

Chapter 9 Answers

9.1. (a) The given integral may be written as

$$\int_0^{\infty} e^{-(5+\sigma)t} e^{j\omega t} dt.$$

If $\sigma < -5$, then the function $e^{-(5+\sigma)t}$ grows towards ∞ with increasing t and the given integral does not converge. But if $\sigma > -5$, then the integral does converge.

(b) The given integral may be written as

$$\int_{-\infty}^0 e^{-(5+\sigma)t} e^{j\omega t} dt.$$

If $\sigma > -5$, then the function $e^{-(5+\sigma)t}$ grows towards ∞ as t decreases towards $-\infty$ and the given integral does not converge. But if $\sigma < -5$, then the integral does converge.

(c) The given integral may be written as

$$\int_{-5}^5 e^{-(5+\sigma)t} e^{j\omega t} dt.$$

Clearly this integral has a finite value for all finite values of σ .

(d) The given integral may be written as

$$\int_{-\infty}^{\infty} e^{-(5+\sigma)t} e^{j\omega t} dt.$$

If $\sigma > -5$, then the function $e^{-(5+\sigma)t}$ grows towards ∞ as t decreases towards $-\infty$ and the given integral does not converge. If $\sigma < -5$, then the function $e^{-(5+\sigma)t}$ grows towards ∞ with increasing t and the given integral does not converge. If $\sigma = 5$, then the integral still does not have a finite value. Therefore, the integral does not converge for any value of σ .

(e) The given integral may be written as

$$\int_{-\infty}^0 e^{-(-5+\sigma)t} e^{j\omega t} dt + \int_0^{\infty} e^{-(5+\sigma)t} e^{j\omega t} dt.$$

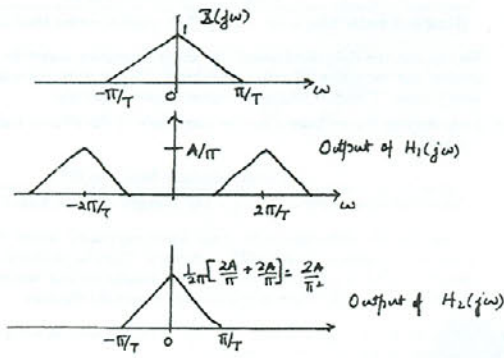
The first integral converges for $\sigma < 5$. The second integral converges if $\sigma > -5$. Therefore, the given integral converges when $|\sigma| < 5$.

(f) The given integral may be written as

$$\int_{-\infty}^0 e^{-(-5+\sigma)t} e^{j\omega t} dt.$$

If $\sigma > 5$, then the function $e^{-(-5+\sigma)t}$ grows towards ∞ as t decreases towards $-\infty$ and the given integral does not converge. But if $\sigma < 5$, then the integral does converge.

Figure S8.49



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9.2. (a)

$$\begin{aligned} X(s) &= \int_{-\infty}^{\infty} e^{-5t} u(t-1) e^{-st} dt \\ &= \int_1^{\infty} e^{-(5+s)t} dt \\ &= \frac{e^{-(5+s)}}{s+5} \end{aligned}$$

As shown in Example 9.1, the ROC will be $\mathcal{R}\{s\} > -5$.

(b) By using eq. (9.3), we can easily show that $g(t) = Ae^{-5t}u(-t-t_0)$ has the Laplace transform

$$G(s) = \frac{Ae^{(s+5)t_0}}{s+5}.$$

The ROC is specified as $\mathcal{R}\{s\} < -5$. Therefore, $A = 1$ and $t_0 = -1$.

9.3. Using an analysis similar to that used in Example 9.3, we know that the given signal has a Laplace transform of the form

$$X(s) = \frac{1}{s+5} + \frac{1}{s+\beta}.$$

The corresponding ROC is $\mathcal{R}\{s\} > \max\{-5, \mathcal{R}\{\beta\}\}$. Since we are given that the ROC is $\mathcal{R}\{s\} > -3$, we know that $\mathcal{R}\{\beta\} = 3$. There are no constraints on the imaginary part of β .

9.4. We know from Table 9.2 that

$$x_1(t) = -e^{-t} \sin(2t) u(t) \xleftrightarrow{\mathcal{L}} X_1(s) = -\frac{2}{(s+1)^2 + 2^2}, \quad \mathcal{R}\{s\} > -1.$$

We also know from Table 9.1 that

$$x(t) = x_1(-t) \xleftrightarrow{\mathcal{L}} X(s) = X_1(-s).$$

The ROC of $X(s)$ is such that if s_0 was in the ROC of $X_1(s)$, then $-s_0$ will be in the ROC of $X(s)$. Putting the two above equations together, we have

$$x(t) = x_1(-t) = -e^{-t} \sin(2t) u(-t) \xleftrightarrow{\mathcal{L}} X(s) = X_1(-s) = -\frac{2}{(s-1)^2 + 2^2}, \quad \mathcal{R}\{s\} < 1.$$

The denominator of $X(s)$ is of the form $s^2 - 2s + 5$. Therefore, the poles of $X(s)$ are $1 + 2j$ and $1 - 2j$.

9.5. (a) The given Laplace transform may be written as

$$X(s) = \frac{2s+4}{(s+1)(s+3)}.$$

Clearly, $X(s)$ has a zero at $s = -2$. Since in $X(s)$ the order of the denominator polynomial exceeds the order of the numerator polynomial by 1, $X(s)$ has a zero at ∞ . Therefore, $X(s)$ has one zero in the finite s -plane and one zero at infinity.

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Chapter 9 Answers

9.1. (a) The given integral may be written as

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If $\sigma < -5$, then the function $e^{-(5+\sigma)t}$ grows towards ∞ with increasing t and the given integral does not converge. But if $\sigma > -5$, then the integral does converge.

(b) The given integral may be written as

$$\int_{-\infty}^0 e^{-(5+\sigma)t} e^{j\omega t} dt.$$

If $\sigma > -5$, then the function $e^{-(5+\sigma)t}$ grows towards ∞ as t decreases towards $-\infty$ and the given integral does not converge. But if $\sigma < -5$, then the integral does converge.

(c) The given integral may be written as

$$\int_{-5}^5 e^{-(5+\sigma)t} e^{j\omega t} dt.$$

Clearly this integral has a finite value for all finite values of σ .

(d) The given integral may be written as

$$\int_{-\infty}^{\infty} e^{-(5+\sigma)t} e^{j\omega t} dt.$$

If $\sigma > -5$, then the function $e^{-(5+\sigma)t}$ grows towards ∞ as t decreases towards $-\infty$ and the given integral does not converge. If $\sigma < -5$, then the function $e^{-(5+\sigma)t}$ grows towards ∞ with increasing t and the given integral does not converge. If $\sigma = 5$, then the integral still does not have a finite value. Therefore, the integral does not converge for any value of σ .

(e) The given integral may be written as

$$\int_{-\infty}^0 e^{-(-5+\sigma)t} e^{j\omega t} dt + \int_0^{\infty} e^{-(5+\sigma)t} e^{j\omega t} dt.$$

The first integral converges for $\sigma < 5$. The second integral converges if $\sigma > -5$. Therefore, the given integral converges when $|\sigma| < 5$.

(f) The given integral may be written as

$$\int_{-\infty}^0 e^{-(-5+\sigma)t} e^{j\omega t} dt.$$

If $\sigma > 5$, then the function $e^{-(-5+\sigma)t}$ grows towards ∞ as t decreases towards $-\infty$ and the given integral does not converge. But if $\sigma < 5$, then the integral does converge.

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(b) The given Laplace transform may be written as

$$X(s) = \frac{s+1}{(s-1)(s+1)} = \frac{1}{s-1}.$$

Clearly, $X(s)$ has no zeros in the finite s -plane. Since in $X(s)$ the order of the denominator polynomial exceeds the order of the numerator polynomial by 1, $X(s)$ has a zero at ∞ . Therefore, $X(s)$ has no zeros in the finite s -plane and one zero at infinity.

(c) The given Laplace transform may be written as

$$X(s) = \frac{(s-1)(s^2+s+1)}{(s^2+s+1)} = s-1.$$

Clearly, $X(s)$ has a zero at $s = 1$. Since in $X(s)$ the order of the numerator polynomial exceeds the order of the denominator polynomial by 1, $X(s)$ has no zeros at ∞ . Therefore, $X(s)$ has one zero in the finite s -plane and no zeros at infinity.

9.6. (a) No. From property 3 in Section 9.2 we know that for a finite-length signal, the ROC is the entire s -plane. Therefore, there can be no poles in the finite s -plane for a finite-length signal. Clearly, in this problem this is not the case.

(b) Yes. Since the signal is absolutely integrable, the ROC must include the $j\omega$ -axis. Furthermore, $X(s)$ has a pole at $s = 2$. Therefore, one valid ROC for the signal would be $\mathcal{R}\{s\} < 2$. From property 5 in Section 9.2 we know that this would correspond to a left-sided signal.

(c) No. Since the signal is absolutely integrable, the ROC must include the $j\omega$ -axis. Furthermore, $X(s)$ has a pole at $s = 2$. Therefore, we can never have an ROC of the form $\mathcal{R}\{s\} > \alpha$. From property 4 in Section 9.2 we know that $x(t)$ cannot be a right-sided signal.

(d) Yes. Since the signal is absolutely integrable, the ROC must include the $j\omega$ -axis. Furthermore, $X(s)$ has a pole at $s = 2$. Therefore, a valid ROC for the signal could be $\alpha < \mathcal{R}\{s\} < 2$ such that $\alpha < 0$. From property 6 in Section 9.2, we know that this would correspond to a two-sided signal.

9.7. We may find different signals with the given Laplace transform by choosing different regions of convergence. The poles of the given Laplace transform are

$$s_0 = -2, \quad s_1 = -3, \quad s_2 = -\frac{1}{2} + \frac{\sqrt{3}}{2}j, \quad s_3 = -\frac{1}{2} - \frac{\sqrt{3}}{2}j.$$

Based on the locations of these poles, we may choose from the following regions of convergence:

- $\mathcal{R}\{s\} > -\frac{1}{2}$
- $-2 < \mathcal{R}\{s\} < -\frac{1}{2}$
- $-3 < \mathcal{R}\{s\} < -2$
- $\mathcal{R}\{s\} < -3$

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Therefore, we may find four different signals with the given Laplace transform.

