] = 2\cos\left(\frac{\pi}{2}(n-2)\right) = 2\cos\left(\frac{\pi}{2}n - \pi\right)$$

(b) The signal $x[n] = \sin(\frac{7\pi}{2}n + \frac{\pi}{4})$ is the same as $-\sin(\frac{\pi}{2}n - \frac{\pi}{4})$. This signal may again be broken up into complex exponentials of frequency $\pi/2$ and $-\pi/2$. We then use an argument similar to the one used in part (a) to argue that the output is

$$\begin{aligned} y[n] &= 2x[n-2] = 2\sin\left(\frac{7\pi}{2}(n-2) + \frac{\pi}{4}\right) \\ &= 2\sin\left(\frac{7\pi}{2}n - 7\pi + \frac{\pi}{4}\right) \\ &= 2\sin\left(\frac{7\pi}{2}n - \pi + \frac{\pi}{4}\right) \\ &= 2\sin\left(\frac{7\pi}{2}n - \frac{3\pi}{4}\right) \end{aligned}$$

6.5. The frequency response $H(j\omega)$ is as shown in Figure S6.5.

(a) Consider the signal $h_1(t) = \sin(\omega_c t)/(\pi t)$. Its Fourier transform $H_1(j\omega)$ is as shown in Figure S6.5.

Clearly,

$$H(j\omega) = H_1(j(\omega - 2\omega_c)) + H_1(j(\omega + 2\omega_c)).$$

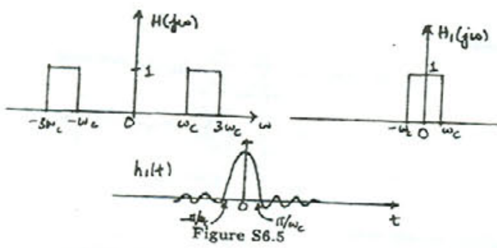


Figure S6.5

Taking the inverse Fourier transform, we have

$$\begin{aligned} h(t) &= h_1(t)e^{j2\omega_c t} + h_1(t)e^{-j2\omega_c t} \\ &= 2h_1(t)\cos(2\omega_c t) \end{aligned}$$

Therefore, $g(t) = \cos(2\omega_c t)$.

(b) The impulse response $h_1(t)$ is as shown in Figure S6.5. As ω_c increases, it is clear that the significant central lobe of $h_1(t)$ becomes more concentrated around the origin. Consequently $h(t) = 2h_1(t)\cos(2\omega_c t)$ also becomes more concentrated about the origin.

6.6. The frequency response $H(e^{j\omega})$ is as shown in Figure S6.6.

(a) Consider the signal $h_1[n] = \sin(\omega_c n)/(\pi n)$. Its Fourier transform $H_1(e^{j\omega})$ is as shown in the figure below.

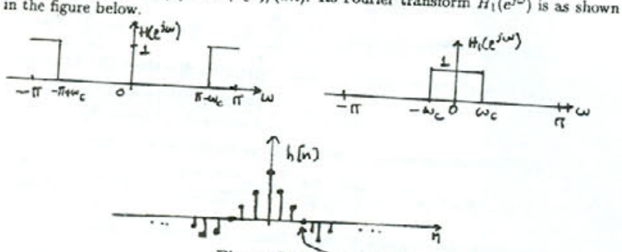


Figure S6.6 $n = \frac{\pi}{\omega_c}$ (assuming $\frac{\pi}{\omega_c} \in I$)

Clearly,

$$H(e^{j\omega}) = H_1(e^{j(\omega-\pi)}).$$

Taking the inverse Fourier transform, we have

$$h[n] = h_1[n]e^{j\pi n} = h_1[n](-1)^n.$$

Therefore, $g[n] = (-1)^n$.

(b) The impulse response $h_1[n]$ is as shown in Figure S6.6. As ω_c increases, it is clear that the significant central lobe of $h_1[n]$ becomes more concentrated around the origin. Consequently $h[n] = h_1[n](-1)^n$ also becomes more concentrated about the origin.

6.7. The frequency response magnitude $|H(j\omega)|$ is as shown in Figure S6.7. The frequency response of the bandpass filter $G(j\omega)$ will be given by

$$\begin{aligned} G(j\omega) &= \mathcal{FT}\{2h(t)\cos(4000\pi t)\} \\ &= H(j(\omega - 4000\pi)) + H(j(\omega + 4000\pi)) \end{aligned}$$

This is as shown in Figure S6.7

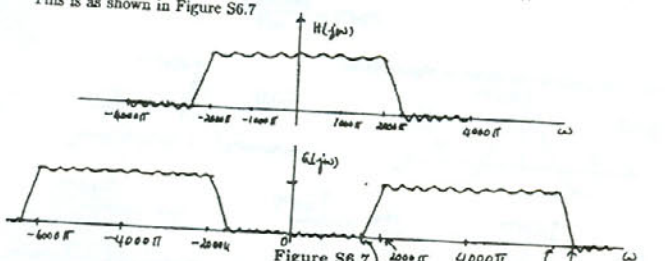


Figure S6.7

(a) From the figure, it is obvious that the passband edges are at 2000π rad/sec and 6000π rad/sec. This translates to 1000 Hz and 3000 Hz, respectively.

(b) From the figure, it is obvious that the stopband edges are at 1600π rad/sec and 6400π rad/sec. This translates to 800 Hz and 3200 Hz, respectively.

6.8. Taking the Fourier transform of both sides of the first difference equation and simplifying, we obtain the frequency response $H(e^{j\omega})$ of the first filter.

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{\sum_{k=0}^{M} b_k e^{-j\omega k}}{1 - \sum_{k=1}^{N} a_k e^{-j\omega k}}.$$

Taking the Fourier transform of both sides of the second difference equation and simplifying, we obtain the frequency response $H_1(e^{j\omega})$ of the second filter.

$$H_1(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{\sum_{k=0}^{M} (-1)^k b_k e^{-j\omega k}}{1 - \sum_{k=1}^{N} (-1)^k a_k e^{-j\omega k}}.$$

This may also be written as

$$H_1(e^{j\omega}) = \frac{\sum_{k=0}^{M} b_k e^{-j(\omega-\pi)k}}{1 - \sum_{k=1}^{N} a_k e^{-j(\omega-\pi)k}} = H(e^{j(\omega-\pi)}).$$

Therefore, the frequency response of the second filter is obtained by shifting the frequency response of the first filter by π. Although the location of the passband changes, the tolerances will be the same in the second filter. The first filter has its passband between $-\omega_p$ and ω_p. Therefore, the second filter will have its passband between $\pi - \omega_p$ and $\pi + \omega_p$.

6.9. Taking the Fourier transform of the given differential equation and simplifying, we obtain the frequency response of the LTI system to be

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{2}{5 + j\omega}.$$

Taking the inverse Fourier transform, we obtain the impulse response to be

$$h(t) = 2e^{-5t}u(t).$$

Using the result derived in Section 6.5.1, we have the step response of the system

$$s(t) = h(t) * u(t) = \frac{2}{5}[1 - e^{-5t}]u(t).$$

The final value of the step response is

$$s(\infty) = \frac{2}{5}.$$

We also have

$$s(t_0) = \frac{2}{5}[1 - e^{-5t_0}].$$

Substituting $s(t_0) = (2/5)[1 - 1/e^2]$, in the above equation, we obtain $t_0 = \frac{2}{5}$ sec.

6.10. We use Example 6.5 to guide us through this problem.

(a) We may rewrite $H_1(j\omega)$ to be

$$H(j\omega) = \left(\frac{1}{j\omega + 40}\right)(j\omega + 0.1).$$

We may then treat each of the two factors as individual first order systems and draw their Bode magnitude plots. The final Bode magnitude plot will then be a sum of these two Bode plots. This is shown in the Figure S6.10.

Mathematically, the straight-line approximation of the Bode magnitude plot is

$$20\log_{10}|H(j\omega)| \approx \begin{cases} -20, & \omega << 0.1 \\ 20\log_{10}(\omega), & 0.1 << \omega << 40 \\ 32, & \omega >> 40 \end{cases}.$$

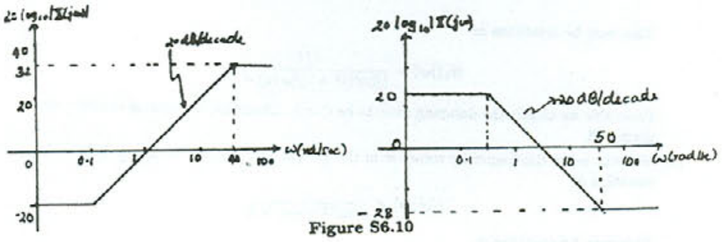


Figure S6.10

(b) Using a similar approach as in part (a), we obtain the Bode plot to be as shown in Figure S6.10.

Mathematically, the straight-line approximation of the Bode magnitude plot is

$$20\log_{10}|H(j\omega)| \approx \begin{cases} 20, & \omega << 0.2 \\ -20\log_{10}(\omega) + 6, & 0.2 << \omega << 50 \\ -28, & \omega >> 50 \end{cases}.$$

6.11. (a) We may rewrite the given frequency response $H_1(j\omega)$ as

$$H_1(j\omega) = \frac{250}{(j\omega)^2 + 50.5j\omega + 25} = \frac{250}{(j\omega + 0.5)(j\omega + 50)}.$$

We may then use an approach similar to the one used in Example 6.5 and in Problem 6.10 to obtain the Bode magnitude plot (with straight line approximations) shown in Figure S6.11.

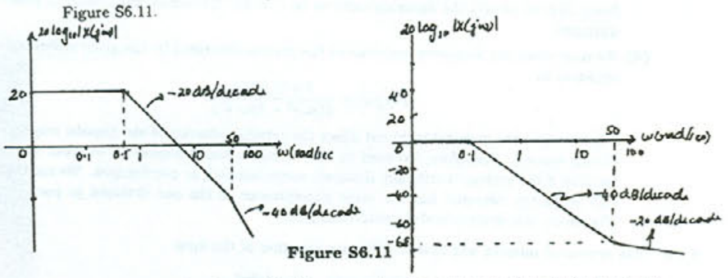


Figure S6.11

Mathematically, the straight-line approximation of the Bode magnitude plot is

$$20\log_{10}|H(j\omega)| \approx \begin{cases} 20, & \omega << 0.5 \\ -20\log_{10}(\omega) + 14, & 0.5 << \omega << 50 \\ -40\log_{10}(\omega) + 48, & \omega >> 50 \end{cases}.$$

(b) We may rewrite the frequency response $H_2(j\omega)$ as

$$H_2(j\omega) = (j\omega + 50)\left(\frac{0.02}{(j\omega)^2 + 0.2j\omega + 1}\right).$$

Again using an approach similar to the one used in Example 6.5, we may draw the Bode magnitude plot by treating the first and second order factors separately. This gives us a Bode magnitude plot (using straight line) approximations as shown below:

Mathematically, the straight-line approximation of the Bode magnitude plot is

$$20\log_{10}|H(j\omega)| \approx \begin{cases} 0, & \omega << 1 \\ -40\log_{10}\omega, & 1 << \omega << 50 \\ -20\log_{10}\omega - 34, & \omega >> 50 \end{cases}.$$

6.12. Using the Bode magnitude plot specified in Figure P6.12(a), we may obtain an expression for $H_1(j\omega)$. The figure shows that $H_1(j\omega)$ has the break frequencies $\omega_1 = 1$, $\omega_2 = 8$, and $\omega_3 = 40$. The frequency response rises at 20 dB/decade after ω_1. At ω_2, this rise is canceled by a -20 dB/decade contribution. Finally, at ω_3, an additional -20 dB/decade contribution results in the subsequent decay at the rate of -20 dB/decade. Therefore, we may conclude that

$$H_1(j\omega) = \frac{A(j\omega + \omega_1)}{(j\omega + \omega_2)(j\omega + \omega_3)}. \quad \text{(S6.12-1)}$$

We now need to find A. Note that when $\omega = 0$, $20\log_{10}|H_1(j0)| = 2$. Therefore, $H_1(j0) = 0.05$. From eq. (S6.12-1), we know that

$$H_1(j0) = A/320.$$

Therefore, $A = 640$. This gives us

$$H_1(j\omega) = \frac{640(j\omega + 1)}{(j\omega + 8)(j\omega + 40)}.$$

Using a similar approach on Figure P6.12(b), we obtain

$$H(j\omega) = \frac{6.4}{(j\omega + 8)^2}.$$

Since the overall system (with frequency response $H(j\omega)$) is constructed by cascading systems with frequency responses $H_1(j\omega)$ and $H_2(j\omega)$,

$$H(j\omega) = H_1(j\omega)H_2(j\omega).$$

Using the previously obtained expressions for $H(j\omega)$ and $H_1(j\omega)$,

$$H_2(j\omega) = \frac{H(j\omega)}{H_1(j\omega)} = \frac{0.01(j\omega + 40)}{(j\omega + 1)(j\omega + 8)}.$$

6.13. Using an approach similar to the one used in the previous problem, we obtain

$$H(j\omega) = \frac{320}{(j\omega + 2)(j\omega + 80)}.$$

(a) Let us assume that we desire to construct this system by cascading two systems with frequency responses $H_1(j\omega)$ and $H_2(j\omega)$, respectively. We require that

$$H(j\omega) = H_1(j\omega)H_2(j\omega).$$

We see that $H_1(j\omega)$ and $H_2(j\omega)$ may be defined in different ways to obtain $H(j\omega)$. For instance

$$H_1(j\omega) = \frac{40}{(j\omega + 2)} \quad \text{and} \quad H_2(j\omega) = \frac{8}{(j\omega + 80)}$$

and

$$H_1(j\omega) = \frac{32}{(j\omega + 2)} \quad \text{and} \quad H_2(j\omega) = \frac{10}{(j\omega + 80)}$$

are both valid combinations.

(b) Let us assume that we desire to construct this system by connecting two systems with frequency responses $H_1(j\omega)$ and $H_2(j\omega)$ in parallel. We require that

$$H(j\omega) = H_1(j\omega) + H_2(j\omega).$$

Using partial fraction expansion on $H(j\omega)$, we obtain

$$H(j\omega) = \frac{160/39}{(j\omega + 2)} - \frac{160/39}{(j\omega + 80)}$$

From the above expression it is clear that we can define $H_1(j\omega)$ and $H_2(j\omega)$ in only one way.

6.14. Using an approach similar to the one used in Problem 6.12, we have

$$H(j\omega) = \frac{50000(j\omega + 0.2)^2}{(j\omega + 50)(j\omega + 10)}.$$

The inverse to this system has a frequency response

$$H_I(j\omega) = \frac{1}{H(j\omega)} = \frac{0.2 \times 10^{-4}(j\omega + 50)(j\omega + 10)}{(j\omega + 0.2)^2}.$$

6.15. We will use the results from Section 6.5 in this problem.

(a) We may write the frequency response of the system described by the given differential equation as

$$H_1(j\omega) = \frac{1}{(j\omega)^2 + 4j\omega + 4}.$$

This may be rewritten as

$$H_1(j\omega) = \frac{1/4}{(j\omega/2)^2 + 2j(\omega/2) + 1}.$$

From this we obtain the damping ratio to be $\zeta = 1$. Therefore, the system is **critically damped**.

(b) We may write the frequency response of the system described by the given differential equation as

$$H_2(j\omega) = \frac{7}{5(j\omega)^2 + 4j\omega + 5}.$$

This may be rewritten as

$$H_2(j\omega) = \frac{7/5}{(j\omega)^2 + 2(2/5)j(\omega) + 1}.$$

From this we obtain the damping ratio to be $\zeta = 2/5$. Therefore, the system is **underdamped**.

(c) We may write the frequency response of the system described by the given differential equation as

$$H_3(j\omega) = \frac{1}{(j\omega)^2 + 20j\omega + 1}.$$

This may be rewritten as

$$H_3(j\omega) = \frac{1}{(j\omega)^2 + 2(10)j(\omega) + 1}.$$

From this we obtain the damping ratio to be $\zeta = 10$. Therefore, the system is **overdamped**.

(d) We may write the frequency response of the system described by the given differential equation as

$$H_3(j\omega) = \frac{7 + (1/3)j\omega}{5(j\omega)^2 + 4j\omega + 5}.$$

The terms in the numerator do not affect the ringing behavior of the impulse response of this system. Therefore, we need to only consider the denominator in order to determine if the system is critically damped, underdamped, or overdamped. We see that this frequency response has the same denominator as the one obtained in part (b). Therefore, this system is still **underdamped**.

6.16. The system of interest will have a difference equation of the form

$$y[n] - ay[n-1] = bx[n].$$

Making slight modifications to the results obtained in Section 6.6.1, we determine the step response of this system to be

$$b\left(\frac{1-a^{n+1}}{1-a}\right)u[n].$$

The final value of the step response will be $b/(1-a)$. The step response exhibits oscillatory behavior only if $|a| < 1$. Using this fact, we may easily show that the maximum overshoot in the step response occurs when $n = 0$. Therefore, the maximum value of the step response is

$$\frac{b}{1-a}(1-a) = b.$$

Since we are given that the maximum, overshoot is 1.5 times the final value, we have

$$1.5\frac{b}{1-a} = b \quad \Rightarrow \quad a = -\frac{1}{2}.$$

Also, since we are given that the final value us 1,

$$\frac{b}{1-a} = 1 \quad \Rightarrow \quad b = \frac{3}{2}.$$

Therefore, the difference equation relating the input and output will be

$$y[n] + \frac{1}{2}y[n-1] = \frac{3}{2}x[n].$$

6.17. We will use the results derived in Section 6.6.2 to solve this problem.

(a) Comparing the given difference equation with eq. (6.56), we obtain

$$r = \frac{1}{2}, \quad \text{and} \quad \cos\theta = -1.$$

Therefore, $\theta = \pi$, and the system has an **oscillatory** step response.

(b) Comparing the given difference equation with eq. (6.56), we obtain

$$r = \frac{1}{2}, \quad \text{and} \quad \cos\theta = 1.$$

Therefore, $\theta = 0$, and the system has a **non-oscillatory** step response.

6.18. Let us first find the differential equation governing the input and output of this circuit.
Current through resistor = Current through capacitor = $C\frac{dy(t)}{dt}$.
Voltage across resistor = $RC\frac{dy(t)}{dt}$.
Total input voltage = Voltage across resistor + Voltage across capacitor
Therefore,

$$x(t) = RC\frac{dy(t)}{dt} + y(t).$$

The frequency response of this circuit is therefore

$$H(j\omega) = \frac{1}{RCj\omega + 1}.$$

Since this is a first order system, the step response has to be non oscillatory.

6.19. Let us first find the differential equation governing the input and output of this circuit.
Current through resistor and inductor = Current through capacitor = $C\frac{dy(t)}{dt}$.
Voltage across resistor = $RC\frac{dy(t)}{dt}$.
Voltage across inductor = $LC\frac{d^2y(t)}{dt^2}$.
Total input voltage = Voltage across inductor + Voltage across resistor + Voltage across capacitor
Therefore,

$$x(t) = LC\frac{d^2y(t)}{dt^2} + RC\frac{dy(t)}{dt} + y(t).$$

The frequency response of this circuit is therefore

$$H(j\omega) = \frac{1}{LC(j\omega)^2 + RCj\omega + 1}.$$

We may rewrite this to be

$$H(j\omega) = \frac{1}{(\frac{j\omega}{1/\sqrt{LC}})^2 + 2(R/2)\sqrt{C/L}\frac{j\omega}{1/\sqrt{LC}} + 1}.$$

Therefore, the damping constant $\zeta = (R/2)\sqrt{C/L}$. In order for the step response to have no oscillations, we must have $\zeta \geq 1$. Therefore, we require

$$R \geq 2\sqrt{\frac{L}{C}}.$$

6.20. Let us call the given impulse response $h[n]$. It is easily observed that the signal $h_1[n] = h[n+2]$ is real and even. Therefore, (using properties of the Fourier transform) we know that the Fourier transform $H_1(e^{j\omega})$ of $h_1[n]$ is real and even. Therefore $H_1(e^{j\omega})$ has zero phase. We also know that the Fourier transform $H(e^{j\omega}) = H_1(e^{j\omega})e^{-2j\omega}$. Since $H_1(e^{j\omega})$ is zero phase, we have

$$\sphericalangle H(e^{j\omega}) = -2\omega.$$

Therefore, the group delay is

$$\tau(\omega) = \frac{d}{d\omega}\sphericalangle H(e^{j\omega}) = 2.$$

6.21. Note that in all parts of this problem $Y(j\omega) = H(j\omega)X(j\omega) = -2j\omega X(j\omega)$. Therefore, $y(t) = -2dx(t)/dt$.

(a) Here, $x(t) = e^{jt}$. Therefore, $y(t) = -2dx(t)/dt = -2je^{jt}$. This part could also have been solved by noting that complex exponentials are Eigen functions of LTI systems. Then, when $x(t) = e^{jt}$, $y(t)$ should be $y(t) = H(j1)e^{jt} = -2je^{jt}$.

(b) Here, $x(t) = \sin(\omega_0 t)u(t)$. Then, $dx(t)/dt = \omega_0\cos(\omega_0 t)u(t) + \sin(\omega_0 t)\delta(t) = \omega_0\cos(\omega_0 t)u(t)$. Therefore, $y(t) = -2dx(t)/dt = -2\omega_0\cos(\omega_0 t)u(t)$.

(c) Here, $Y(j\omega) = X(j\omega)H(j\omega) = -2/(6+j\omega)$. Taking the inverse Fourier transform we obtain $y(t) = -2e^{-6t}u(t)$.

(d) Here, $X(j\omega) = 1/(2+j\omega)$. From this we obtain $x(t) = e^{-2t}u(t)$. Therefore, $y(t) = -2dx(t)/dt = 4e^{-2t}u(t) - 2\delta(t)$.

6.22. Note that

$$H(j\omega) = \begin{cases} \frac{j\omega}{3\pi}, & -3\pi \leq \omega \leq 3\pi \\ 0, & \text{otherwise} \end{cases}.$$

(a) Since $x(t) = \cos(2\pi t + \theta)$, $X(j\omega) = e^{j\theta}\pi\delta(\omega - 2\pi) + e^{-j\theta}\pi\delta(\omega + 2\pi)$. This is zero outside the region $-3\pi < \omega < 3\pi$. Thus, $Y(j\omega) = H(j\omega)X(j\omega) = (j\omega/3\pi)X(j\omega)$. This implies that $y(t) = (1/3\pi)dx(t)/dt = (-2/3)\sin(2\pi t + \theta)$.

(b) Since $x(t) = \cos(4\pi t + \theta)$, $X(j\omega) = e^{j\theta}\pi\delta(\omega - 4\pi) + e^{-j\theta}\pi\delta(\omega + 4\pi)$. Therefore, the nonzero portions of $X(j\omega)$ lie outside the range $-3\pi < \omega < 3\pi$. This implies that $Y(j\omega) = X(j\omega)H(j\omega) = 0$. Therefore, $y(t) = 0$.

(c) The Fourier series coefficients of the signal $x(t)$ are given by

$$a_k = \frac{1}{T_0}\int_{<T_0>} x(t)e^{-jk\omega_0 t},$$

where $T_0 = 1$ and $\omega_0 = 2\pi/T_0 = 2\pi$. Also,

$$X(j\omega) = 2\pi\sum_{k=-\infty}^{\infty} a_k\delta(\omega - k\omega_0).$$

The only impulses of $X(j\omega)$ which lie in the region $-3\pi < \omega < 3\pi$ are at $\omega = 0$, 2π, and 2π. Defining the signal $x_{lp}(t) = a_0 + a_1e^{j2\pi t} + a_{-1}e^{-j2\pi t}$, we note that $y(t) = (1/3\pi)dx_{lp}(t)/dt$. We can also easily show that $a_0 = 1/\pi$, $a_1 = a_{-1}^* = -1/(4j)$. Putting these into the expression for $x_{lp}(t)$ we obtain $x_{lp}(t) = (1/\pi) + (1/2)\sin(2\pi t)$. Finally, $y(t) = (1/3\pi)dx_{lp}(t)/dt = (1/3)\cos(2\pi t)$.

6.23. (a) From the given information, we have

$$H_a(j\omega) = \begin{cases} 1, & |\omega| \leq \omega_c \\ 0, & \text{otherwise} \end{cases}.$$

Using Table 4.2, we get

$$h_a(t) = \frac{\sin(\omega_c t)}{\pi t}.$$

(b) Here,

$$H_b(j\omega) = H_a(j\omega)e^{j\omega T}.$$

Using Table 4.1, we get

$$h_b(t) = h_a(t + T).$$

Therefore,

$$h_b(t) = \frac{\sin[\omega_c(t+T)]}{\pi(t+T)}.$$

(c) Let us consider a frequency response $H_0(j\omega)$ given by

$$H_0(j\omega) = \begin{cases} 1, & |\omega| \le \omega_c/2 \\ 0, & \text{otherwise} \end{cases}.$$

Clearly,

$$H_c(j\omega) = \frac{1}{2\pi}[H_0(j\omega) * W(j\omega)],$$

where

$$W(j\omega) = j2\pi\delta(\omega - \omega_c/2) - j2\pi\delta(\omega + \omega_c/2).$$

Therefore, from Table 4.1

$$h_c(t) = h_0(t)w(t) = \left[\frac{\sin(\omega_c t/2)}{\pi t}\right][-2\sin(\omega_c t/2)].$$

6.24. If $\tau(\omega) = k_1$, where k_1 is a constant, then

$$\sphericalangle H(j\omega) = -k_1\omega + k_2 \qquad \text{(S6.24-1)}$$

where k_2 is another constant.

(a) Note that if $h(t)$ is real, the phase of the Fourier transform $\sphericalangle H(j\omega)$ has to be an odd function. Therefore, the value of k_2 in eq. (S6.24-1) will be zero.