9.8. From Table 9.1, we know that

$$g(t) = e^{2t}x(t) \xleftrightarrow{\mathcal{L}} G(s) = X(s-2).$$

The ROC of $G(s)$ is the ROC of $X(s)$ shifted to the right by 2.

We are also given that $X(s)$ has exactly 2 poles, located at $s = -1$ and $s = -3$. Since $G(s) = X(s-2)$, $G(s)$ also has exactly two poles, located at $s = -1+2 = 1$ and $s = -3+2 = -1$. Since we are given $G(j\omega)$ exists, we may infer that the $j\omega$ -axis lies in the ROC of $G(s)$. Given this fact and the locations of the poles, we may conclude that $g(t)$ is a two-sided sequence. Obviously $x(t) = e^{-2t}g(t)$ will also be two-sided.

9.9. Using partial fraction expansion

$$X(s) = \frac{4}{s+4} - \frac{2}{s+3}.$$

Taking the inverse Laplace transform,

$$x(t) = 4e^{-4t}u(t) - 2e^{-3t}u(t).$$

9.10. The pole-zero plots for each of the three Laplace transforms is as shown in Figure S9.10.

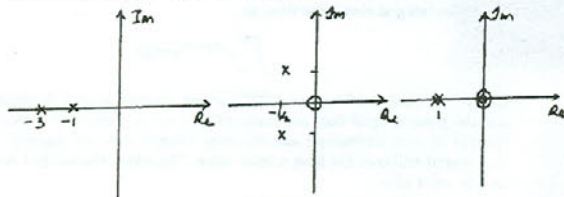


Figure S9.10

(a) From Section 9.4 we know that the magnitude of the Fourier transform may be expressed as

$$\frac{1}{(\text{Length of vector from } \omega \text{ to } -1)(\text{Length of vector from } \omega \text{ to } -2)}$$

We see that the right-hand side of the above expression is maximum for $\omega = 0$ and decreases as ω becomes increasingly more positive or more negative. Therefore $|H_1(j\omega)|$ is approximately lowpass.

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Therefore,

$$G(s) = \beta \left[\frac{1-s+\alpha s+\alpha}{1-s^2} \right].$$

Comparing with the given equation for $G(s)$,

$$\alpha = -1, \quad \beta = \frac{1}{2}.$$

9.14. Since $X(s)$ has 4 poles and no zeros in the finite s -plane, we may assume that $X(s)$ is of the form

$$X(s) = \frac{A}{(s-a)(s-b)(s-c)(s-d)}.$$

Since $x(t)$ is real, the poles of $X(s)$ must occur in conjugate reciprocal pairs. Therefore, we may assume that $b = a^*$ and $d = c^*$. This results in

$$X(s) = \frac{A}{(s-a)(s-a^*)(s-c)(s-c^*)}.$$

Since the signal $x(t)$ is also even, the Laplace transform $X(s)$ must also be even. This implies that the poles have to be symmetric about the $j\omega$ -axis. Therefore, we may assume that $c = -a^*$. This results in

$$X(s) = \frac{A}{(s-a)(s-a^*)(s+a^*)(s+a)}.$$

We are given that the location of one of the poles is $(1/2)e^{j\pi/4}$. If we assume that this pole is a , we have

$$X(s) = \frac{A}{(s - \frac{1}{2}e^{j\pi/4})(s - \frac{1}{2}e^{-j\pi/4})(s + \frac{1}{2}e^{-j\pi/4})(s + \frac{1}{2}e^{j\pi/4})}.$$

This gives us

$$X(s) = \frac{A}{(s^2 - \frac{j}{\sqrt{2}} + \frac{1}{4})(s^2 + \frac{j}{\sqrt{2}} + \frac{1}{4})}.$$

Also, we are given that

$$\int_{-\infty}^{\infty} x(t)dt = X(0) = 4.$$

Substituting in the above expression for $X(s)$, we have $A = 1/4$. Therefore,

$$X(s) = \frac{(1/4)}{(s^2 - \frac{j}{\sqrt{2}} + \frac{1}{4})(s^2 + \frac{j}{\sqrt{2}} + \frac{1}{4})}.$$

9.15. Taking the Laplace transforms of both sides of the two differential equations, we have

$$sX(s) = -2Y(s) + 1 \quad \text{and} \quad sY(s) = 2X(s).$$

Solving for $X(s)$ and $Y(s)$, we obtain

$$X(s) = \frac{s}{s^2 + 4} \quad \text{and} \quad Y(s) = 2s^2 + 4.$$

The region of convergence for both $X(s)$ and $Y(s)$ is $\mathcal{R}\{s\} > 0$ because both are right-sided signals.

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(b) From Section 9.4 we know that the magnitude of the Fourier transform may be expressed as

$$\frac{(\text{Length of vector from } \omega \text{ to } 0)}{(\text{Length of vector from } \omega \text{ to } -\frac{1}{2} + j\frac{\sqrt{3}}{2})(\text{Length of vector from } \omega \text{ to } -\frac{1}{2} - j\frac{\sqrt{3}}{2})}$$

We see that the right-hand side of the above expression is zero for $\omega = 0$. It then increases with increasing $|\omega|$ until $|\omega|$ reaches $\frac{1}{2}$. Then it starts decreasing as $|\omega|$ increases even further. Therefore $|H_2(j\omega)|$ is approximately bandpass.

(c) From Section 9.4 we know that the magnitude of the Fourier transform may be expressed as

$$\frac{(\text{Length of vector from } \omega \text{ to } 0)^2}{(\text{Length of vector from } \omega \text{ to } -\frac{1}{2} + j\frac{\sqrt{3}}{2})(\text{Length of vector from } \omega \text{ to } -\frac{1}{2} - j\frac{\sqrt{3}}{2})}$$

We see that the right-hand side of the above expression is zero for $\omega = 0$. It then increases with increasing $|\omega|$ until $|\omega|$ reaches $\frac{1}{2}$. Then $|\omega|$ increases, $|H_3(j\omega)|$ decreases towards a value of 1 (because all the vector lengths become almost identical and the ratio becomes 1). Therefore $|H_3(j\omega)|$ is approximately highpass.

9.11. $X(s)$ has poles at $s = -\frac{1}{2} + j\frac{\sqrt{3}}{2}$ and $-\frac{1}{2} - j\frac{\sqrt{3}}{2}$. $X(s)$ has zeros at $s = \frac{1}{2} + j\frac{\sqrt{3}}{2}$ and $\frac{1}{2} - j\frac{\sqrt{3}}{2}$. From Section 9.4, we know that $|X(j\omega)|$ is

$$\frac{(\text{Length of vector from } \omega \text{ to } \frac{1}{2} + j\frac{\sqrt{3}}{2})(\text{Length of vector from } \omega \text{ to } \frac{1}{2} - j\frac{\sqrt{3}}{2})}{(\text{Length of vector from } \omega \text{ to } -\frac{1}{2} + j\frac{\sqrt{3}}{2})(\text{Length of vector from } \omega \text{ to } -\frac{1}{2} - j\frac{\sqrt{3}}{2})}$$

The terms in the numerator and denominator of the right-hand side of the above expression cancel out giving us $|X(j\omega)| = 1$

9.12. (a) If $X(s)$ has only one pole, then $x(t)$ would be of the form Ae^{-at} . Clearly such a signal violates condition 2. Therefore, this statement is inconsistent with the given information.

(b) If $X(s)$ has only two poles, then $x(t)$ would be of the form $Ae^{-at}\sin(\omega_0 t)$. Clearly such a signal could be made to satisfy all three conditions (Example: $\omega_0 = 80\pi$, $a = 19200$). Therefore, this statement is consistent with the given information.

(c) If $X(s)$ has more than two poles (say 4 poles), then $x(t)$ could be assumed to be of the form $Ae^{-at}\sin(\omega_0 t) + Be^{-bt}\sin(\omega_0 t)$. Clearly such a signal could still be made to satisfy all three conditions. Therefore, this statement is consistent with the given information.

9.13. We have

$$X(s) = \frac{\beta}{s+1}, \quad \mathcal{R}\{s\} > -1.$$

Also,

$$G(s) = X(s) + \alpha X(-s), \quad -1 < \mathcal{R}\{s\} < 1.$$

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9.16. Taking the Laplace transform of both sides of the given differential equation, we obtain

$$Y(s)[s^3 + (1+\alpha)s^2 + \alpha(\alpha+1)s + \alpha^2] = X(s).$$

Therefore,

$$H(s) = \frac{Y(s)}{X(s)} = \frac{1}{s^3 + (1+\alpha)s^2 + \alpha(\alpha+1)s + \alpha^2}.$$

(a) Taking the Laplace transform of both sides of the given equation, we have

$$G(s) = sH(s) + H(s).$$

Substituting for $H(s)$ from above,

$$G(s) = \frac{(s+1)}{s^3 + (1+\alpha)s^2 + \alpha(\alpha+1)s + \alpha^2} = \frac{1}{s^2 + \alpha s + \alpha^2}.$$

Therefore, $G(s)$ has 2 poles.

(b) We know that

$$H(s) = \frac{1}{(s+1)(s^2 + \alpha s + \alpha^2)}.$$

Therefore, $H(s)$ has poles at -1 , $\alpha(-\frac{1}{2} + j\frac{\sqrt{3}}{2})$, and $\alpha(-\frac{1}{2} - j\frac{\sqrt{3}}{2})$. If the system has to be stable, then the real part of the poles has to be less than zero. For this to be true, we require that $-\alpha/2 < 0$, i.e., $\alpha > 0$.