Also, let us define $H_0(j\omega) = |H(j\omega)|$. Then,

$$h_0(t) = \frac{\sin(200\pi t)}{\pi t}.$$

(i) Here $k_1 = 5$. Hence, $\sphericalangle H(j\omega) = -5\omega$. Then,

$$H(j\omega) = |H(j\omega)|e^{j\sphericalangle H(j\omega)} = H_0(j\omega)e^{-j5\omega}.$$

Therefore,

$$h(t) = h_0(t-5) = \frac{\sin[200\pi(t-5)]}{\pi(t-5)}.$$

(ii) Here $k_1 = 5/2$. Hence, $\sphericalangle H(j\omega) = -(5/2)\omega$. Then,

$$H(j\omega) = |H(j\omega)|e^{j\sphericalangle H(j\omega)} = H_0(j\omega)e^{-j(5/2)\omega}.$$

Therefore,

$$h(t) = h_0(t-5/2) = \frac{\sin[200\pi(t-5/2)]}{\pi(t-5/2)}.$$

(iii) Here $k_1 = -5/2$. Hence, $\sphericalangle H(j\omega) = (5/2)\omega$. Then,

$$H(j\omega) = |H(j\omega)|e^{j\sphericalangle H(j\omega)} = H_0(j\omega)e^{j(5/2)\omega}.$$

Therefore,

$$h(t) = h_0(t+5/2) = \frac{\sin[200\pi(t+5/2)]}{\pi(t+5/2)}.$$

(b) If $h(t)$ is not specified to be real, then $\sphericalangle H(j\omega)$ does not have to be an odd function. Therefore, the value of k_2 in eq. (S6.24-1) does not have to be zero. Given only $|H(j\omega)|$ and $\tau(\omega)$, k_2 cannot be determined uniquely. Therefore, $h(t)$ cannot be determined uniquely.

6.25. (a) We may write $H_a(j\omega)$ as

$$H_a(j\omega) = \frac{(1-j\omega)}{(1+j\omega)(1-j\omega)} = \frac{1-j\omega}{2}.$$

Therefore,

$$\sphericalangle H_a(j\omega) = \tan^{-1}[-\omega].$$

and

$$\tau_a(\omega) = -\frac{d\sphericalangle H_a(j\omega)}{d\omega} = \frac{1}{1+\omega^2}.$$

Since $\tau_a(0) = 1 \ne 2 = \tau_a(1)$, $\tau_a(\omega)$ is not a constant for all ω. Therefore, the frequency response has nonlinear phase.

(b) In this case, $H_b(j\omega)$ is the frequency response of a system which is a cascade combination of two systems, each of which has a frequency response $H_a(j\omega)$. Therefore,

$$\sphericalangle H_b(j\omega) = \sphericalangle H_a(j\omega) + \sphericalangle H_a(j\omega)$$

and

$$\tau_b(\omega) = -2\frac{d\sphericalangle H_a(j\omega)}{d\omega} = \frac{2}{1+\omega^2}.$$

Since $\tau_b(0) = 2 \ne 4 = \tau_b(1)$, $\tau_b(\omega)$ is not a constant for all ω. Therefore, the frequency response has nonlinear phase.

(c) In this case, $H_c(j\omega)$ is again the frequency response of a system which is a cascade combination of two systems. The first system has a frequency response $H_a(j\omega)$, while the second system has a frequency response $H_0(j\omega) = 1/(2+j\omega)$. Therefore.

$$\sphericalangle H_b(j\omega) = \sphericalangle H_a(j\omega) + \sphericalangle H_0(j\omega)$$

and

$$\tau_c(\omega) = -\frac{d\sphericalangle H_a(j\omega)}{d\omega} - \frac{d\sphericalangle H_0(j\omega)}{d\omega} = \frac{1}{1+\omega^2} + \frac{2}{4+\omega^2}.$$

Since $\tau_c(0) = (3/2) \ne (3/5) = \tau_c(1)$, $\tau_b(\omega)$ is not a constant for all ω. Therefore, the frequency response has nonlinear phase.

6.26. (a) Note that $H(j\omega) = 1 - H_0(j\omega)$, where $H_0(j\omega)$ is

$$H_0(j\omega) = \begin{cases} 1, & 0 \le |\omega| \le \omega_c \\ 0, & \text{otherwise} \end{cases}.$$

Therefore,

$$h(t) = \delta(t) - h_0(t).$$

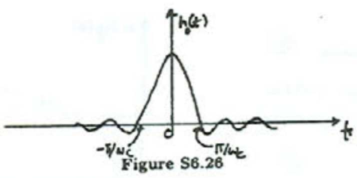


Figure S6.26

From Table 4.2, we have

$$h_0(t) = \frac{\sin(\omega_c t)}{\pi t}.$$

Therefore,

$$h(t) = \delta(t) - \frac{\sin(\omega_c t)}{\pi t}.$$

(b) A sketch of $h_0(t)$ is Figure S6.26. Clearly, as ω_c increases, $h(t)$ becomes more concentrated about the origin.

(c) Note that the step response is given by

$$s(t) = h(t) * u(t) = u(t) - u(t) * h_0(t).$$

Also, note that $h_0(t)$ is the impulse response of an ideal lowpass filter. If $s_0(t) = u(t) * h_0(t)$ denotes the step response of the lowpass filter, we know from Figure 6.14 that $s_0(0) = 0$ and $s_\infty = 1$. Therefore,

$$s(0+) = u(0+) - s_0(0+) = 1 - (1/2) = 1/2$$

and

$$s(\infty) = u(\infty) - s_0(\infty) = 0.$$

6.27. (a) Taking the Fourier transform of both sides of the given differential equation, we obtain

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{1}{2+j\omega}.$$

The Bode plot is as shown in Figure S6.27.

(b) From the expression for $H(j\omega)$ we obtain

$$\sphericalangle H(j\omega) = -\tan^{-1}(\omega/2).$$

Therefore,

$$\tau(\omega) = -\frac{d\sphericalangle H(j\omega)}{d\omega} = \frac{2}{4+\omega^2}.$$

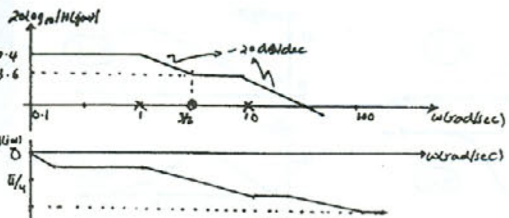


Figure S6.27

(c) Since $x(t) = e^{-t}u(t)$,

$$X(j\omega) = \frac{1}{1+j\omega}.$$

Therefore,

$$Y(j\omega) = X(j\omega)H(j\omega) = \frac{1}{(1+j\omega)(2+j\omega)}.$$

(d) Taking the inverse Fourier transform of the partial fraction expansion of $Y(j\omega)$, we obtain

$$y(t) = e^{-t}u(t) - e^{-2t}u(t).$$

(e) (i) Here,

$$Y(j\omega) = \frac{1+j\omega}{(2+j\omega)^2}.$$

Taking the inverse Fourier transform of the partial fraction expansion of $Y(j\omega)$, we obtain

$$y(t) = e^{-2t}u(t) - te^{-2t}u(t).$$

(ii) Here,

$$Y(j\omega) = \frac{1}{(1+j\omega)}.$$

Taking the inverse Fourier transform of $Y(j\omega)$, we obtain

$$y(t) = e^{-t}u(t).$$

(iii) Here,

$$Y(j\omega) = \frac{1}{(1+j\omega)(2+j\omega)^2}.$$

Taking the inverse Fourier transform of the partial fraction expansion of $Y(j\omega)$, we obtain

$$y(t) = e^{-t}u(t) + \frac{1}{2}e^{-2t}u(t) - te^{-2t}u(t).$$

6.28. (a) The Bode plots are as shown below

(b) We may write the frequency response of (iv) as

$$H(j\omega)=\frac{11/10}{1+j\omega}-\frac{1}{10}.$$

Therefore,

$$h(t)=\frac{11}{10}e^{-t}u(t)-\frac{1}{10}\delta(t)$$

and

$$s(t)=h(t)*u(t)=\frac{11}{10}(1-e^{-t})u(t)-\frac{1}{10}u(t).$$

Both $h(t)$ and $s(t)$ are as shown in Figure S6.28.

We may write the frequency response of (vi) as

$$H(j\omega)=\frac{9/10}{1+j\omega}+\frac{1}{10}.$$

Therefore,

$$h(t)=\frac{9}{10}e^{-t}u(t)+\frac{1}{10}\delta(t)$$

and

$$s(t)=h(t)*u(t)=\frac{9}{10}(1-e^{-t})u(t)+\frac{1}{10}u(t).$$

Both $h(t)$ and $s(t)$ are as shown in Figure S6.28.

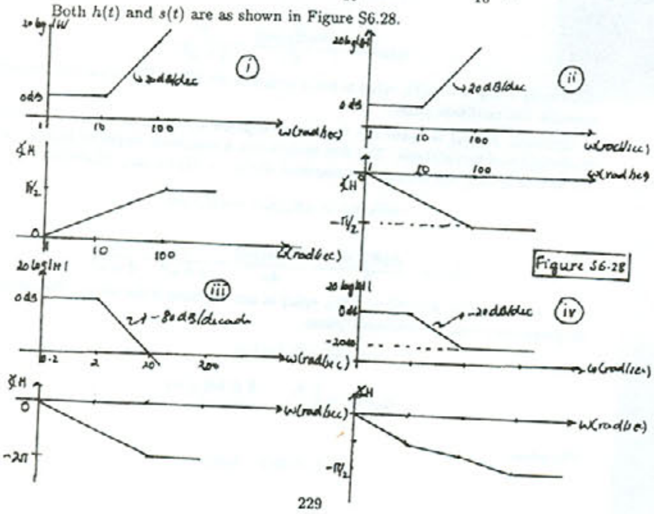
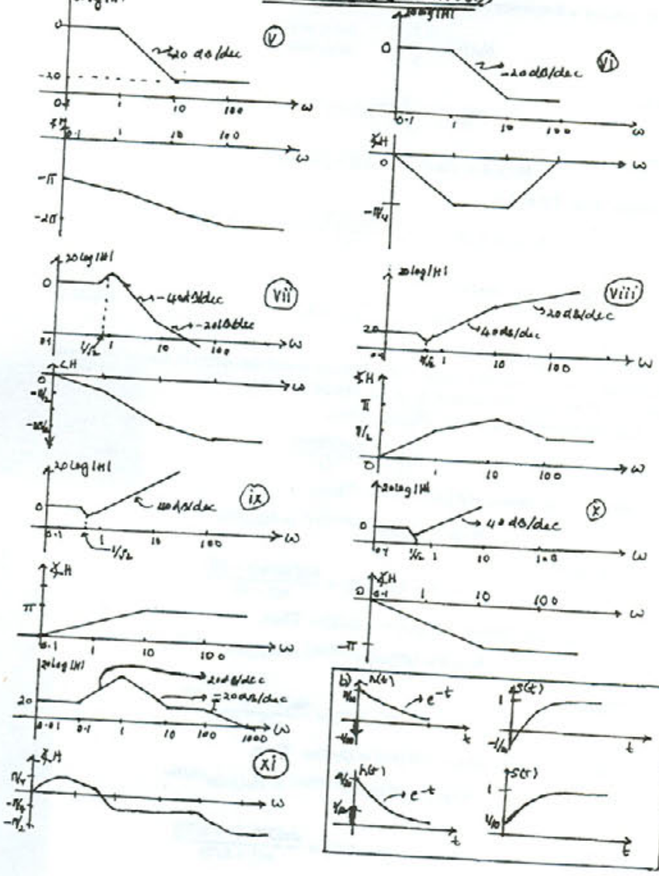


Figure S6.28

FIGURE S6.28 (CONTINUED)



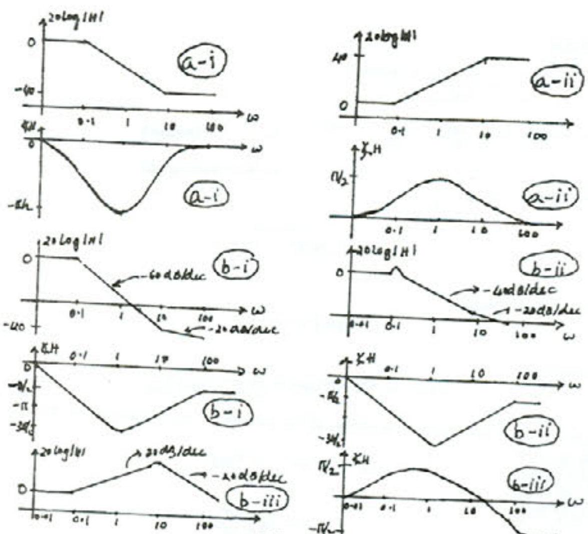


Figure S6.29

6.29. (a) (i) The Bode plot is as shown in Figure S6.29. Clearly, the system has phase lag. It also has no amplification at any frequencies (i.e., $|H(j\omega)|$ never exceeds 0 dB).

(ii) The Bode plot is as shown in Figure S6.29. Clearly, the system has phase lead. It has amplification at approximately frequencies which exceed 0.1 rad/sec.