9.17. The overall system shown in Figure P9.17 may be treated as two feedback systems of the form shown in Figure 9.31 connected in parallel. By carrying out an analysis similar to that described in Section 9.8.1, we find the system function of the upper feedback system to be

$$H_1(s) = \frac{2/s}{1 + 4(2/s)} = \frac{2}{s+8}.$$

Similarly, the system function of the lower feedback system is

$$H_2(s) = \frac{1/s}{1 + 2(1/2)} = \frac{1}{s+2}.$$

The system function of the overall system is now

$$H(s) = H_1(s) + H_2(s) = \frac{3s+12}{s^2+10s+16}.$$

Since $H(s) = Y(s)/X(s)$, we may write

$$Y(s)[s^2 + 10s + 16] = X(s)[3s + 12].$$

Taking the inverse Laplace transform, we obtain

$$\frac{d^2y(t)}{dt^2} + 10\frac{dy(t)}{dt} + 16y(t) = 12x(t) + 3\frac{dx(t)}{dt}.$$

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- 9.18. (a) From Problem 3.20, we know that the differential equation relating the input and output of the RLC circuit is

$$\frac{d^2 y(t)}{dt^2} + \frac{dy(t)}{dt} + y(t) = x(t).$$

Taking the Laplace transform of this (while noting that the system is causal and stable), we obtain

$$Y(s)[s^2 + s + 1] = X(s).$$

Therefore,

$$H(s) = \frac{Y(s)}{X(s)} = \frac{1}{s^2 + s + 1}, \quad \mathcal{R}\{s\} > -\frac{1}{2}.$$

- (b) We note that $H(s)$ has two poles at $s = -\frac{1}{2} - j\frac{\sqrt{3}}{2}$ and $s = -\frac{1}{2} + j\frac{\sqrt{3}}{2}$. It has no zeros in the finite s -plane. From Section 9.4 we know that the magnitude of the Fourier transform may be expressed as

$$\frac{1}{(\text{Length of vector from } \omega \text{ to } -\frac{1}{2} - j\frac{\sqrt{3}}{2})(\text{Length of vector from } \omega \text{ to } -\frac{1}{2} + j\frac{\sqrt{3}}{2})}$$

We see that the right-hand side of the above expression increases with increasing $|\omega|$ until $|\omega|$ reaches $\frac{1}{2}$. Then it starts decreasing as $|\omega|$ increases even further. It finally reaches 0 for $|\omega| = \infty$. Therefore $H_2(j\omega)$ is approximately lowpass.

- (c) By repeating the analysis carried out in Problem 3.20 and part (a) of this problem with $R = 10^{-3}\Omega$, we can show that

$$H(s) = \frac{Y(s)}{X(s)} = \frac{1}{s^2 + 10^{-3}s + 1}, \quad \mathcal{R}\{s\} > -0.0005.$$

- (d) We have

$$\frac{1}{(\text{Vect. Len. from } \omega \text{ to } -0.0005 + j\frac{\sqrt{3}}{2})(\text{Vect. Len. from } \omega \text{ to } -0.0005 - j\frac{\sqrt{3}}{2})}$$

We see that when $|\omega|$ is in the vicinity 0.0005, the right-hand side of the above equation takes on extremely large values. On either side of this value of $|\omega|$ the value of $|H(j\omega)|$ rolls off rapidly. Therefore, $H(s)$ may be considered to be approximately bandpass.

- 9.19. (a) The unilateral Laplace transform is

$$\begin{aligned} X(s) &= \int_0^\infty e^{-2t}u(t+1)e^{-st}dt \\ &= \int_0^\infty e^{-2t}e^{-st}dt \\ &= \frac{1}{s+2} \end{aligned}$$

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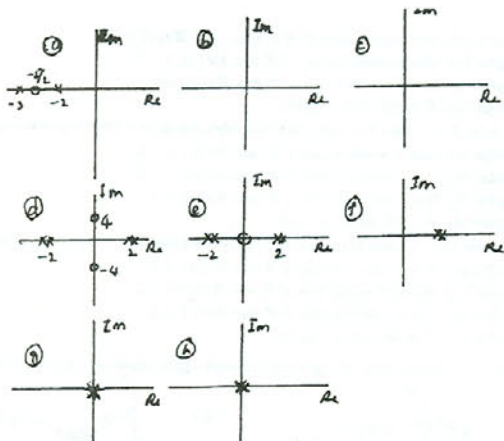


Figure S9.21

- (c) The total response is the sum of the zero-state and zero-input responses. This is

$$y(t) = 2e^{-t}u(t) - e^{-2t}u(t).$$

- 9.21. The pole-zero plots for all the subparts are shown in Figure S9.21.

- (a) The Laplace transform of $x(t)$ is

$$\begin{aligned} X(s) &= \int_0^\infty (e^{-2t} + e^{-3t})e^{-st}dt \\ &= \left[-\frac{e^{-(s+2)t}}{(s+2)} \right]_0^\infty + \left[-\frac{e^{-(s+3)t}}{(s+3)} \right]_0^\infty \\ &= \frac{1}{s+2} + \frac{1}{s+3} = \frac{2s+5}{s^2+5s+6} \end{aligned}$$

The region of convergence (ROC) is $\mathcal{R}\{s\} > -2$.

- (b) Using an approach similar to that shown in part (a), we have

$$e^{-4t}u(t) \xrightarrow{\mathcal{L}} \frac{1}{s+4}, \quad \mathcal{R}\{s\} > -4.$$

Also,

$$e^{-5t}e^{j5t}u(t) \xrightarrow{\mathcal{L}} \frac{1}{s+5-j5}, \quad \mathcal{R}\{s\} > -5.$$

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- (b) The unilateral Laplace transform is

$$\begin{aligned} X(s) &= \int_0^\infty [\delta(t+1) + \delta(t) + e^{-2(t+3)}u(t+1)]e^{-st}dt \\ &= \int_0^\infty [\delta(t) + e^{-2(t+3)}]e^{-st}dt \\ &= 1 + \frac{e^{-6}}{s+2} \end{aligned}$$

- (c) The unilateral Laplace transform is

$$\begin{aligned} X(s) &= \int_0^\infty [e^{-2t}u(t)e^{-4t}u(t)]e^{-st}dt \\ &= \int_0^\infty [e^{-2t} + e^{-4t}]e^{-st}dt \\ &= \frac{1}{s+2} + \frac{1}{s+4} \end{aligned}$$

- 9.20. In Problem 3.19, we showed that the input and output of the RL circuit are related by

$$\frac{dy(t)}{dt} + y(t) = x(t).$$

Applying the unilateral Laplace transform to this equation, we have

$$sY(s) - y(0^-) + Y(s) = X(s).$$

- (a) For the zero-state response, set $y(0^-) = 0$. Also we have

$$X(s) = \mathcal{UL}\{e^{-2t}u(t)\} = \frac{1}{s+2}.$$

Therefore,

$$Y(s)(s+1) = \frac{1}{s+2}.$$

Computing the partial fraction expansion of the right-hand side of the above equation and then taking its inverse unilateral Laplace transform, we have

$$y(t) = e^{-t}u(t) - e^{-2t}u(t).$$

- (b) For the zero-input response, assume that $x(t) = 0$. Since we are given that $y(0^-) = 1$,

$$sY(s) - 1 + Y(s) = 0 \Rightarrow Y(s) = \frac{1}{s+1}.$$

Taking the inverse unilateral Laplace transform we have

$$y(t) = e^{-t}u(t).$$

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and

$$e^{-5t}e^{-j5t}u(t) \xrightarrow{\mathcal{L}} \frac{1}{s+5-j5}, \quad \mathcal{R}\{s\} > -5.$$

From this we obtain

$$e^{-5t}\sin(5t)u(t) = \frac{1}{2j}[e^{-5t}e^{j5t} - e^{-5t}e^{-j5t}]u(t) \xrightarrow{\mathcal{L}} \frac{5}{(s+5)^2 + 25},$$

where $\mathcal{R}\{s\} > -5$. Therefore,

$$e^{-4t}u(t) + e^{-5t}\sin(5t)u(t) \xrightarrow{\mathcal{L}} \frac{s^2 + 15s + 70}{s^3 + 14s^2 + 90s + 100}, \quad \mathcal{R}\{s\} > -5.$$

- (c) The Laplace transform of $x(t)$ is

$$\begin{aligned} X(s) &= \int_0^\infty (e^{2t} + e^{3t})e^{-st}dt \\ &= \left[-\frac{e^{-(s-2)t}}{(s-2)} \right]_0^\infty + \left[-\frac{e^{-(s-3)t}}{(s-3)} \right]_0^\infty \\ &= \frac{1}{s-2} + \frac{1}{s-3} = \frac{2s-5}{s^2-5s+6} \end{aligned}$$

The region of convergence (ROC) is $\mathcal{R}\{s\} < 2$.

- (d) Using an approach along the lines of part (a), we obtain

$$e^{-2t}u(t) \xrightarrow{\mathcal{L}} \frac{1}{s+2}, \quad \mathcal{R}\{s\} > -2. \quad (\text{S9.21-1})$$

Using an approach along the lines of part (c), we obtain

$$e^{2t}u(-t) \xrightarrow{\mathcal{L}} \frac{1}{s-2}, \quad \mathcal{R}\{s\} < 2. \quad (\text{S9.21-2})$$

From these we obtain

$$e^{-2|t|} = e^{-2t}u(t) + e^{2t}u(-t) \xrightarrow{\mathcal{L}} \frac{2s}{s^2-4}, \quad -2 < \mathcal{R}\{s\} < 2.$$

Using the differentiation in the s -domain property, we obtain

$$te^{-2|t|} \xrightarrow{\mathcal{L}} -\frac{d}{ds} \left[\frac{2s}{s^2-4} \right] = -\frac{2s^2+8}{(s^2-4)^2}, \quad -2 < \mathcal{R}\{s\} < 2.$$

- (e) Using the differentiation in the s -domain property on eq. (S9.21-1), we get

$$te^{-2t}u(t) \xrightarrow{\mathcal{L}} -\frac{d}{ds} \left[\frac{1}{s+2} \right] = \frac{1}{(s+2)^2}, \quad \mathcal{R}\{s\} > -2.$$

Using the differentiation in the s -domain property on eq. (S9.21-2), we get

$$-te^{2t}u(-t) \xrightarrow{\mathcal{L}} \frac{d}{ds} \left[\frac{1}{s-2} \right] = -\frac{1}{(s-2)^2}, \quad \mathcal{R}\{s\} < 2.$$

Therefore,

$$|t|e^{-2|t|} = te^{-2t}u(t) + -te^{2t}u(-t) \xrightarrow{\mathcal{L}} \frac{-4s}{(s+2)^2(s-2)^2}, \quad -2 < \mathcal{R}\{s\} < 2.$$

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(f) From the previous part, we have

$$|t|e^{2t}u(-t) \xleftrightarrow{\mathcal{L}} -\frac{1}{(s-2)^2}, \quad \mathcal{R}\{s\} < 2.$$

(g) Note that the given signal may be written as $x(t) = u(t) - u(t-1)$. Note that

$$u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s}, \quad \mathcal{R}\{s\} > 0.$$

Using the time shifting property, we get

$$u(t-1) \xleftrightarrow{\mathcal{L}} \frac{e^{-s}}{s}, \quad \mathcal{R}\{s\} > 0.$$

Therefore,

$$u(t) - u(t-1) \xleftrightarrow{\mathcal{L}} \frac{1-e^{-s}}{s}, \quad \text{All } s.$$

Note that in this case, since the signal is finite duration, the ROC is the entire s -plane.