(b) (i) The Bode plot is as shown in Figure S6.29. Clearly, the system has phase lag. It also has no amplification at any frequencies (i.e., $|H(j\omega)|$ never exceeds 0 dB).

(ii) The Bode plot is as shown in Figure S6.29. Clearly, the system has phase lag. It has some amplification at frequencies near 0.1 rad/sec.

(iii) The Bode plot is as shown in Figure S6.29. Clearly, the system has both phase lag and phase lead. It also has amplification for a band of frequencies.

6.30. We know that

$$10x(10t)\overset{FT}{\longleftrightarrow}X\left(j\frac{\omega}{10}\right).$$

Therefore, the Bode plot shifts by 1 decade to the left. The shape remains unaltered.

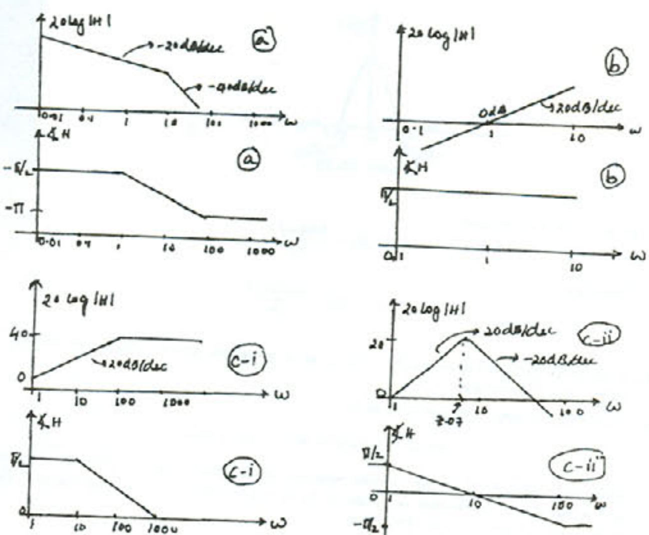


Figure S6.31

6.31. (a) The Bode plot is as shown in Figure S6.31.

(b) Since

$$\frac{dx(t)}{dt}\overset{FT}{\longleftrightarrow}j\omega X(j\omega),$$

the frequency response of a differentiator is $H(j\omega)=j\omega$. Therefore, its Bode plot is as shown in the figure below.

(c) (i) The Bode plot is as shown in Figure S6.31.

(ii) Here, $\omega_n=10$ and $\zeta=\frac{1}{2}$. The Bode plot is as shown in Figure S6.31.

6.32. (a) One possible choice for the compensator frequency response is

$$H_c(j\omega)=\frac{50(\frac{j\omega}{50}+1)}{(\frac{j\omega}{100}+1)^2}.$$

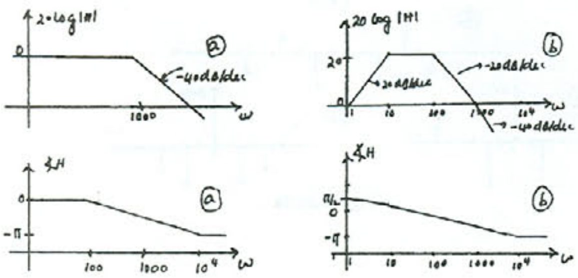


Figure S6.32

Therefore, the overall frequency response is

$$H(j\omega)=\frac{1}{(1+\frac{j\omega}{100})^2}.$$

The Bode plot for this frequency response is as shown in Figure S6.32.

(b) One possible choice for the compensator frequency response is

$$H_c(j\omega)=\frac{50j\omega(\frac{j\omega}{50}+1)}{(\frac{j\omega}{10}+1)(\frac{j\omega}{100}+1)(\frac{j\omega}{1000}+1)}.$$

Therefore, the overall frequency response is

$$H(j\omega)=\frac{j\omega}{(\frac{j\omega}{10}+1)(\frac{j\omega}{100}+1)(\frac{j\omega}{1000}+1)}.$$

The Bode plot for this frequency response is as shown in Figure S6.32.

6.33. (a) From Figure P6.33, we may write

$$Y(j\omega)=X(j\omega)-H(j\omega)H(j\omega)=H_{ov}(j\omega)X(j\omega).$$

Therefore,

$$H_{ov}(j\omega)=1-H(j\omega) \qquad \text{(S6.33-1)}$$

If $H(j\omega)$ corresponds to an ideal lowpass filter with cutoff frequency ω_{lp}, then $H_{ov}(j\omega)$ is as shown in Figure S6.33.

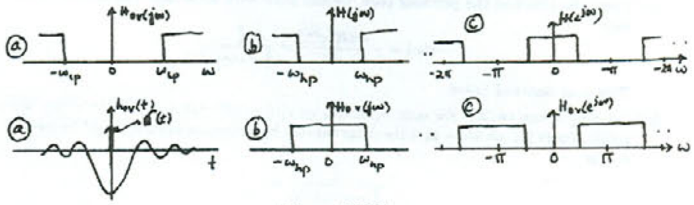


Figure S6.33

Clearly, $H_{ov}(j\omega)$ corresponds to an ideal highpass filter with cutoff frequency ω_{lp}. Also,

$$h_{ov}(t)=\delta(t)-h(t)=\delta(t)-\frac{\sin(\omega_{lp}t)}{\pi t}.$$

This is as shown in Figure S6.33.

(b) If $H(j\omega)$ corresponds to an ideal highpass filter with cutoff frequency ω_{hp}, then from eq.(S6.33-1) it is clear that $H_{ov}(j\omega)$ is as shown in Figure S6.33. Clearly, $H_{ov}(j\omega)$ corresponds to an ideal lowpass filter with cutoff frequency ω_{hp}.

(c) If we replace $H(j\omega)$ with a discrete-time lowpass filter with frequency response $H(e^{j\omega})$ as shown in Figure S6.33, then the overall frequency response still is

$$H_{ov}(e^{j\omega})=1-H(e^{j\omega}).$$

Therefore, $H(e^{j\omega})$ is as shown in Figure S6.33. Clearly, it is highpass.

6.34. (a) From the previous problem,

$$H_{ov}(j\omega)=1-H(j\omega).$$

This is sketched in Figure S6.34. Clearly, it is approximately highpass.

(b) We have $H(j\omega)=H_1(j\omega)e^{j\theta(\omega)}$. Therefore, $|H(j\omega)|=|H_1(j\omega)|$. Therefore, it is still lowpass.

(c) We have

$$H_{ov}(j\omega)=1-H(j\omega)=1-H_1(j\omega)e^{j\theta(\omega)}.$$

Therefore,

$$|H_{ov}(j\omega)|=|1-H_1(j\omega)e^{j\theta(\omega)}|.$$

We also have

$$1-|H_1(j\omega)|\le|1-H_1(j\omega)e^{j\theta(\omega)}|\le 1+|H_1(j\omega)|.$$

Therefore, $H_{ov}(j\omega)$ is between the two curves sketched in Figure S6.34.

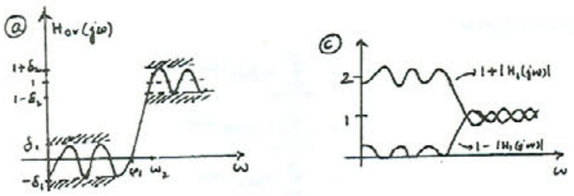


Figure S6.34

(d) From the tolerances derived in the previous part, it is clear that $H_{ov}(j\omega)$ is not necessarily highpass.

6.35. Since $x[n]=\cos(\omega_0 n+\theta)$, we have

$$X(e^{j\omega})=\pi\sum_{l=-\infty}^{\infty}[e^{j\theta}\delta(\omega-\omega_0-2\pi l)+e^{-j\theta}\delta(\omega+\omega_0-2\pi l)].$$

Let ω_0' be the principal value of ω_0 in $[-\pi,\pi]$. Then

$$Y(e^{j\omega})=X(e^{j\omega})H(e^{j\omega})=\pi\sum_{l=-\infty}^{\infty}[e^{j\theta}j\omega_0'\delta(\omega-\omega_0-2\pi l)-e^{-j\theta}j\omega_0'\delta(\omega+\omega_0-2\pi l)].$$

It follows that

$$y[n]=-\omega_0'\sin(\omega_0 n+\theta).$$

If $-\pi\le\omega_0\le\pi$, then

$$y[n]=-\omega_0\sin(\omega_0 n+\theta).$$

6.36. Let $H_1(e^{j\omega})=|H(e^{j\omega})|$. Then from Table 5.2 we know that

$$h_1[n]=\frac{\sin(\pi n/2)}{\pi n}.$$

If $\tau(\omega)=-\frac{d}{d\omega}\sphericalangle H(e^{j\omega})=k$ (where k is a constant), then $\sphericalangle H(e^{j\omega})=-k\omega+k_1$, where k_1 is a constant. If $h[n]$ is real, then $\sphericalangle H(e^{j\omega})$ is an odd function, and therefore we may conclude that $k_1=0$. Therefore,

$$H(e^{j\omega})=|H(e^{j\omega})|e^{j\sphericalangle H(e^{j\omega})}=H_1(e^{j\omega})e^{-jk\omega}.$$

Taking the inverse Fourier transform we obtain

$$h[n]=h_1[n-k]=\frac{\sin[\pi(n-k)/2]}{\pi(n-k)}.$$

(a) If $\tau(\omega)=5$, then from the above result,

$$h[n]=\frac{\sin[\pi(n-5)/2]}{\pi(n-5)}.$$

(b) If $\tau(\omega)=5/2$, then from the result derived at the beginning of this problem

$$h[n]=\frac{\sin[\pi(n-5/2)/2]}{\pi(n-5/2)}.$$

(c) If $\tau(\omega)=-5/2$, then from the result derived at the beginning of this problem

$$h[n]=\frac{\sin[\pi(n+5/2)/2]}{\pi(n+5/2)}.$$

The results of all the parts of this problem are sketched in Figure S6.36.

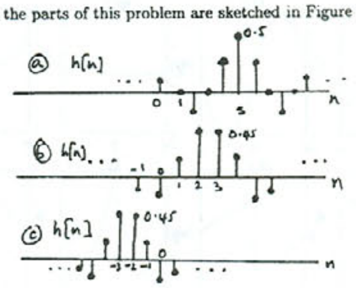


Figure S6.36

6.37. (a) We have

$$|H(e^{j\omega})|=\frac{|1-\frac{1}{2}e^{j\omega}|}{|1-\frac{1}{2}e^{-j\omega}|}=1.$$

(b) We have

$$\begin{aligned}\sphericalangle H(e^{j\omega}) &= \sphericalangle[e^{-j\omega}]+\sphericalangle\left[1-\frac{1}{2}e^{j\omega}\right]-\sphericalangle\left[1-\frac{1}{2}e^{-j\omega}\right]\\ &= \sphericalangle[e^{-j\omega}]+\sphericalangle\left[1-\frac{1}{2}\cos(\omega)-\frac{j}{2}\sin(\omega)\right]-\sphericalangle\left[1-\frac{1}{2}\cos(\omega)+\frac{j}{2}\sin(\omega)\right]\\ &= -\omega-2\tan^{-1}\left[\frac{\frac{1}{2}\sin(\omega)}{1-\frac{1}{2}\cos(\omega)}\right]\end{aligned}$$

(c) Using the result of the previous part, we can show with some algebraic manipulation that

$$\tau(\omega) = -\frac{d\sphericalangle H(e^{j\omega})}{d\omega} = \frac{\frac{3}{4}}{\frac{5}{4} - \cos\omega}.$$

This is as sketched below

(d) Let $x[n] = \cos(\pi n/3)$. We may write this as $x[n] = e^{j\pi n/3}/2 + e^{-j\pi n/3}/2$. From the result of part (c), we know that the delay suffered by a complex exponential of frequency $\pi/3$ is

$$\frac{\frac{3}{4}}{\frac{5}{4} - \cos(\pi/3)} = 1.$$

Similarly, we know that the delay suffered by a complex exponential of frequency $-\pi/3$ is also 1. Therefore, the output of the system is $y[n] = e^{j\pi(n-1)/3}/2 + e^{-j\pi(n-1)/3}/2 = \cos(\pi(n-1)/3)$.

6.38. We may express $H(e^{j\omega})$ as

$$H(e^{j\omega}) = \frac{1}{2\pi}\left[H_1(e^{j\omega}) * \{2\pi\delta(\omega - \pi/2) + 2\pi\delta(\omega + \pi/2)\}\right],$$

and

$$H_1(e^{j\omega}) = \begin{cases} 1, & |\omega| < \omega_c \\ 0, & \omega_c < |\omega| < \pi \end{cases}$$

Using the properties of the Fourier transform, we obtain

$$h[n] = h_1[n]\left[2\cos(\pi n/2)\right],$$

where

$$h_1[n] = \frac{\sin(\omega_c n)}{\pi n}.$$

(a) When $\omega_c = \pi/5$, $h[n] = 2\frac{\sin(\pi n/5)}{\pi n}\cos(\pi n/2)$. This is as shown in Figure S6.38.

(b) When $\omega_c = \pi/4$, $h[n] = 2\frac{\sin(\pi n/4)}{\pi n}\cos(\pi n/2)$. This is as shown in Figure S6.38.

(c) When $\omega_c = \pi/3$, $h[n] = 2\frac{\sin(\pi n/3)}{\pi n}\cos(\pi n/2)$. This is as shown in Figure S6.38. As ω_c increases, $h[n]$ becomes more concentrated about the origin.

6.39. The plots are as shown in Figure S6.39.

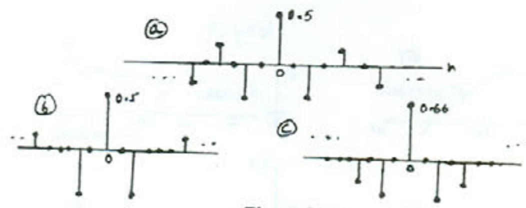


Figure S6.38

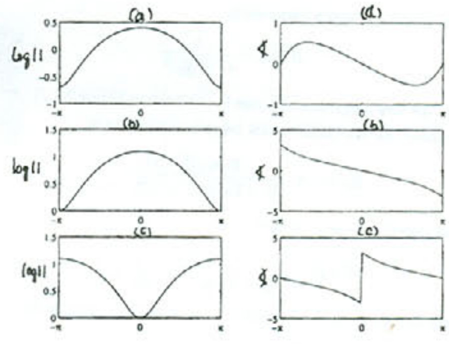


FIGURE S6.39

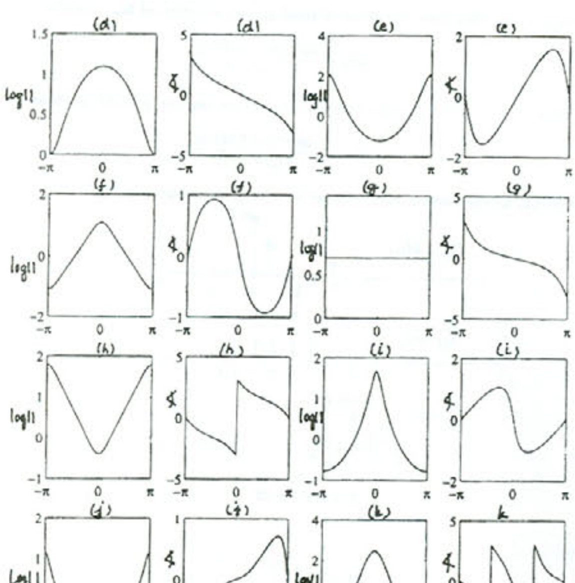


FIGURE S6.39 (CONTD.)

6.40. We may write $h_1[n]$ as

$$\begin{aligned} H_1(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} h_1[n]e^{-j\omega n} \\ &= \sum_{n=-\infty}^{\infty} h_1[2n]e^{-j2\omega n} \\ &= \sum_{n=-\infty}^{\infty} h[n]e^{-j2\omega n} \\ &= H(e^{j2\omega}) \end{aligned}$$

Therefore, $H_1(e^{j\omega})$ is $H(e^{j\omega})$ compressed by a factor of two. This is as shown in Figure S6.40.

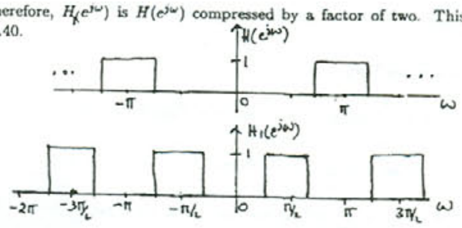


Figure S6.40

Therefore, $H_1(e^{j\omega})$ corresponds to a band-stop filter.

6.41. (a) Taking the Fourier transform of both sides of the given difference equation, we obtain

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1 - e^{-j\omega}}{1 - \frac{1}{\sqrt{2}}e^{-j\omega} + \frac{1}{4}e^{-2j\omega}}.$$

Taking the inverse Fourier transform of $H(e^{j\omega})$ we obtain

$$h[n] = \left(\frac{1}{2}\right)^n \cos(\pi n/4)u[n] - (2\sqrt{2} - 1)\left(\frac{1}{2}\right)^n \sin(\pi n/4)u[n].$$

(b) The log-magnitude and phase of the frequency response are as shown in Figure S6.41.

6.42. (a) We get

$$|H_1(e^{j\omega})| = |H_2(e^{j\omega})| = \frac{5/4 + \cos\omega}{17/6 + (1/2)\cos\omega}$$

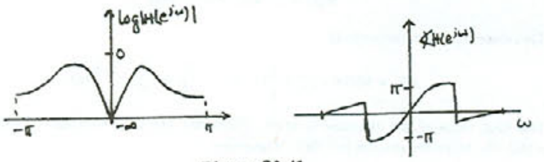


Figure S6.41

and

$$\sphericalangle H_1(e^{j\omega}) = \tan^{-1}\left(\frac{(1/2)\sin\omega}{1+(1/2)\cos(\omega)}\right) \quad \text{and} \quad \sphericalangle H_2(e^{j\omega}) = \tan^{-1}\left(\frac{\sin\omega}{1 \text{ [illegible]}}\right)$$

Comparing tangents of these angle in the range $0 \le \omega \le \pi$, we get

$$\sphericalangle H_2(e^{j\omega}) > \sphericalangle H_1(e^{j\omega}).$$

(b) We get

$$h_1[n] = \left(-\frac{1}{4}\right)^n u[n] + \frac{1}{2}\left(-\frac{1}{4}\right)^{n-1} u[n-1]$$

and

$$h_2[n] = \frac{1}{2}\left(-\frac{1}{4}\right)^n u[n] + \left(-\frac{1}{4}\right)^{n-1} u[n-1].$$

This is as sketched in Figure S6.42.

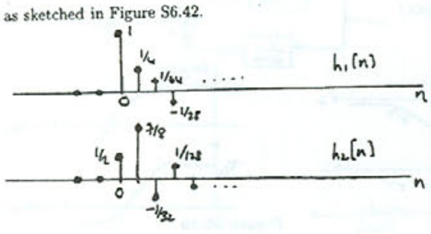


Figure S6.42

(c) We get

$$H_2(e^{j\omega}) = \left(\frac{1/2 + e^{-j\omega}}{1+(1/2)e^{-j\omega}}\right) H_1(e^{j\omega}).$$

Therefore,

$$G(e^{j\omega}) = \left(\frac{1/2 + e^{-j\omega}}{1+(1/2)e^{-j\omega}}\right)$$

and

$$|G(e^{j\omega})| = \frac{(5/4)+\cos\omega}{(5/4)+\cos\omega} = 1.$$

6.43. (a) If $h_{hp}[n] = (-1)^n h_{lp}[n] = e^{j\pi n} h_{lp}[n]$, then

$$H_{hp}(e^{j\omega}) = H_{lp}(e^{j(\omega-\pi)}).$$

Therefore, $H_{hp}(e^{j\omega})$ is as shown in Figure S6.43. Clearly, it corresponds to a highpass filter.

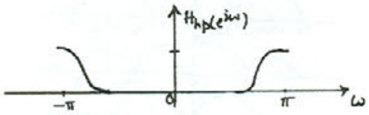


Figure S6.43

(b) Now let us define $h[n] = (-1)^n h_{hp}[n]$, where $h_{hp}[n]$ is the impulse response of a highpass filter. Then

$$H(e^{j\omega}) = H_{hp}(e^{j(\omega-\pi)}).$$

Therefore, if $H_{hp}(e^{j\omega})$ is as shown in Figure S6.43, then $H(e^{j\omega})$ is lowpass.

6.44. (a) Note that $(-1)^n = e^{j\pi n}$. From the figure we have

$$y[n] = \left(x[n]e^{j\pi n} * h_{lp}[n]\right) e^{j\pi n}.$$

We may write this as

$$y[n] = a[n]e^{j\pi n},$$

where $a[n] = \left(x[n]e^{j\pi n} * h_{lp}[n]\right)$. Taking the Fourier transform of $a[n]$, we obtain

$$A(e^{j\omega}) = X(e^{j(\omega-\pi)}) H_{lp}(e^{j\omega}).$$

Suppose that the input to the system is now $x[n-n_0]$. Let the corresponding output be $y_1[n]$. Then we may write

$$y_1[n] = b[n]e^{j\pi n},$$

where $b[n] = \left(x[n-n_0]e^{j\pi n} * h_{lp}[n]\right)$. Taking the Fourier transform of $b[n]$, we obtain

$$B(e^{j\omega}) = X(e^{j(\omega-\pi)}) H_{lp}(e^{j\omega}) e^{-j\omega n_0} = A(e^{j\omega})e^{-j\omega n_0}.$$

Therefore,

$$b[n] = a[n-n_0].$$

Consequently, $y_1[n] = y[n-n_0]$. Therefore, the system in time invariant.

(b) Since

$$y[n] = a[n]e^{j\pi n}$$

and

$$A(e^{j\omega}) = X(e^{j(\omega-\pi)}) H_{lp}(e^{j\omega}),$$

we obtain

$$Y(e^{j\omega}) = H_{lp}(e^{j(\omega-\pi)}) X(e^{j\omega}).$$

Therefore, the frequency response of the overall system is $H_{lp}(e^{j(\omega-\pi)})$. If $H_{lp}(e^{j\omega})$ is lowpass, then $H_{lp}(e^{j(\omega-\pi)})$ is highpass.

6.45. (i) All three first order factors in this frequency response are of the form $\frac{1}{1-\alpha e^{-j\omega}}$, $\alpha > 0$. Therefore, none of these factors contributes an oscillatory component to the step response. Therefore, the step response of the overall system is non oscillatory.

(ii) The factor $\frac{1}{1+\frac{1}{2}e^{-j\omega}}$ contributes an oscillatory component to the step response. Therefore, the step response of the overall system is oscillatory.

(iii) Consider the second order factor $\frac{1}{1-\frac{3}{4}e^{-j\omega}+\frac{9}{16}e^{-2j\omega}}$. For this, we get $r = \frac{3}{4}$ and $\cos\theta = \frac{1}{2}$. Since $\theta \neq 0$, this second order factor contributes an oscillatory component to the step response. Therefore, the step response of the overall system is oscillatory.

6.46. (a) We have

$$\begin{aligned} H(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n} \\ &= h[0] + h[1]e^{-j\omega} + \cdots + h[\frac{N-1}{2}]e^{-j\omega(N-1)/2} + \cdots + h[N-1]e^{-j\omega(N-1)} \end{aligned}$$

Since $h[\frac{N-1}{2}+n] = h[\frac{N-1}{2}-n]$, we may write

$$\begin{aligned} H(e^{j\omega}) &= e^{-j\omega(N-1)/2}\Big[h[0]e^{j\omega(N-1)/2} + h[1]e^{j\omega(\frac{N-1}{2}-1)} + \cdots + h[\frac{N-1}{2}] \\ &\quad + \cdots + h[1]e^{-j\omega(\frac{N-1}{2}-1)} + h[0]e^{-j\omega(\frac{N-1}{2})}\Big] \\ &= e^{-j\omega(N-1)/2}\Big[2h[0]\cos(\omega(N-1)/2) + 2h[1]\cos[\omega(\frac{N-1}{2}-1)] \\ &\quad + \cdots + h[\frac{N-1}{2}]\Big] \\ &= e^{-j\omega(N-1)/2}A(\omega) \end{aligned}$$

where

$$A(\omega) = \left[2h[0]\cos(\omega(N-1)/2) + 2h[1]\cos[\omega(\frac{N-1}{2}-1)] + \cdots + h[\frac{N-1}{2}]\right]$$

is a real-valued function.

(b) One such example is $h[n] = \delta[n] + 2\delta[n-1] + 3\delta[n-2] + 2\delta[n-3] + \delta[n-4]$.

(c) We have

$$\begin{aligned} H(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n} \\ &= h[0] + h[1]e^{-j\omega} + \cdots + h[\frac{N}{2}-1]e^{-j\omega(\frac{N}{2}-1)} + h[\frac{N}{2}]e^{-j\omega N/2} \\ &\quad + \cdots + h[N-1]e^{-j\omega N-1} \end{aligned}$$

Since $h[\frac{N}{2}+n] = h[\frac{N}{2}-n-1]$, we may write

$$\begin{aligned} H(e^{j\omega}) &= e^{-j\omega(N-1)/2}\Big[h[0]e^{j\omega(N-1)/2} + h[1]e^{j\omega(\frac{N-1}{2}-1)} \\ &\quad + \cdots + h[\frac{N}{2}-1]e^{-j\omega/2} + h[\frac{N}{2}-1]e^{j\omega} \\ &\quad + \cdots + h[1]e^{-j\omega(\frac{N-1}{2}-1)} + h[0]e^{-j\omega(\frac{N-1}{2})}\Big] \\ &= e^{-j\omega(N-1)/2}\Big[2h[0]\cos(\omega(N-1)/2) + 2h[1]\cos[\omega(\frac{N-1}{2}-1)] \\ &\quad + \cdots + 2h[\frac{N}{2}-1]\cos(\omega/2)\Big] \\ &= e^{-j\omega(N-1)/2}A(\omega) \end{aligned}$$

where

$$A(\omega) = \left[2h[0]\cos(\omega(N-1)/2) + 2h[1]\cos[\omega(\frac{N-1}{2}-1)] + \cdots + 2h[\frac{N}{2}-1]\cos(\omega/2)\right]$$

is a real-valued function.