(h) Consider the signal $x_1(t) = t[u(t) - u(t-1)]$. Note that the signal $x(t)$ may be expressed as $x(t) = x_1(t) + x_1(-t+2)$. We have from the previous part

$$u(t) - u(t-1) \xleftrightarrow{\mathcal{L}} \frac{1-e^{-s}}{s}, \quad \text{All } s.$$

Using the differentiation in s -domain property, we have

$$x_1(t) = t[u(t) - u(t-1)] \xleftrightarrow{\mathcal{L}} \frac{d}{ds} \left[\frac{1-e^{-s}}{s} \right] = \frac{se^{-s} - 1 + e^{-s}}{s^2}, \quad \text{All } s.$$

Using the time-scaling property, we obtain

$$x_1(-t) \xleftrightarrow{\mathcal{L}} \frac{-se^s - 1 + e^s}{s^2}, \quad \text{All } s.$$

Then, using the shift property, we have

$$x_1(-t+2) \xleftrightarrow{\mathcal{L}} e^{-2s} \frac{-se^s - 1 + e^s}{s^2}, \quad \text{All } s.$$

Therefore,

$$x(t) = x_1(t) + x_1(-t+2) \xleftrightarrow{\mathcal{L}} \frac{se^{-s} - 1 + e^{-s}}{s^2} + e^{-2s} \frac{-se^s - 1 + e^s}{s^2}, \quad \text{All } s.$$

(i) The Laplace transform of $x(t) = \delta(t) + u(t)$ is $X(s) = 1 + 1/s$, $\mathcal{R}\{s\} > 0$.

(j) Note that $\delta(3t) + u(3t) = \delta(t) + u(t)$. Therefore, the Laplace transform is the same as the result of the previous part.

9.22. (a) From Table 9.2, we have

$$x(t) = \frac{1}{3} \sin(3t)u(t).$$

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(g) We may rewrite $X(s)$ as

$$X(s) = 1 - \frac{3s}{(s+1)^2}.$$

From Table 9.2, we know that

$$tu(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s^2}, \quad \mathcal{R}\{s\} > 0.$$

Using the shifting property, we obtain

$$e^{-t}tu(t) \xleftrightarrow{\mathcal{L}} \frac{1}{(s+1)^2}, \quad \mathcal{R}\{s\} > -1.$$

Using the differentiation property,

$$\frac{d}{dt}[e^{-t}tu(t)] = e^{-t}u(t) - te^{-t}u(t) \xleftrightarrow{\mathcal{L}} \frac{s}{(s+1)^2}, \quad \mathcal{R}\{s\} > -1.$$

Therefore,

$$x(t) = \delta(t) - 3e^{-t}u(t) - 3te^{-t}u(t).$$

9.23. The four pole-zero plots shown may have the following possible ROCs:

- Plot (a): $\mathcal{R}\{s\} < -2$ or $-2 < \mathcal{R}\{s\} < 2$ or $\mathcal{R}\{s\} > 2$.
- Plot (b): $\mathcal{R}\{s\} < -2$ or $\mathcal{R}\{s\} > -2$.
- Plot (c): $\mathcal{R}\{s\} < 2$ or $\mathcal{R}\{s\} > 2$.
- Plot (d): Entire s -plane.

Also, suppose that the signal $x(t)$ has a Laplace transform $X(s)$ with ROC R .

(1) We know from Table 9.1 that

$$e^{-3t}x(t) \xleftrightarrow{\mathcal{L}} X(s+3).$$

The ROC R_1 of this new Laplace transform is R shifted by 3 to the left. If $x(t)e^{-3t}$ is absolutely integrable, then R_1 must include the $j\omega$ axis.

- For plot (a), this is possible only if R was $\mathcal{R}\{s\} > 2$.
- For plot (b), this is possible only if R was $\mathcal{R}\{s\} > -2$.
- For plot (c), this is possible only if R was $\mathcal{R}\{s\} > 2$.
- For plot (d), R is the entire s -plane.

(2) We know from Table 9.2 that

$$e^{-t}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+1}, \quad \mathcal{R}\{s\} > -1.$$

Also, from Table 9.1 we obtain

$$x(t) * [e^{-t}u(t)] \xleftrightarrow{\mathcal{L}} \frac{X(s)}{s+1}, \quad R_2 = R \cap \{\mathcal{R}\{s\} > -1\}.$$

If $e^{-t}u(t) * x(t)$ is absolutely integrable, then R_2 must include the $j\omega$ -axis.

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(b) From Table 9.2 we know that

$$\cos(3t)u(t) \xleftrightarrow{\mathcal{L}} \frac{s}{s^2+9}, \quad \mathcal{R}\{s\} > 0.$$

Using the time scaling property, we obtain

$$\cos(3t)u(-t) \xleftrightarrow{\mathcal{L}} -\frac{s}{s^2+9}, \quad \mathcal{R}\{s\} < 0.$$

Therefore, the inverse Laplace transform of $X(s)$ is

$$x(t) = -\cos(3t)u(-t).$$

(c) From Table 9.2 we know that

$$e^t \cos(3t)u(t) \xleftrightarrow{\mathcal{L}} \frac{s-1}{(s-1)^2+9}, \quad \mathcal{R}\{s\} > 1.$$

Using the time scaling property, we obtain

$$e^{-t} \cos(3t)u(-t) \xleftrightarrow{\mathcal{L}} -\frac{s+1}{(s+1)^2+9}, \quad \mathcal{R}\{s\} < -1.$$

Therefore, the inverse Laplace transform of $X(s)$ is

$$x(t) = -e^{-t} \cos(3t)u(-t).$$

(d) Using partial fraction expansion on $X(s)$, we obtain

$$X(s) = \frac{2}{s+4} - \frac{1}{s+3}.$$

From the given ROC, we know that $x(t)$ must be a two-sided signal. Therefore,

$$x(t) = 2e^{-4t}u(t) + e^{-3t}u(-t).$$

(e) Using partial fraction expansion on $X(s)$, we obtain

$$X(s) = \frac{2}{s+3} - \frac{1}{s+2}.$$

From the given ROC, we know that $x(t)$ must be a two-sided signal. Therefore,

$$x(t) = 2e^{-3t}u(t) + e^{-2t}u(-t).$$

(f) We may rewrite $X(s)$ as

$$\begin{aligned} X(s) &= 1 + \frac{3s}{s^2-s+1} \\ &= 1 + \frac{3s}{(s-1/2)^2 + (\sqrt{3}/2)^2} \\ &= 1 + 3 \frac{s-1/2}{(s-1/2)^2 + (\sqrt{3}/2)^2} + \frac{3/2}{(s-1/2)^2 + (\sqrt{3}/2)^2} \end{aligned}$$

Using Table 9.2, we obtain

$$x(t) = \delta(t) + 3e^{-t/2} \cos(\sqrt{3}t/2)u(t) + \sqrt{3}e^{-t/2} \sin(\sqrt{3}t/2)u(t).$$

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- For plot (a), this is possible only if R was $-2 < \mathcal{R}\{s\} < 2$.
- For plot (b), this is possible only if R was $\mathcal{R}\{s\} > -2$.
- For plot (c), this is possible only if R was $\mathcal{R}\{s\} < 2$.
- For plot (d), R is the entire s -plane.

(3) If $x(t) = 0$ for $t > 1$, then the signal is a left-sided signal or a finite-duration signal.

- For plot (a), this is possible only if R was $\mathcal{R}\{s\} < -2$.
- For plot (b), this is possible only if R was $\mathcal{R}\{s\} < -2$.
- For plot (c), this is possible only if R was $\mathcal{R}\{s\} < 2$.
- For plot (d), R is the entire s -plane.

(4) If $x(t) = 0$ for $t < -1$, then the signal is a right-sided signal or a finite-duration signal.

- For plot (a), this is possible only if R was $\mathcal{R}\{s\} > 2$.
- For plot (b), this is possible only if R was $\mathcal{R}\{s\} > -2$.
- For plot (c), this is possible only if R was $\mathcal{R}\{s\} > 2$.
- For plot (d), R is the entire s -plane.

9.24. (a) The pole-zero diagram with the appropriate markings is shown in Figure S9.24.

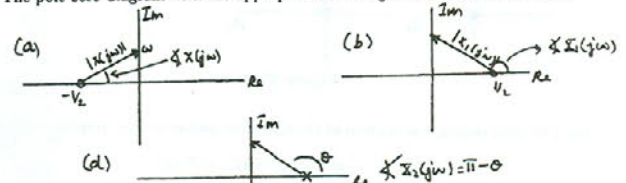


Figure S9.24

(b) By inspecting the pole-zero diagram of part (a), it is clear that the pole-zero diagram shown in Figure S9.24 will also result in the same $|X(j\omega)|$. This would correspond to the Laplace transform

$$X_1(s) = s - \frac{1}{2}, \quad \mathcal{R}\{s\} < \frac{1}{2}.$$

(c) $\angle X(j\omega) = \pi - \angle X_1(j\omega)$.

(d) $X_2(s)$ with the pole-zero diagram shown below in Figure S9.24 would have the property that $\angle X_2(j\omega) = \angle X(j\omega)$. Here, $X_2(s) = s - \frac{1}{2}$.

(e) $|X_2(j\omega)| = 1/|X(j\omega)|$.