(d) One such example is $h[n] = \delta[n] + 2\delta[n-1] + 2\delta[n-2] + \delta[n-3]$.

6.47. (a) Taking the Fourier transform of both sides of the given difference equation, we have

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = b[1 + 2a\cos\omega].$$

(b) We want $H(e^{j0}) = b[1+2a] = 1$. Therefore, $b = 1/(1+2a)$.

(c) If $a = 1/2$, then $b = 1/2$. Therefore, $H(e^{j\omega}) = \frac{1}{2}[1+\cos\omega]$. This is plotted in Figure S6.47.

6.48. (a) Here,

$$H(e^{j\omega}) = b_1 e^{-j\omega} + b_2 e^{-2j\omega} = 2b_1 e^{-j3\omega/2}\cos(\omega/2).$$

Therefore,

$$|H(e^{j\omega})| = 2|b_1||\cos(\omega/2)|.$$

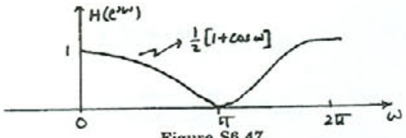


Figure S6.47

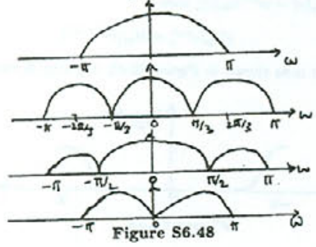


Figure S6.48

(b) Here,

$$H(e^{j\omega}) = b_0 + b_3 e^{-3j\omega} = 2b_0 e^{-j3\omega/2}\cos(3\omega/2).$$

Therefore,

$$|H(e^{j\omega})| = 2|b_0||\cos(3\omega/2)|.$$

(c) Here,

$$H(e^{j\omega}) = b_0 + b_1 e^{-j\omega} + b_2 e^{-j2\omega} + b_3 e^{-j3\omega} = 2b_0 e^{-j3\omega/2}\cos(\omega)\cos(\omega/2).$$

Therefore,

$$|H(e^{j\omega})| = 2|b_0||\cos(\omega)||\cos(\omega/2)|.$$

(d) Here,

$$H(e^{j\omega}) = b_0 + b_1 e^{-j\omega} + b_2 e^{-j2\omega} + b_3 e^{-j3\omega} = -2b_0 e^{-j3\omega/2}\sin(\omega)\sin(\omega/2).$$

Therefore,

$$|H(e^{j\omega})| = 2|b_0||\sin(\omega)||\sin(\omega/2)|.$$

The plots for the frequency response magnitudes are shown in Figure S6.48.

6.49. (a) Taking the Fourier transform of both sides of the given differential equation, we obtain

$$H(j\omega) = \frac{9}{-\omega^2 + 11j\omega + 10}.$$

Taking the inverse Fourier transform of the partial fraction expansion of $H(j\omega)$, we obtain the impulse response to be

$$h(t) = e^{-t}u(t) - e^{-10t}u(t).$$

Therefore, the step response is

$$s(t) = h(t) * u(t) = \left[1 - e^{-t} - \frac{1}{10} + \frac{1}{10}e^{-10t}\right]u(t).$$

The final value of this response is 9/10. Therefore, the time-constant τ is the time at which the response reaches $9/(10e)$. Therefore,

$$\left[\frac{9}{10} - e^{-\tau} + \frac{1}{10}e^{-10\tau}\right] = \frac{9}{10e}$$

is the equation that we need to solve.

(b) We may write $H(j\omega)$ as

$$H(j\omega) = \frac{1}{1+j\omega} - \frac{1}{10+j\omega} = H_1(j\omega) - H_2(j\omega).$$

Therefore, $H(j\omega)$ may be viewed as the parallel interconnection shown in Figure S6.49.

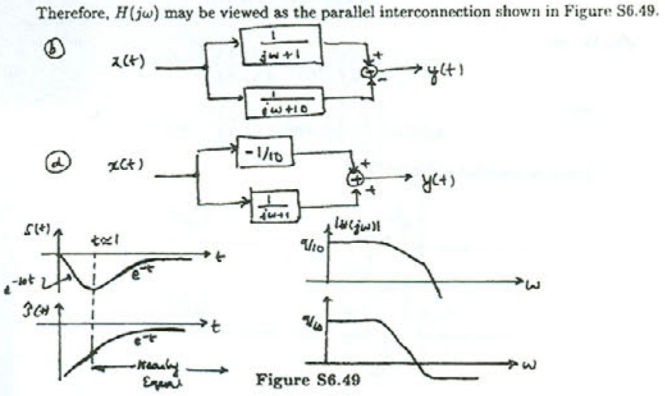


Figure S6.49

The first time constant is $\tau_1 = 1$ and the second time constant is $\tau_2 = \frac{1}{10}$.

(c) Dominant time constant is $\tau = 1$. This approximately satisfies the equation of part (a).

(d) The approximate frequency response may be expressed as

$$\hat{H}(j\omega) = H_1(j\omega) - \hat{H}_2(j\omega) = \frac{1}{1+j\omega} - \frac{1}{10}.$$

The differential equation relating the input and output of the approximate system is

$$\frac{dy(t)}{dt} + y(t) = \frac{1}{10}\frac{dx(t)}{dt} + \frac{9}{10}x(t).$$

The magnitude of the frequency responses of the exact and approximate systems are plotted in Figure S6.49. Clearly, they are identical for low frequencies. The step responses of the exact and approximate systems are also plotted in Figure S6.49. Clearly, they are identical for t approximately greater than 1.

6.50. (a) We have

$$Y(j\omega) = X(j\omega)H(j\omega) = [S(j\omega) + W(j\omega)]\,H(j\omega).$$

Therefore,

$$\epsilon(\omega) = |S(j\omega) - Y(j\omega)|^2 = |S(j\omega) - [S(j\omega) + W(j\omega)]\,H(j\omega)|^2.$$

(b) From part (a), we obtain

$$\begin{aligned}\epsilon(\omega) &= |S(j\omega)|^2 + H^2(j\omega)|S(j\omega) + W(j\omega)|^2 - 2\mathcal{R}e\{S^*(j\omega)[S(j\omega) + W(j\omega)]\}H(j\omega)\\ &= |S(j\omega)|^2 + H^2(j\omega)|S(j\omega) + W(j\omega)|^2 - 2H(j\omega)\left[|S(j\omega)|^2 + \mathcal{R}e\{S^*(j\omega)W(j\omega)\}\right]\end{aligned}$$

Therefore,

$$\frac{\partial\epsilon(\omega)}{\partial H(j\omega)} = 2H(j\omega)|S(j\omega) + W(j\omega)|^2 - 2\left[|S(j\omega)|^2 + \mathcal{R}e\{S^*(j\omega)W(j\omega)\}\right]$$

If $\frac{\partial\epsilon(\omega)}{\partial H(j\omega)} = 0$, then

$$H(j\omega) = \frac{\left[|S(j\omega)|^2 + \mathcal{R}e\{S^*(j\omega)W(j\omega)\}\right]}{|S(j\omega) + W(j\omega)|^2}.$$

Note that is $S(j\omega_0) + W(j\omega_0) = 0$, then $X(j\omega_0) = 0$ and $Y(j\omega_0) = 0$ no matter what the value of $H(j\omega_0)$.

(c) If $S(j\omega)$ and $W(j\omega)$ are non-overlapping, then $\mathcal{R}e\{S^*(j\omega)W(j\omega)\} = 0$ for all ω and so

$$H(j\omega) = \begin{cases} \frac{|S(j\omega)|^2}{|S(j\omega)-0|^2} = 1, & \text{for } W(j\omega) = 0, S(j\omega) \neq 0 \\ \frac{0}{|0-W(j\omega)|^2} = 0, & \text{for } W(j\omega) \neq 0, S(j\omega) = 0 \\ 0(\text{arbitrarily}), & \text{for } W(j\omega) = 0, S(j\omega) = 0 \end{cases}$$

Clearly, this is an ideal frequency selective filter.

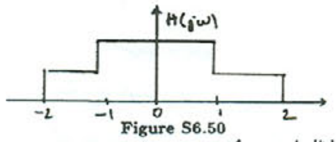


Figure S6.50

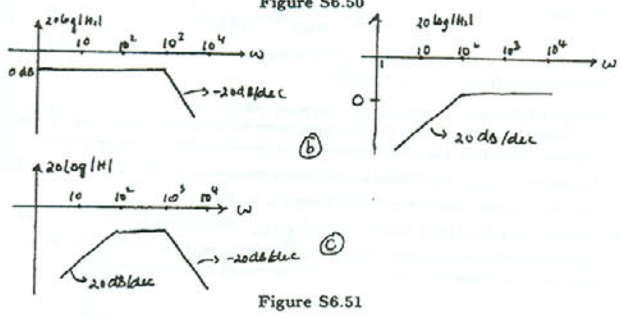


Figure S6.51

(d) In this case,

$$H(j\omega) = \begin{cases} 1, & |\omega| \le 1 \\ \frac{1}{2}, & 1 < |\omega| < 2 \\ 0, & |\omega| \ge 2 \end{cases}$$

This is as shown in Figure S6.50.

6.51. (a) We may write $H(j\omega)$ as

$$H(j\omega) = H_{lp}(j\omega) * [\delta(\omega - \omega_0) + \delta(\omega + \omega_0)],$$

where $H_{lp}(j\omega)$ is the frequency response of an ideal lowpass filter with cutoff frequency $\frac{w}{2}$. Therefore,

$$h(t) = 2h_{lp}(t)\cos(\omega_0 t),$$

where

$$h_{lp}(t) = \frac{\sin(wt/2)}{\pi t}.$$

(b) We have

$$H_1(j\omega) = \frac{1}{1 + j\frac{\omega}{10^3}} \quad \text{and} \quad H_2(j\omega) = \frac{j\omega/10^2}{1 + j\frac{\omega}{10^2}}.$$

Therefore the Bode diagrams for these two filters are as shown in Figure S6.51.

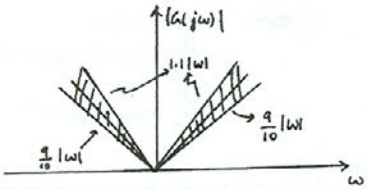


Figure S6.52

(c) Since $H(j\omega) = H_1(j\omega)H_2(j\omega)$,

$$20\log_{10}|H(j\omega)| = 20\log_{10}|H_1(j\omega)| + 20\log_{10}|H_2(j\omega)|.$$

Therefore, the Bode diagram for the bandpass filter is the sum of the two Bode diagrams sketched in part (b).

6.52. (a) Since

$$-0.1|H(j\omega)| \le [|G(j\omega)| - |H(j\omega)|] \le 0.1|H(j\omega)|,$$

we have

$$0.9|H(j\omega)| \le |G(j\omega)| \le 1.1|H(j\omega)|.$$

Therefore,

$$0.9|\omega| \le |G(j\omega)| \le 1.1|\omega|.$$

This is sketched in Figure S6.52.

(b) From Figure P6.52(b) we have

$$y(t) = \frac{1}{T}\left[x(t) - x(t-T)\right].$$

Therefore,

$$Y(j\omega) = \frac{1}{T}\left[X(j\omega) - e^{-j\omega T}X(j\omega)\right]$$

and

$$G(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{1}{T}[1 - e^{-j\omega T}] = \frac{2}{T}e^{-j\omega T/2}\sin(\omega T/2).$$

Therefore,

$$|G(j\omega)| = \frac{2}{T}|\sin(\omega T/2)|,$$

and

$$\frac{|G(j\omega)|}{|\omega|} = \frac{|\sin(\omega T/2)|}{|\omega T/2|}.$$

For $|G(j\omega)|$ to be within $\pm 10\%$ of $|\omega|$, we require the above ratio to be greater than 0.9. It can be easily shown that for $T = 10^{-2}$, the above ratio falls below 0.9 for $\omega T/2 = \pi/20$, that is, $\omega \approx 31.4$ rad/sec. Therefore, the magnitude of the frequency response of the approximate system remains within $\pm 10\%$ of the ideal differentiator for $|\omega| < 31.4$ rad/sec.

6.53. If $s(t)$ denotes the step response and $h(t)$ the impulse response, then

$$h(t) = \frac{ds(t)}{dt}.$$

If $h(t) \ge 0$, then $\frac{ds(t)}{dt} \ge 0$. This implies that $s(t)$ is a monotonically non-decreasing function.