(f) From the result of part (b), it is clear that $X_1(s)$ may be obtained by reflecting the poles and zeros in the right-half of the s -plane to the left-half of the s -plane. Therefore,

$$X_1(s) = \frac{s+1/2}{s+2}.$$

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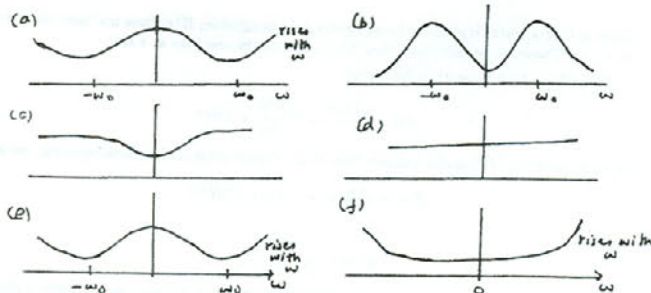


Figure S9.25

From part (d), it is clear that $X_2(s)$ may be obtained by reflecting the poles (zeros) in the right-half of the s -plane to the left-half and simultaneously changing them to zeros (poles). Therefore,

$$X_2(s) = \frac{(s+1)^2}{(s+1/2)(s+2)}$$

9.25. The plots are as shown in Figure S9.25.

9.26. From Table 9.2 we have

$$x_1(t) = e^{-2t}u(t) \xleftrightarrow{\mathcal{L}} X_1(s) = \frac{1}{s+2}, \quad \mathcal{R}\{s\} > -2$$

and

$$x_1(t) = e^{-3t}u(t) \xleftrightarrow{\mathcal{L}} X_1(s) = \frac{1}{s+3}, \quad \mathcal{R}\{s\} > -3.$$

Using the time-shifting time-scaling properties from Table 9.1, we obtain

$$x_1(t-2) \xleftrightarrow{\mathcal{L}} e^{-2s}X_1(s) = \frac{e^{-2s}}{s+2}, \quad \mathcal{R}\{s\} > -2$$

and

$$x_2(-t+3) \xleftrightarrow{\mathcal{L}} e^{-3s}X_2(-s) = \frac{e^{-3s}}{3-s}, \quad \mathcal{R}\{s\} > -3.$$

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(b) Since $y(t) = x(t) * h(t)$, we may use the convolution property to obtain

$$Y(s) = X(s)H(s) = \frac{1}{(s+1)(s+2)}.$$

The ROC of $Y(s)$ is $\mathcal{R}\{s\} > -1$.

(c) Performing partial fraction expansion on $Y(s)$, we obtain

$$Y(s) = \frac{1}{s+1} - \frac{1}{s+2}.$$

Taking the inverse Laplace transform, we get

$$y(t) = e^{-t}u(t) - e^{-2t}u(t).$$

(d) Explicit convolution of $x(t)$ and $h(t)$ gives us

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau \\ &= \int_0^{\infty} e^{-2\tau}e^{-(t-\tau)}u(t-\tau)d\tau \\ &= e^{-t} \int_0^t e^{-\tau}d\tau \quad \text{for } t > 0 \\ &= [e^{-t} - e^{-2t}]u(t). \end{aligned}$$

9.30. For the input $x(t) = u(t)$, the Laplace transform is

$$X(s) = \frac{1}{s}, \quad \mathcal{R}\{s\} > 0.$$

The corresponding output $y(t) = [1 - e^{-t} - te^{-t}]u(t)$ has the Laplace transform

$$Y(s) = \frac{1}{s} - \frac{1}{s+1} - \frac{1}{(s+1)^2} = \frac{1}{s(s+1)^2}, \quad \mathcal{R}\{s\} > 0.$$

Therefore,

$$H(s) = \frac{Y(s)}{X(s)} = \frac{1}{(s+1)^2}, \quad \mathcal{R}\{s\} > 0.$$

Now, the output $y_1(t) = [2 - 3e^{-t} + e^{-3t}]u(t)$ has the Laplace transform

$$Y_1(s) = \frac{2}{s} - \frac{3}{s+1} + \frac{1}{s+3} = \frac{6}{s(s+1)(s+3)}, \quad \mathcal{R}\{s\} > 0.$$

Therefore, the Laplace transform of the corresponding input will be

$$X_1(s) = \frac{Y_1(s)}{H(s)} = \frac{6(s+1)}{s(s+3)}, \quad \mathcal{R}\{s\} > 0.$$

Taking the inverse Laplace transform of the partial fraction expansion of $X_1(s)$, we obtain

$$x_1(t) = 2u(t) + 4e^{-3t}u(t).$$

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Therefore, using the convolution property we obtain

$$y(t) = x_1(t-2) * x_2(-t+3) \xleftrightarrow{\mathcal{L}} Y(s) = \left[\frac{e^{-2s}}{s+2} \right] \left[\frac{e^{-3s}}{3-s} \right].$$

9.27. From clues 1 and 2, we know that $X(s)$ is of the form

$$X(s) = \frac{A}{(s+a)(s+b)}.$$

Furthermore, we are given that one of the poles of $X(s)$ is $-1+j$. Since $x(t)$ is real, the poles of $X(s)$ must occur in conjugate reciprocal pairs. Therefore, $a = 1-j$ and $b = 1+j$ and

$$H(s) = \frac{A}{(s+1-j)(s+1+j)}.$$

From clue 5, we know that $X(0) = 8$. Therefore, we may deduce that $A = 16$ and

$$H(s) = \frac{16}{s^2 + 2s + 2}.$$

Let R denote the ROC of $X(s)$. From the pole locations we know that there are two possible choices of R . R may either be $\mathcal{R}\{s\} < -1$ or $\mathcal{R}\{s\} > -1$. We will now use clue 4 to pick one. Note that

$$y(t) = e^{2t}x(t) \xleftrightarrow{\mathcal{L}} Y(s) = X(s-2).$$

The ROC of $Y(s)$ is R shifted by 2 to the right. Since it is given that $y(t)$ is not absolutely integrable, the ROC of $Y(s)$ should not include the $j\omega$ -axis. This is possible only if R is $\mathcal{R}\{s\} > -1$.

9.28. (a) The possible ROCs are

- $\mathcal{R}\{s\} < -2$.
- $-2 < \mathcal{R}\{s\} < -1$.
- $-1 < \mathcal{R}\{s\} < 1$.
- $\mathcal{R}\{s\} > 1$.

- (i) Unstable and anticausal.
- (ii) Unstable and non causal.
- (iii) Stable and non causal.
- (iv) Unstable and causal.

9.29. (a) Using Table 9.2, we obtain

$$X(s) = \frac{1}{s+1}, \quad \mathcal{R}\{s\} > -1$$

and

$$H(s) = \frac{1}{s+2}, \quad \mathcal{R}\{s\} > -2.$$

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9.31. (a) Taking the Laplace transform of both sides of the given differential equation and simplifying, we obtain

$$H(s) = \frac{Y(s)}{X(s)} = \frac{1}{s^2 - s - 2}.$$

The pole-zero plot for $H(s)$ is as shown in Figure S9.31.

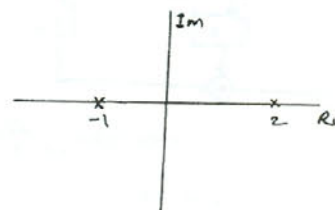


Figure S9.31

(b) The partial fraction expansion of $H(s)$ is

$$H(s) = \frac{1/3}{s-2} - \frac{1/3}{s+1}.$$

(i) If the system is stable, the ROC for $H(s)$ has to be $-1 < \mathcal{R}\{s\} < 2$. Therefore,

$$h(t) = \frac{1}{3}e^{2t}u(-t) - \frac{1}{3}e^{-t}u(-t).$$

(ii) If the system is causal, the ROC for $H(s)$ has to be $\mathcal{R}\{s\} > 2$. Therefore,

$$h(t) = \frac{1}{3}e^{2t}u(t) - \frac{1}{3}e^{-t}u(t).$$

(iii) If the system is neither stable nor causal, the ROC for $H(s)$ has to be $\mathcal{R}\{s\} < -1$. Therefore,

$$h(t) = -\frac{1}{3}e^{2t}u(-t) + \frac{1}{3}e^{-t}u(-t).$$

9.32. If $x(t) = e^{2t}$ produces $y(t) = (1/6)e^{2t}$, then $H(2) = 1/6$. Also, by taking the Laplace transform of both sides of the given differential equation we get

$$H(s) = \frac{s+b(s+4)}{s(s+4)(s+2)}.$$

Since $H(2) = 1/6$, we may deduce that $b = 1$. Therefore,

$$H(s) = \frac{2(s+2)}{s(s+4)(s+2)} = \frac{2}{s(s+4)}.$$

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9.33. Since $x(t) = e^{-|t|} = e^{-t}u(t) + e^t u(-t)$,

$$X(s) = \frac{1}{s+1} - \frac{1}{s-1} = \frac{-2}{(s+1)(s-1)}, \quad -1 < \operatorname{Re}\{s\} < 1.$$

We are also given that

$$H(s) = \frac{s+1}{s^2+2s+2}.$$

Since the poles of $H(s)$ are at $-1 \pm j$, and since $h(t)$ is causal, we may conclude that the ROC of $H(s)$ is $\operatorname{Re}\{s\} > -1$. Now,

$$Y(s) = H(s)X(s) = \frac{-2}{(s^2+2s+2)(s-1)}.$$

The ROC of $Y(s)$ will be the intersection of the ROCs of $X(s)$ and $H(s)$. This is $-1 < \operatorname{Re}\{s\} < 1$.

We may obtain the following partial fraction expansion for $Y(s)$:

$$Y(s) = -\frac{2/5}{s-1} + \frac{2s/5+6/5}{s^2+2s+2}.$$

We may rewrite this as

$$Y(s) = -\frac{2/5}{s-1} + \frac{2}{5} \left[\frac{s+1}{(s+1)^2+1} \right] + \frac{4}{5} \left[\frac{1}{(s+1)^2+1} \right].$$

Noting that the ROC of $Y(s)$ is $-1 < \operatorname{Re}\{s\} < 1$ and using Table 9.2, we obtain

$$y(t) = \frac{2}{5} e^t u(-t) + \frac{2}{5} e^{-t} \cos tu(t) + \frac{4}{5} e^{-t} \sin tu(t).$$

9.34. We know that

$$x_1(t) = u(t) \xrightarrow{\mathcal{L}} X_1(s) = \frac{1}{s}, \quad \operatorname{Re}\{s\} > 0.$$

Therefore, $X_1(s)$ has a pole at $s = 0$. Now, the Laplace transform of the output $y_1(t)$ of the system with $x_1(t)$ as the input is

$$Y_1(s) = H(s)X_1(s).$$

Since in clue 2, $Y_1(s)$ is given to be absolutely integrable, $H(s)$ must have a zero at $s = 0$ which cancels out the pole of $X_1(s)$ at $s = 0$.

We also know that

$$x_2(t) = tu(t) \xrightarrow{\mathcal{L}} X_2(s) = \frac{1}{s^2}, \quad \operatorname{Re}\{s\} > 0.$$

Therefore, $X_2(s)$ has two poles at $s = 0$. Now, the Laplace transform of the output $y_2(t)$ of the system with $x_2(t)$ as the input is

$$Y_2(s) = H(s)X_2(s).$$

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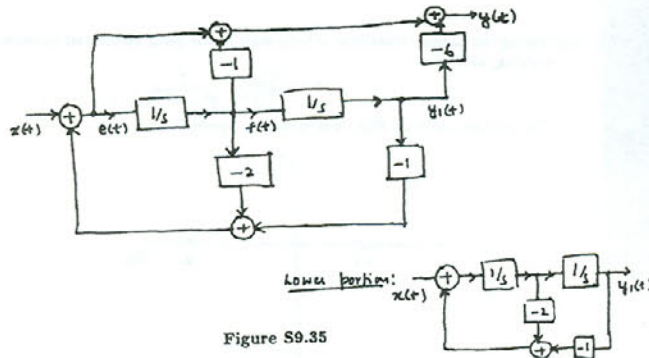


Figure S9.35

Therefore, $f(t) = dy_1(t)/dt$. Similarly, $e(t) = df(t)/dt$. Therefore, $e(t) = d^2y_1(t)/dt^2$. From the block diagram it is clear that

$$y(t) = e(t) - f(t) - 6y_1(t) = \frac{d^2y_1(t)}{dt^2} - \frac{dy_1(t)}{dt} - 6y_1(t).$$

Therefore,

$$Y(s) = s^2Y_1(s) - sY_1(s) - 6Y_1(s). \quad (\text{S9.35-1})$$

Now, let us determine the relationship between $y_1(t)$ and $x(t)$. This may be done by concentrating on the lower half of the above figure. We redraw this in Figure S9.36.

From Example 9.30, it is clear that $y_1(t)$ and $x(t)$ must be related by the following differential equation:

$$\frac{d^2y_1(t)}{dt^2} + 2\frac{dy_1(t)}{dt} + y_1(t) = x(t).$$

Therefore,

$$Y_1(s) = \frac{X(s)}{s^2+2s+1}.$$

Using this in conjunction with eq (S9.35-1), we get

$$Y(s) = \frac{s^2-s-6}{s^2+2s+1} X(s).$$

Taking the inverse Laplace transform, we obtain

$$\frac{d^2y(t)}{dt^2} + 2\frac{dy(t)}{dt} + y(t) = \frac{d^2x(t)}{dt^2} - \frac{dx(t)}{dt} - 6x(t).$$

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Since in clue 3, $Y_2(s)$ is given to be not absolutely integrable, $H(s)$ does not have two zeros at $s = 0$. Therefore, we conclude that $H(s)$ has exactly one zero at $s = 0$.

From Clue 4 we know that the signal

$$p(t) = \frac{d^2h(t)}{dt^2} + 2\frac{dh(t)}{dt} + 2h(t)$$

is finite duration. Taking the Laplace transform of both sides of the above equation, we get

$$P(s) = s^2H(s) + 2sH(s) + 2H(s).$$

Therefore,

$$H(s) = \frac{P(s)}{s^2+2s+2}.$$

Since $p(t)$ is of finite duration, we know that $P(s)$ will have no poles in the finite s -plane. Therefore, $H(s)$ is of the form

$$H(s) = \frac{A \prod_{i=1}^N (s - z_i)}{s^2+2s+2},$$

where $z_i, i = 1, 2, \dots, N$ represent the zeros of $P(s)$. Here, A is some constant.

From Clue 5 we know that the denominator polynomial of $H(s)$ has to have a degree which is exactly one greater than the degree of the numerator polynomial. Therefore,

$$H(s) = \frac{A(s - s_1)}{s^2+2s+2}.$$

Since we already know that $H(s)$ has a zero at $s = 0$, we may rewrite this as

$$H(s) = \frac{As}{s^2+2s+2}.$$

From Clue 1 we know that $H(1)$ is 0.2. From this, we may easily show that $A = 1$. Therefore,

$$H(s) = \frac{s}{s^2+2s+2}.$$

Since the poles of $H(s)$ are at $-1 \pm j$ and since $h(t)$ is causal and stable, the ROC of $H(s)$ is $\operatorname{Re}\{s\} > -1$.

9.35. (a) We may redraw the given block diagram as shown in Figure S9.35.

From the figure, it is clear that

$$\frac{F(s)}{s} = Y_1(s).$$

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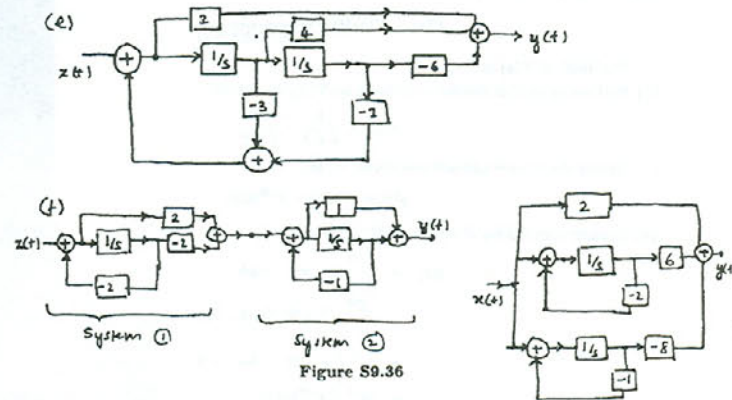


Figure S9.36

(b) The two poles of the system are at -1 . Since the system is causal, the ROC must be to the right of $s = -1$. Therefore, the ROC must include the $j\omega$ -axis. Hence, the system is stable.

9.36. (a) We know that $Y_1(s)$ and $Y(s)$ are related by

$$Y(s) = (2s^2 + 4s - 6)Y_1(s).$$

Taking the inverse Laplace transform, we get

$$y(t) = 2\frac{d^2y_1(t)}{dt^2} + 4\frac{dy_1(t)}{dt} - 6y_1(t).$$

(b) Since $Y_1(s) = F(s)/s$, $f(t) = dy_1(t)/dt$.

(c) Since $F(s) = E(s)/s$, $e(t) = df(t)/dt = d^2y_1(t)/dt^2$.

(d) From part (a), $y(t) = 2e(t) + 4f(t) - 6y_1(t)$.

(e) The extended block diagram is as shown in Figure S9.36.

(f) The block diagram is as shown in Figure S9.36.

(g) The block diagram is as shown in Figure S9.36.

The three subsystems may be connected in parallel as shown in the figure above to obtain the overall system

9.37. The block diagrams are shown in Figure S9.37.

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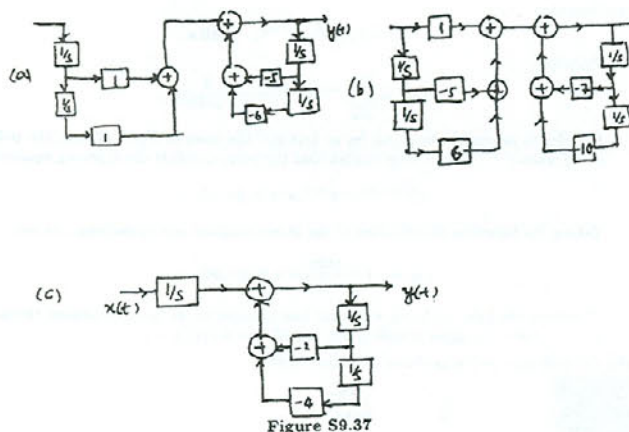


Figure S9.37

9.38. (a) We may rewrite $H(s)$ as

$$H(s) = \left[\frac{1}{s+1} \right] \left[\frac{1}{s+1} \right] \left[\frac{1}{s - \frac{1}{2} + \frac{j\sqrt{3}}{2}} \right] \left[\frac{1}{s - \frac{1}{2} - \frac{j\sqrt{3}}{2}} \right]$$

$H(s)$ clearly may be treated as the cascade combination of four first order subsystems. Consider one of these subsystems with the system function

$$H_1(s) = \left[\frac{1}{s - \frac{1}{2} + \frac{j\sqrt{3}}{2}} \right]$$

The block diagram for this is as shown in Figure S9.38. Clearly, it contains multiplications with coefficients that are not real.

(b) We may write $H(s)$ as

$$H(s) = \left[\frac{1}{s^2 + 2s + 1} \right] \left[\frac{1}{s^2 - s + 1} \right] = H_1(s)H_2(s)$$

The block diagram for $H(s)$ may be constructed as a cascade of the block diagrams of $H_1(s)$ and $H_2(s)$ as shown in Figure S9.38.

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(c) We have

$$\begin{aligned} G(s) &= X_1(s)X_2(s) = \frac{e^s}{(s+2)(s+3)} \\ &= e^s \left[\frac{1}{s+2} - \frac{1}{s+3} \right] \end{aligned}$$

Taking the inverse Laplace transform, we obtain

$$g(t) = e^{-2(t+1)}u(t+1) - e^{-3(t+1)}u(t+1)$$

(d) We have

$$\begin{aligned} \mathcal{R}(s) &= X_1(s)X_2(s) = \frac{e^{-3}}{(s+2)(s+3)} \\ &= e^{-3} \left[\frac{1}{s+2} - \frac{1}{s+3} \right] \end{aligned}$$

Taking the inverse unilateral Laplace transform, we obtain

$$r(t) = e^{-2t-3}u(t) - e^{-3t-3}u(t)$$

Clearly, $r(t) \neq g(t)$ for $t > 0^-$.