6.54. (a) The cutoff frequency $2\pi \times 10^2$ rad/sec in $H_{lp}(j\omega)$ maps to the frequency $\omega_c = 2\pi \times 10^2/a$ rad/sec in $H_0(j\omega)$. Therefore,

$$a = \frac{2\pi \times 10^2}{\omega_c}.$$

(b) We know from Table 4.1 that

$$x(at) \overset{FT}{\longleftrightarrow} \frac{1}{a}X(j\frac{\omega}{a}).$$

Therefore,

$$h_{lp}(t) = \frac{1}{a}h_0(t/a) = \frac{\omega_c}{2\pi \times 10^2}h_0\left(\frac{\omega_c t}{2\pi \times 10^2}\right).$$

(c) We know that

$$s_0(t) = \int_{-\infty}^{t} h_0(\tau)d\tau.$$

Also,

$$s_{lp}(t) = \int_{-\infty}^{t} h_{lp}(t)d\tau.$$

Therefore,

$$s_{lp}(t) = \frac{1}{a}\int_{-\infty}^{t} h_0(\tau/a)d\tau.$$

Let $\tau' = \tau/a$. Then,

$$s_{lp}(t) = \int_{-\infty}^{t/a} h_0(\tau')d\tau' = s_0(t/a) = s_0(t\omega_c/(2\pi \times 10^2)).$$

(d) Let

$$\lim_{t\to\infty} s_0(t) = A.$$

Then,

$$\tau_r = t_1 - t_0,$$

where $s_0(t_0) = A/10$ and $s_0(t_1) = 9A/10$. Now,

$$\lim_{t\to\infty} s_{lp}(t) = \lim_{t\to\infty} s_{lp}(t/a) = A.$$

We now need to find the times t_2 and t_3 at which $s_{lp}(t)$ is $A/10$ and $9A/10$, respectively. If $s_{lp}(t_2) = A/10$, then $s_0(t_2/a) = A/10$. This implies that $t_2 = at_0$. Also, if $s_{lp}(t_3) = 9A/10$, then $s_0(t_3/a) = 9A/10$. This implies that $t_3 = at_1$. Therefore, the new rise-time is

$$\tau_r' = t_3 - t_2 = a(t_1 - t_0) = a\tau_r = \frac{2\pi}{\omega_c}.$$

τ_r' is sketched in Figure S6.54 as a function of ω_c.

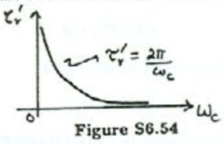


Figure S6.54

6.55. We have

$$|B(j\omega)|^2 = \frac{1}{1 + (\omega/\omega_c)^{2N}} \qquad \text{(S6.55-1)}$$

Also, $|B(j0)|^2 = 1$. Therefore, $|B(j\omega_p)|^2 = 1/2$. From eq.(S6.55-1), we conclude that

$$\left(\frac{\omega_p}{\omega_c}\right)^{2N} = 1 \quad \Rightarrow \quad \omega_p = \omega_c.$$

Also, since $|B(j\omega_s)|^2 = 1/100$, we may use eq.(S6.55-1) to conclude that

$$\left(\frac{\omega_s}{\omega_c}\right)^{2N} = 99 \quad \Rightarrow \quad \omega_s = (99)^{1/2N}\omega_c.$$

Therefore, the transition ratio is

$$\frac{\omega_s}{\omega_p} = (99)^{1/2N} \approx 10^{1/N}.$$

This is sketched in Figure S6.55.

6.56. (a) The conditioning system with frequency response $H_1(j\omega)$ boosts the frequencies that are going to be most affected by the noise. Therefore, its frequency response is chosen to have a magnitude plot as shown in Figure 6.56(a). Therefore,

$$H_1(j\omega) = \frac{\left(1 + \frac{j\omega}{\omega_0}\right)^2}{\left(1 + \frac{j\omega}{\omega_1}\right)^2},$$

where $\omega_0 = 2\pi(5000)$ rad/sec and $\omega_1 = 2\pi(10000)$ rad/sec.

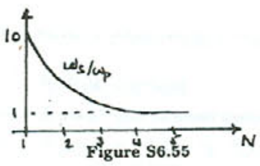


Figure S6.55

(b) The higher frequencies would appear boosted. This would make it sound like the "treble" was higher.

(c) The system with frequency response $H_2(j\omega)$ should undo the effects of $H_1(j\omega)$. Therefore, it has to be the inverse system of $H_1(j\omega)$. The Bode plot for $H_2(j\omega)$ would be as shown in Figure S6.56.

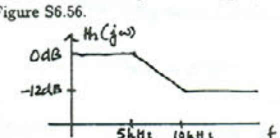


Figure S6.56

Therefore,

$$H_2(j\omega) = \frac{\left(1 + \frac{j\omega}{\omega_1}\right)^2}{\left(1 + \frac{j\omega}{\omega_0}\right)^2},$$

where $\omega_0 = 2\pi(5000)$ rad/sec and $\omega_1 = 2\pi(10000)$ rad/sec. The input $x(t)$ and the output $y(t)$ of $H_2(j\omega)$ are related by the following differential equation

$$\frac{1}{\omega_0^2}\frac{dy^2(t)}{dt^2} + \frac{2}{\omega_0}\frac{dy(t)}{dt} + y(t) = \frac{1}{\omega_1^2}\frac{dx^2(t)}{dt^2} + \frac{2}{\omega_1}\frac{dx(t)}{dt} + x(t).$$

6.57. If $s[n]$ denotes the step response and $h[n]$ the impulse response, then

$$h[n] = s[n] - s[n-1].$$

If $h[n] \ge 0$, then $s[n] \ge s[n-1]$. This implies that $s[n]$ is a monotonically non-decreasing function.

6.58. (a) The sequence of operations shown in Figure 6.58(a) may be interpreted as follows:

$$\begin{aligned}
G(e^{j\omega}) &= H(e^{j\omega})X(e^{j\omega}) \\
R(e^{j\omega}) &= G(e^{-j\omega})H(e^{j\omega}) = H(e^{-j\omega})X(e^{-j\omega})H(e^{j\omega}) \\
S(e^{j\omega}) &= R(e^{-j\omega}) = H(e^{j\omega})X(e^{j\omega})H(e^{-j\omega}) = H_1(e^{j\omega})X(e^{j\omega})
\end{aligned}$$

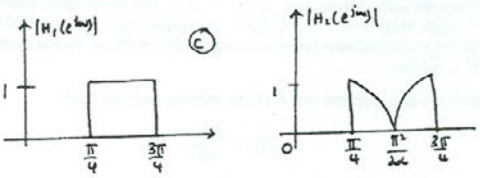


Figure S6.58

Therefore,

$$H_1(e^{j\omega}) = H(e^{j\omega})H(e^{-j\omega}).$$

If $h[n]$ is real, then $H(e^{j\omega}) = H^*(e^{-j\omega})$. Then

$$H_1(e^{j\omega}) = |H(e^{j\omega})|^2.$$

Therefore,

$$h_1[n] = h[n] * h[-n].$$

Also,

$$|H_1(e^{j\omega})| = |H(e^{j\omega})|^2 \quad \text{and} \quad \sphericalangle H_1(e^{j\omega}) = 0.$$

(b) The sequence of operations shown in Figure 6.58(a) may be interpreted as follows:

$$\begin{aligned} G(e^{j\omega}) &= H(e^{j\omega})X(e^{j\omega}) \\ R(e^{j\omega}) &= X(e^{-j\omega})H(e^{j\omega}) \\ Y(e^{j\omega}) &= G(e^{j\omega}) + R(e^{-j\omega}) = X(e^{j\omega})[H(e^{j\omega}) + H(e^{-j\omega})] \end{aligned}$$

Therefore,

$$H_2(e^{j\omega}) = H(e^{j\omega}) + H(e^{-j\omega}).$$

If $h[n]$ is real, then $H(e^{j\omega}) = H^*(e^{-j\omega})$. Then

$$H_1(e^{j\omega}) = 2\mathcal{R}e\{H(e^{j\omega})\} = 2|H(e^{j\omega})|\cos(\sphericalangle H(e^{j\omega})).$$

Therefore,

$$h_2[n] = \frac{h[n] + h[-n]}{2}.$$

Also,

$$|H_2(e^{j\omega})| = 2|H(e^{j\omega})||\cos(\sphericalangle H(e^{j\omega}))|.$$

(c) The plots for $|H_1(e^{j\omega})|$ and $|H_1(e^{j\omega})|$ are shown in Figure S6.58.

Clearly, Method A is preferable because the magnitude of the zero-phase filter does not depend on the phase of $h[n]$.

6.59. (a) We have

$$\begin{aligned} E(e^{j\omega}) &= H_d(e^{j\omega}) - H(e^{j\omega}) \\ &= \sum_{n=-\infty}^{\infty} [h_d[n]e^{-j\omega n} - h[n]e^{-j\omega n}] \\ &= \sum_{n=-\infty}^{\infty} (h_d[n] - h[n])e^{-j\omega n} \end{aligned}$$

Therefore, $e[n] = h_d[n] - h[n]$.

(b) Noting that $E(e^{j\omega})$ is the Fourier transform of $e[n]$, we may use Parseval's theorem to obtain

$$\epsilon^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |E(e^{j\omega})|^2 d\omega = \sum_{n=-\infty}^{\infty} |e[n]|^2.$$

(c) We have

$$\begin{aligned} \epsilon^2 &= \sum_{n=-\infty}^{\infty} |e[n]|^2 \\ &= \sum_{n=-\infty}^{\infty} |h_d[n] - h[n]|^2 \\ &= \sum_{n=0}^{N-1} |h_d[n] - h[n]|^2 + \sum_{n=-\infty}^{0} |h_d[n]|^2 + \sum_{n=N}^{\infty} |h_d[n]|^2 \end{aligned}$$

The last two terms in the right-hand side of the above equation are constant. The only variable term $\sum_{n=0}^{N-1} |h_d[n] - h[n]|^2$ is minimized when $h_d[n] = h[n]$ in the range $0 \le n \le N-1$.

6.60. The development is identical to that in Problem 6.50. We have

$$\begin{aligned} \epsilon(e^{j\omega}) &= |S(e^{j\omega}) - Y(e^{j\omega})|^2 \\ &= |S(e^{j\omega}) - H(e^{j\omega})[S(e^{j\omega}) + W(e^{j\omega})]|^2 \\ &= |S(e^{j\omega})|^2 + H^2(e^{j\omega})|S(e^{j\omega}) + W(e^{j\omega})|^2 \\ &\quad -2H(e^{j\omega})[|S(e^{j\omega})|^2 + \mathcal{R}e\{S^*(e^{j\omega})W(e^{j\omega})\}] \end{aligned}$$

where $H(e^{j\omega})$ is assumed to be real. With $\partial\epsilon(e^{j\omega})/\partial H(e^{j\omega}) = 0$, we obtain

$$H(e^{j\omega}) = \frac{[|S(e^{j\omega})|^2 + \mathcal{R}e\{S^*(e^{j\omega})W(e^{j\omega})\}]}{|S(e^{j\omega}) + W(e^{j\omega})|^2}.$$

If for some ω_0, $S(e^{j\omega_0}) = W(e^{j\omega_0}) = 0$, then $Y(e^{j\omega_0}) = 0$ regardless of the value of $H(e^{j\omega_0})$.

6.61. (a) We have

$$G(e^{j\omega}) = H(e^{j\omega})H(e^{j\omega}) = |H(e^{j\omega})|^2 e^{j2\sphericalangle H(e^{j\omega})}.$$

Therefore,

$$|G(e^{j\omega})| = |H(e^{j\omega})|^2.$$

It follows that the tolerance limits on $|G(e^{j\omega})|$ are given by

$$\begin{aligned} (1-\delta_1)^2 &\le |G(e^{j\omega})| \le (1+\delta_1)^2, \quad 0 \le \omega \le \omega_1 \\ 0 &\le |G(e^{j\omega})| \le \delta_2^2, \quad \omega_2 \le \omega \le \pi \end{aligned}$$

(b) If $\delta_1 << 1$ and $\delta_2 << 1$, then $(1-\delta_1)^2 \approx 1 - 2\delta_1$ and $(1+\delta_1)^2 \approx 1 + 2\delta_1$. Also, $\delta_2^2 < \delta_2$. Therefore, the passband ripple *increases* and the stopband ripple *decreases*.