9.40. Taking the unilateral Laplace transform of both sides of the given differential equation, we get

$$\begin{aligned} s^3 Y(s) - s^2 y(0^-) - s y'(0^-) - y''(0^-) + 6s^2 Y(s) - 6s y(0^-) \\ - 6y'(0^-) + 11s Y(s) - 11y(0^-) + 6Y(s) = X(s) \end{aligned} \quad (S9.40-1)$$

(a) For the zero state response, assume that all the initial conditions are zero. Furthermore, from the given $x(t)$ we may determine

$$X(s) = \frac{1}{s+4}, \quad \mathcal{R}\{s\} > -4$$

From eq. (S9.40-1), we get

$$Y(s)[s^3 + 6s^2 + 11s + 6] = \frac{1}{s+4}$$

Therefore,

$$Y(s) = \frac{1}{(s+4)(s^3 + 6s^2 + 11s + 6)}$$

Taking the inverse unilateral Laplace transform of the partial fraction expansion of the above equation, we get

$$y(t) = \frac{1}{6}e^{-t}u(t) - \frac{1}{6}e^{-4t}u(t) + \frac{1}{2}e^{-2t}u(t) - \frac{1}{2}e^{-3t}u(t)$$

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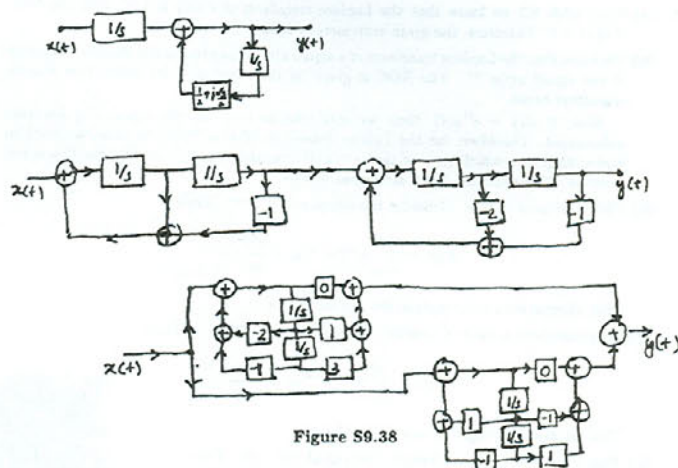


Figure S9.38

(c) We may rewrite $H(s)$ as

$$H(s) = \frac{1}{3} \left[\frac{s+3}{s^2+2s+1} \right] + \frac{1}{3} \left[\frac{1-s}{s^2-s+1} \right] = H_3(s) + H_4(s)$$

The block diagram for $H(s)$ may be constructed as a parallel combination of the block diagrams of $H_3(s)$ and $H_4(s)$ as shown in Figure S9.38.

9.39. (a) For $x_1(t)$, the unilateral and bilateral Laplace transforms are identical.

$$X_1(s) = X_1(s) = \frac{1}{s+2}, \quad \mathcal{R}\{s\} > -2$$

(b) Here, using Table 9.2 and the time shifting property we get

$$X_2(s) = \frac{e^s}{s+3}, \quad \mathcal{R}\{s\} > -3$$

The unilateral Laplace transform is

$$X_2(s) = e^{-3} \frac{1}{s+3}, \quad \mathcal{R}\{s\} > -3$$

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(b) For the zero-input response, we assume that $X(s) = 0$. Assuming that the initial conditions are as given, we obtain from (S9.40-1)

$$Y(s) = \frac{s^2 + 5s + 6}{s^3 + 6s^2 + 11s + 6} = \frac{1}{s+1}$$

Taking the inverse unilateral Laplace transform of the above equation, we get

$$y(t) = e^{-t}u(t)$$

(c) The total response is the sum of the zero-state and zero-input responses.

$$y(t) = \frac{7}{6}e^{-t}u(t) - \frac{1}{6}e^{-4t}u(t) + \frac{1}{2}e^{-2t}u(t) - \frac{1}{2}e^{-3t}u(t)$$

9.41. Let us first find the Laplace transform of the signal $y(t) = x(-t)$. We have

$$\begin{aligned} Y(s) &= \int_{-\infty}^{\infty} x(-t)e^{-st}dt \\ &= \int_{\infty}^{-\infty} x(t)e^{st}dt \\ &= X(-s) \end{aligned}$$

(a) Since $x(t) = x(-t)$ for an even signal, we can conclude that $\mathcal{L}\{x(t)\} = \mathcal{L}\{x(-t)\}$. Therefore, $X(s) = X(-s)$.

(b) Since $x(t) = -x(-t)$ for an odd signal, we can conclude that $\mathcal{L}\{x(t)\} = -\mathcal{L}\{x(-t)\}$. Therefore, $X(s) = -X(-s)$.

(c) First of all note that for a signal to be even, it must be either two-sided or finite duration. Therefore, if $X(s)$ has poles, the ROC must be a strip in the s -plane.

From plot (a), we get

$$X(s) = \frac{As}{(s+1)(s-1)}$$

Therefore,

$$X(-s) = \frac{-As}{(s-1)(s+1)} = -X(s)$$

Therefore, $x(t)$ is not even (in fact it is odd).

For plot (b), we note that the ROC cannot be chosen to correspond to a two-sided function $x(t)$. Therefore, this signal is not even.

From plot (c), we get

$$X(s) = \frac{A(s-j)(s+j)}{(s+1)(s-1)} = \frac{A(s^2+1)}{s^2-1}$$

Therefore,

$$X(-s) = \frac{A(s^2+1)}{s^2-1} = X(s)$$

Therefore, $x(t)$ is even provided the ROC is chosen to be $-1 < \mathcal{R}\{s\} < 1$.

For plot (d), we note that the ROC cannot be chosen to correspond to a two-sided function $x(t)$. Therefore, this signal is not even.

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9.42. (a) From table 9.2 we know that the Laplace transform of $t^2 u(t)$ is $1/s^3$ with the ROC $\mathcal{R}\{s\} > 0$. Therefore, the given statement is false.

(b) We know that the Laplace transform of a signal $x(t)$ is the same as the Fourier transform of the signal $x(t)e^{-\sigma t}$. The ROC is given by the range of σ for which this Fourier transform exists.

Now, if $x(t) = e^{t^2} u(t)$, then we note that as $t \rightarrow \infty$, the signal $x(t)$ becomes unbounded. Therefore, for the Fourier transform of $e^{-\sigma t} x(t)$ to exist, we need to find a range of σ which ensures that $e^{-\sigma t} x(t)$ is bounded as $t \rightarrow \infty$. Clearly, this is not possible. Therefore, the given statement is true.

(c) This statement is true. Consider the signal $x(t) = e^{j\omega_0 t}$. Then

$$X(s) = \int_{-\infty}^{\infty} e^{j\omega_0 t} e^{-st} dt = \frac{e^{t(j\omega_0 - s)}}{j\omega_0 - s} \Big|_{-\infty}^{\infty}$$

This integral does not converge for any value of s .

(d) This statement is false. Consider the signal $x(t) = e^{j\omega_0 t} u(t)$. Then

$$X(s) = \int_0^{\infty} e^{j\omega_0 t} e^{-st} dt = \frac{e^{t(j\omega_0 - s)}}{j\omega_0 - s} \Big|_0^{\infty}$$

This integral converges for any value of $s > 0$.

(e) This statement is false. Consider the signal $x(t) = |t|$. Then

$$X(s) = \int_0^{\infty} te^{-st} dt + \int_{-\infty}^0 -te^{-st} dt$$

Both integrals on the right-hand side converge for any value of $s > 0$.

9.43. We are given that $h(t)$ is causal and stable. Therefore, all poles are in the left half of the s -plane.

(a) Note that

$$g(t) = \frac{dh(t)}{dt} \xrightarrow{\mathcal{L}} G(s) = sH(s)$$

Now, $G(s)$ has the same poles as $H(s)$ and hence the ROC for $G(s)$ remains the same. Therefore, $g(t)$ is also guaranteed to be causal and stable.

(b) Note that

$$r(t) = \int_{-\infty}^t h(\tau) d\tau \xrightarrow{\mathcal{L}} R(s) = \frac{H(s)}{s}$$

Note that $R(s)$ does not have a pole at $s = 0$ only if $H(s)$ has a zero at $s = 0$. Therefore, we cannot guarantee that $r(t)$ is always causal and unstable.

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Now,

$$H(s) = \frac{Y(s)}{X(s)} = \frac{s}{(s+2)(s+1)}$$

We know that the ROC of $Y(s)$ has to be the intersection of the ROCs of $X(s)$ and $H(s)$. This leads us to conclude that the ROC of $H(s)$ is $\mathcal{R}\{s\} > -1$.

(b) The partial fraction expansion of $H(s)$ is

$$H(s) = \frac{2}{s+2} - \frac{1}{s+1}$$

Therefore,

$$h(t) = 2e^{-2t}u(t) - e^{-t}u(t)$$

(c) e^{3t} is an Eigen function of the LTI system. Therefore,

$$y(t) = H(3)e^{3t} = \frac{3}{20}e^{3t}$$

9.46. Since $y(t)$ is real, the third input must be of the form $e^{j\omega_0 t}$. Since $x(t)$ is of the form $\delta(t) + e^{j\omega_0 t} + e^{j\omega_0 t}$ and the output is $y(t) = -6e^{-t}u(t) + \frac{4}{3}e^{j\omega_0 t} \cos(3t) + \frac{15}{34}e^{j\omega_0 t} \sin(3t)$, we may conclude that $H(4 \pm j3) = \frac{4}{34} \pm j\frac{15}{34}$.

Let us try $h(t) = \delta(t) - 6e^{-t}u(t)$. Then

$$H(s) = \frac{s-5}{s+1}$$

We may easily show that $H(4 \pm j3) = \frac{4}{34} \pm j\frac{15}{34}$. Therefore, $H(s)$ as given above is consistent with the given information.

9.47. (a) Taking the Laplace transform of $y(t)$, we obtain

$$Y(s) = \frac{1}{s+2}, \quad \mathcal{R}\{s\} > -2$$

Therefore,

$$X(s) = \frac{Y(s)}{H(s)} = \frac{s+1}{(s-1)(s+2)}$$

The pole-zero diagram for $X(s)$ is as shown in Figure S9.47. Now, the ROC of $H(s)$ is $\mathcal{R}\{s\} > -1$. We know that the ROC of $Y(s)$ is at least the intersection of the ROCs of $X(s)$ and $H(s)$. Note that the ROC can be larger if some poles are canceled out by zeros at the same location. In this case, we can choose the ROC of $X(s)$ to be either $-2 < \mathcal{R}\{s\} < 1$ or $\mathcal{R}\{s\} > 1$. In both cases, we get the same ROC for $Y(s)$ because the poles at $s = -1$ and $s = 1$ in $H(s)$ and $X(s)$, respectively are canceled out by zeros.