(c) If N filters are cascaded, then the overall frequency response is

$$G(e^{j\omega}) = |H(e^{j\omega})|^N e^{jN\sphericalangle H(e^{j\omega})}.$$

Therefore,

$$|G(e^{j\omega})| = |H(e^{j\omega})|^N.$$

The tolerance limits are now:

$$\begin{aligned} (1-\delta_1)^N &\le |G(e^{j\omega})| \le (1+\delta_1)^N, \quad 0 \le \omega \le \omega_1 \\ 0 &\le |G(e^{j\omega})| \le \delta_2^N, \quad \omega_2 \le \omega \le \pi \end{aligned}$$

If $\delta_1 << 1$, then $(1-\delta_1)^N \approx 1 - N\delta_1$ and $(1+\delta_1)^N \approx 1 + N\delta_1$. Therefore, the tolerance limits on $|G(e^{j\omega})|$ are given by

$$\begin{aligned} 1 - N\delta_1 &\le |G(e^{j\omega})| \le 1 + N\delta_1, \quad 0 \le \omega \le \omega_1 \\ 0 &\le |G(e^{j\omega})| \le \delta_2^N, \quad \omega_2 \le \omega \le \pi \end{aligned}$$

6.62. (a) From Figure P6.62(a) we have

$$W(e^{j\omega}) = [2X(e^{j\omega}) - X(e^{j\omega})H(e^{j\omega})]\, H(e^{j\omega}).$$

Therefore,

$$G(e^{j\omega}) = \frac{W(e^{j\omega})}{X(e^{j\omega})} = [2 - H(e^{j\omega})]\, H(e^{j\omega}).$$

Let $H(e^{j\omega}) = 1 + \delta_1$. Then $G(e^{j\omega}) = [2 - 1 - \delta_1][1 + \delta_1] = 1 - \delta_1^2$. Let $H(e^{j\omega}) = 1 - \delta_1$. Then $G(e^{j\omega}) = [2 - 1 + \delta_1][1 - \delta_1] = 1 - \delta_1^2$. Therefore,

$$1 - \delta_1^2 \le G(e^{j\omega}) \le 1, \quad 0 \le \omega \le \omega_p.$$

Therefore, $A = 1 - \delta_1^2$ and $B = 1$. Let $H(e^{j\omega}) = -\delta_2$. Then $G(e^{j\omega}) = [2 + \delta_2][-\delta_2] = -2\delta_2 - \delta_2^2$. Let $H(e^{j\omega}) = \delta_2$. Then $G(e^{j\omega}) = [2 - \delta_2][\delta_2] = 2\delta_2 - \delta_2^2$. Therefore,

$$-2\delta_2 - \delta_2^2 \le G(e^{j\omega}) \le 2\delta_2 - \delta_2^2, \quad \omega_s \le \omega \le \pi.$$

Therefore, $C = -2\delta_2 - \delta_2^2$ and $D = 2\delta_2 - \delta_2^2$.

(b) If $\delta_1 << 1$ and $\delta_2 << 1$, then $A \approx 1 - \delta_1^2$, $B \approx 1 + \delta_1^2$, $C \approx -2\delta_2$ and $D \approx 2\delta_2$. Therefore, the passband ripple is smaller and the stopband ripple is larger.

(c) From part (a), we have

$$|G(e^{j\omega})| = |2 - H(e^{j\omega})||H(e^{j\omega})|.$$

Since $|2 - H(e^{j\omega})| \le 2 + |H(e^{j\omega})|$ and $|2 - H(e^{j\omega})| \ge 2 - |H(e^{j\omega})|$, we may write

$$[2 - |H(e^{j\omega})|]\, |H(e^{j\omega})| \le G(e^{j\omega}) \le [2 + |H(e^{j\omega})|]\, |H(e^{j\omega})| \qquad \text{(S6.62-1)}$$

If $H(e^{j\omega}) \approx 1$, then from the above equation we obtain

$$1 \le G(e^{j\omega}) \le 3.$$

If $H(e^{j\omega}) \approx 0$, then from the eq. (S6.62-1) we obtain

$$0 \le G(e^{j\omega}) \le 0.$$

Therefore, the filter is a good approximation of a lowpass filter in the stopband. But in the passband, for some $\theta(\omega)$ it is possible to obtain extremely large ripple. Therefore, overall it is not a good approximation for a lowpass filter.

(d) In Figure P6.62(a) if we attach a N point delay to $H(e^{j\omega})$, then the equivalent filter will be a real filter that is a good approximation to a lowpass filter. We have seen that in such a case the overall system is also a good approximation to lowpass.

6.63. (a) Let $g[n] = nh[n]$. Then,

$$G(e^{j\omega}) = j\frac{dH(e^{j\omega})}{d\omega}.$$

Using Parseval's theorem (an also noting that $g[n]$ is real)

$$\sum_{n=-\infty}^{\infty} g^2[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} |G(e^{j\omega})|^2 d\omega.$$

Therefore,

$$D = \sum_{n=-\infty}^{\infty} n^2 h^2[n] = \frac{1}{2\pi}\left|\frac{dH(e^{j\omega})}{d\omega}\right|^2 d\omega.$$

(b) Replacing $H(e^{j\omega})$ by $|H(e^{j\omega})|e^{j\theta(\omega)}$ in the result of part (a),

$$\begin{aligned} D &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left| e^{j\theta(\omega)}\frac{d|H(e^{j\omega})|}{d\omega} + |H(e^{j\omega})|e^{j\theta(\omega)}\frac{d\theta(\omega)}{d\omega}\right|^2 d\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|\frac{d|H(e^{j\omega})|}{d\omega} + |H(e^{j\omega})|\frac{d\theta(\omega)}{d\omega}\right|^2 d\omega \end{aligned}$$

Let $M(\omega) = |H(e^{j\omega})|$ and $\theta'(\omega) = \frac{d\theta(\omega)}{d\omega}$. Also note that $M(\omega) = M(-\omega)$, $M'(\omega) = M'(-\omega)$ and $\theta'(\omega) = \theta'(-\omega)$. Therefore,

$$D = \frac{1}{2\pi}\int_0^{\pi} \left\{|M'(\omega) + M(\omega)\theta'(\omega)|^2 + |M'(\omega) - M(\omega)\theta'(\omega)|^2\right\} d\omega.$$

Now since the integrand is positive for all ω, it is sufficient to minimize the integrand to minimize D. Therefore,

$$\frac{d}{d\theta'(\omega)}\left\{|M'(\omega) + M(\omega)\theta'(\omega)|^2 + |M'(\omega) - M(\omega)\theta'(\omega)|^2\right\} = 0.$$

Simplifying this, we obtain

$$2M^2(\omega)\theta'(\omega) = 0 \quad \Rightarrow \quad \theta'(\omega) = 0.$$

However, since $\theta(\omega)$ is odd, the only function that satisfies $\theta'(\omega) = 0$ is $\theta(\omega) = 0$.

6.64. (a) From Table 5.1 we know that when a signal is real and even, then its Fourier transform is also real and even. Therefore, using duality, we may say that if the Fourier transform of a signal is real and even, then the signal is real and even. Therefore, $h_r[n] = h_r[-n]$.
By using the time shift property, we know that if $H(e^{j\omega}) = H_r(e^{j\omega})e^{-j\omega M}$, then

$$h[n] = h_r[n - M].$$

(b) We have

$$h[M + n] = h_r[M + n - M] = h_r[n].$$

Also,

$$h[M - n] = h_r[M - n - M] = h_r[-n].$$

Since $h_r[n] = h_r[-n]$,

$$h[M + n] = h[M - n].$$

(c) Since $h[n]$ is causal, $h[-k] = 0$ for $k > 0$. But due to the symmetry property,

$$h[-k] = h_r[-k - M] = h_r[k + M] = h[k + 2M].$$

Therefore,

$$h[k + 2M] = 0 \quad \text{for } k > 0.$$

It follows that

$$h[n] = 0 \quad \text{for } n > 2M.$$

6.65. (a) We have

$$|B(e^{j\omega})|^2 = \frac{1}{1 + \tan^2(\omega/2)} = \frac{1}{\sec^2(\omega/2)} = \cos^2(\omega/2).$$

(b) If $B(e^{j\omega}) = a\cos(\omega/2)$, then

$$|B(e^{j\omega})|^2 = aa^* \cos^2(\omega/2).$$

If we want this to be the same as part (a), then $aa^* = 1$. Therefore,

$$a = e^{j\theta(\omega)}.$$

(c) Taking the Fourier transform of the given difference equation we obtain

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \alpha + \beta e^{-j\omega\gamma} = e^{-j\omega\gamma/2}[\alpha e^{j\omega\gamma/2} + \beta e^{-j\omega\gamma/2}].$$

Comparing with

$$B(e^{j\omega}) = e^{-j\theta(\omega)}\left[\frac{1}{2}e^{j\omega/2} + \frac{1}{2}e^{-j\omega/2}\right],$$

we find that $H(e^{j\omega}) = B(e^{j\omega})$ when

$$\alpha = \beta = \frac{1}{2}, \qquad \gamma = 1.$$

6.66. (a) Since $h_k[n] = e^{j2\pi nk/N}h_0[n]$, we have

$$H_k(e^{j\omega}) = H_0(e^{j(\omega - 2\pi k/N)}).$$

Below are shown the sketches of $H_k(e^{j\omega})$ for $N = 16$ in Figure S6.66.

(b) Overall frequency response of the system is $H_{ov}(e^{j\omega}) = \sum_{k=0}^{N-1} H_k(e^{j\omega})$. For this to be an identity system, we require that $H_{ov}(e^{j\omega}) = 1$ for all ω. Therefore, we want the non-zero portions of the $H_k(e^{j\omega})$s to be non-overlapping and yet cover the region from $-\pi$ to π. We see that this is achieved by having $\omega_c = \pi/N$.

(c) Since $H_{ov}(e^{j\omega}) = \sum_{k=0}^{N-1} H_k(e^{j\omega})$, we have

$$h_{ov}[n] = \sum_{k=0}^{N-1} h_k[n] = \sum_{k=0}^{N-1} h_0[n]e^{j2\pi kn/N} = h_0[n]\sum_{k=0}^{N-1} e^{j2\pi kn/N}.$$

Therefore,

$$r[n] = \sum_{k=0}^{N-1} e^{j2\pi kn/N} = \begin{cases} N, & n = 0, \pm N, \pm 2N, \cdots \\ 0, & \text{otherwise} \end{cases}$$

Therefore, $r[n] = N\sum_{k=-\infty}^{\infty}\delta[n - kN]$ and is as sketched in Figure S6.66.

Figure S6.66

(d) In order for $h[n]$ to be the impulse response of an identity system, we require that $h[n] = \delta[n]$. From part (c), we know that

$$h[n] = h_0[n]\sum_{k=-\infty}^{\infty}\delta[n - kN].$$

Therefore, the necessary and sufficient condition for $h[n]$ to be $\delta[n]$ is

$$h_0[0] = \frac{1}{N} \quad \text{and} \quad h_0[kN] = 0 \quad \text{for } k = \pm 1, \pm 2, \cdots.$$

Chapter 7 Answers

7.1. From the Nyquist sampling theorem, we know that only if $X(j\omega) = 0$ for $|\omega| > \omega_s/2$ will be signal be recoverable from its samples. Therefore, $X(j\omega) = 0$ for $|\omega| > 5000\pi$.

7.2. From the Nyquist theorem, we know that the sampling frequency in this case must be at least $\omega_s = 2000\pi$. In other words, the sampling period should be at most $T = 2\pi/(\omega_s) = 1 \times 10^{-3}$. Clearly, only (a) and (c) satisfy this condition.

7.3. (a) We can easily show that $X(j\omega) = 0$ for $|\omega| > 4000\pi$. Therefore, the Nyquist rate for this signal is $\omega_N = 2(4000\pi) = 8000\pi$.

(b) From Table 4.2 we know that, $X(j\omega)$ is a rectangular pulse for which $X(j\omega) = 0$ for $|\omega| > 4000\pi$. Therefore, the Nyquist rate for this signal is $\omega_N = 2(4000\pi) = 8000\pi$.

(c) From Tables 4.1 and 4.2, we know that $X(j\omega)$ is the convolution of two rectangular pulses each of which is zero for $|\omega| > 4000\pi$. Therefore, $X(j\omega) = 0$ for $|\omega| > 8000\pi$ and the Nyquist rate for this signal is $\omega_N = 2(8000\pi) = 16000\pi$.

7.4. If the signal $x(t)$ has a Nyquist rate of ω_0, then its Fourier transform $X(j\omega) = 0$ for $|\omega| > \omega_0/2$.

(a) From chapter 4,

$$y(t) = x(t) + x(t-1) \stackrel{FT}{\longleftrightarrow} Y(j\omega) = X(j\omega) + e^{-j\omega t}X(j\omega).$$

Clearly, we can only guarantee that $Y(j\omega) = 0$ for $|\omega| > \omega_0/2$. Therefore, the Nyquist rate for $y(t)$ is also ω_0.

(b) From chapter 4,

$$y(t) = \frac{dx(t)}{dt} \stackrel{FT}{\longleftrightarrow} Y(j\omega) = j\omega X(j\omega).$$

Clearly, we can only guarantee that $Y(j\omega) = 0$ for $|\omega| > \omega_0/2$. Therefore, the Nyquist rate for $y(t)$ is also ω_0.

(c) From chapter 4,

$$y(t) = x^2(t) \stackrel{FT}{\longleftrightarrow} Y(j\omega) = (1/2\pi)[X(j\omega) * X(j\omega).$$

Clearly, we can guarantee that $Y(j\omega) = 0$ for $|\omega| > \omega_0$. Therefore, the Nyquist rate for $y(t)$ is $2\omega_0$.

(d) From chapter 4,

$$y(t) = x(t)\cos(\omega_0 t) \stackrel{FT}{\longleftrightarrow} Y(j\omega) = (1/2)X(j(\omega - \omega_0)) + (1/2)X(j(\omega + \omega_0)).$$

Clearly, we can guarantee that $Y(j\omega) = 0$ for $|\omega| > \omega_0 + \omega_0/2$. Therefore, the Nyquist rate for $y(t)$ is $3\omega_0$.