The partial fraction expansion of $X(s)$ is

$$X(s) = \frac{2/3}{s-1} + \frac{1/3}{s-2}$$

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9.44. (a) Note that

$$\delta(t - nT) \xrightarrow{\mathcal{L}} e^{-snT}, \quad \text{All } s.$$

Therefore,

$$X(s) = \sum_{n=0}^{\infty} e^{-nT} e^{-snT} = \frac{1}{1 - e^{-T(1+s)}}$$

In order to determine the ROC, let us first find the poles of $X(s)$. Clearly, the pole occurs when $e^{-T(1+s)} = 1$. This implies that the poles s_k satisfy the following equation

$$e^{-T(1+s_k)} = e^{jk2\pi}, \quad k = 0, \pm 1, \pm 2, \dots$$

Taking the logarithm of both sides of the above equation and simplifying, we get

$$s_k = -1 + \frac{j2\pi k}{T}, \quad k = 0, \pm 1, \pm 2, \dots$$

Therefore, the poles all lie on a vertical line (parallel to the $j\omega$ -axis) passing through $s = -1$. Since the signal is right-sided, the ROC is $\mathcal{R}\{s\} > -1$.

(b) The pole-zero plot is as shown in Figure S9.44.

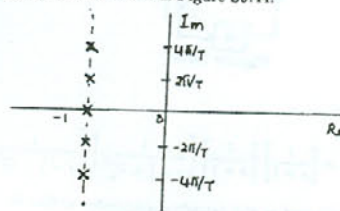


Figure S9.44

(c) The magnitude of the Fourier transform $X(j\omega)$ is given by the product of the reciprocals of the lengths of the vectors from the poles to the point $j\omega$. The phase of $X(j\omega)$ is given by the negative of the sum of the angles of these vectors. Clearly from the pole-zero plot above it is clear that both the magnitude and phase have to vary periodically with a period of $2\pi/T$.

9.45. (a) Taking the Laplace transform of the signal $x(t)$, we get

$$Y(s) = \frac{2/3}{s-2} + \frac{1/3}{s+1} = \frac{s}{(s-2)(s-1)}$$

The ROC is $-1 < \mathcal{R}\{s\} < 2$. Also, note that since $x(t)$ is a left-sided signal, the ROC for $X(s)$ is $\mathcal{R}\{s\} < 2$.

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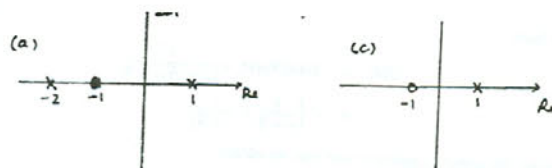


Figure S9.47

Taking the ROC of $X(s)$ to be $-2 < \mathcal{R}\{s\} < 1$, we get

$$x(t) = -\frac{2}{3}e^{-t}u(-t) + \frac{1}{3}e^{-2t}u(t)$$

Taking the ROC of $X(s)$ to be $\mathcal{R}\{s\} > 1$, we get

$$x(t) = \frac{2}{3}e^{-t}u(t) + \frac{1}{3}e^{-2t}u(t)$$

(b) Since it is given that $x(t)$ is absolutely integrable, we can conclude that the ROC of $X(s)$ must include the $j\omega$ -axis. Therefore, the first choice of $x(t)$ given above is the one we want.

(c) We need to first find a $H(s)$ such that $H(s)Y(s) = X(s)$. Clearly,

$$H(s) = \frac{X(s)}{Y(s)} = \frac{s+1}{s-1}$$

The pole-zero plot for $H(s)$ is as shown in Figure S9.47. Since $h(t)$ is given to be stable, the ROC of $H(s)$ has to be $\mathcal{R}\{s\} < 1$. The partial fraction expansion of $H(s)$ is

$$H(s) = 1 + \frac{2}{s-1}$$

Therefore,

$$h(t) = \delta(t) - 2e^{-t}u(-t)$$

Also, $Y(s)$ has the ROC $\mathcal{R}\{s\} > -2$. Therefore, $X(s)$ must have the ROC $-2 < \mathcal{R}\{s\} < 1$ (the intersection of the ROCs of $Y(s)$ and $H(s)$). From this we get (as shown in part (a))

$$x(t) = -\frac{2}{3}e^{-t}u(-t) + \frac{1}{3}e^{-2t}u(t)$$

Verification: Now,

$$\begin{aligned} h(t) * y(t) &= [\delta(t) - 2e^{-t}u(-t)] * [-\frac{2}{3}e^{-t}u(-t) + \frac{1}{3}e^{-2t}u(t)] \\ &= e^{-2t}u(t) - 2 \int_0^{\infty} e^{-2\tau} e^{-t-\tau} u(\tau-t) d\tau \end{aligned}$$

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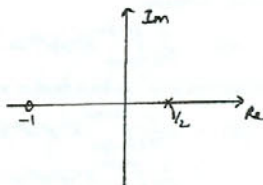


Figure S9.48

For $t > 0$, the integral in the above equation is

$$e^t \int_t^\infty e^{-3\tau} d\tau = \frac{1}{3} e^{-2t}.$$

For $t < 0$, the integral in the above equation is

$$e^t \int_0^\infty e^{-3\tau} d\tau = \frac{1}{3} e^t.$$

Therefore,

$$h(t) * y(t) = -\frac{2}{3} e^t u(-t) + \frac{1}{3} e^{-2t} u(t) = x(t).$$

9.48. (a) $H_1(s) = 1/H(s)$.

(b) From the above relationship it is clear that the poles of the inverse system will be the zeros of original system. Also, the zeros of the inverse system will be the poles of the original system. Therefore, the pole-zero plot for $H_1(s)$ is as sketched in Figure S9.48.

9.49. If a system is causal and stable, then the poles of its transfer function must all be in the left half of the s -plane. This is because the ROC of a causal system is to the right of the right-most pole. For the ROC to contain the $j\omega$ -axis, the right-most pole must be in the left-half of the s -plane.

Now, if the inverse system is also causal and stable, then its poles must also all lie in the left half of the s -plane. But we know that the poles of the inverse system are the zeros of the original system. Therefore, the zeros of the original system must also lie in the left-half of the s -plane.

9.50. (a) False. Counter-example: $H(s) = 1/(s-2)$, $\mathcal{R}\{s\} < 2$.

(b) True. If the system function has more poles than zeros, then $h(t)$ does not have an impulse at $t = 0$. Since we know that $h(t)$ is the derivative of the step response, we may conclude that the step response has no discontinuities at $t = 0$.

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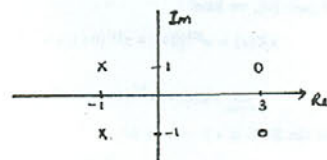


Figure S9.51

(c) False. Causality plays no part in the argument of part (b).

(d) False. Counter-example: $H(s) = (s-1)/(s+2)$, $\mathcal{R}\{s\} > -2$.

9.51. Since $h(t)$ is real, its poles and zeros must occur in complex conjugate pairs. Therefore, the known poles and zeros of $H(s)$ are as shown in Figure S9.51. Since $H(s)$ has exactly 2 zeros at infinity, $H(s)$ has at least two more unknown finite poles. In case $H(s)$ has more than 4 poles, then it will have a zero at some location for every additional pole. Furthermore, since $h(t)$ is causal and stable, all poles of $H(s)$ must lie in the left half of the s -plane and the ROC must include the $j\omega$ -axis.

(a) True. Consider

$$g(t) = h(t)e^{-3t} \xleftrightarrow{\mathcal{L}} G(s) = H(s+3).$$

The ROC of $G(s)$ will be the ROC of $H(s)$ shifted by 3 to the left. Clearly this ROC will still include the $j\omega$ -axis. Therefore, $g(t)$ has to be stable.

(b) Insufficient information. As mentioned earlier, $H(s)$ has some unknown poles. So we do not know which the rightmost pole is in $H(s)$. Therefore, we cannot determine what its exact ROC is.

(c) True. Since $H(s)$ is rational, $H(s)$ may be expressed as a ratio of two polynomials in s . Furthermore, since $h(t)$ is real, the coefficients of these polynomials will be real. Now,

$$H(s) = \frac{Y(s)}{X(s)} = \frac{P(s)}{Q(s)}.$$

Here, $P(s)$ and $Q(s)$ are polynomials in s . The differential equation relating $x(t)$ and $y(t)$ is obtained by taking the inverse Laplace transform of $Y(s)Q(s) = X(s)P(s)$. Clearly, this differential equation has to have only real coefficients.

(d) False. We are given that $H(s)$ has 2 zeros at $s = \infty$. Therefore, $\lim_{s \rightarrow \infty} H(s) = 0$.

(e) True. See the reasoning at the beginning of the problem.

(f) Insufficient information. $H(s)$ may have other zeros. See reasoning at the beginning of the problem.

(g) False. We know that $e^{3t} \sin(t) = (1/2j)e^{(3+j)t} - (1/2j)e^{(3-j)t}$. Both $e^{(3+j)t}$ and $e^{(3-j)t}$ are Eigen functions of the LTI system. Therefore, the response of the system to these exponentials is $H(3+j)e^{(3+j)t}$ and $H(3-j)e^{(3-j)t}$, respectively. Since $H(s)$ has zeros

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at $3 \pm j$, we know that the output of the system to the two exponentials has to be zero. Hence, the response of the system to $e^{3t} \sin(t)$ has to be zero.

9.52. (a) Consider the signal $y(t) = x(t-t_0)$. Now,

$$Y(s) = \int_{-\infty}^{\infty} x(t-t_0)e^{-st} dt.$$

Replacing $t-t_0$ by τ , we get

$$\begin{aligned} Y(s) &= \int_{-\infty}^{\infty} x(\tau)e^{-s(\tau+t_0)} d\tau \\ &= e^{-st_0} \int_{-\infty}^{\infty} x(\tau)e^{-s\tau} d\tau \\ &= e^{-st_0} X(s) \end{aligned}$$

This obviously converges when $X(s)$ converges because e^{-st_0} has no poles. Therefore, the ROC of $Y(s)$ is the same as the ROC of $X(s)$.

(b) Consider the signal $y(t) = e^{s_0 t} x(t)$. Now,

$$\begin{aligned} Y(s) &= \int_{-\infty}^{\infty} x(t)e^{s_0 t} e^{-st} dt \\ &= \int_{-\infty}^{\infty} x(t)e^{-(s-s_0)t} dt \\ &= X(s-s_0) \end{aligned}$$

If $X(s)$ converges in the range $a < \mathcal{R}\{s\} < b$, then $X(s-s_0)$ converges in the range $a+s_0 < s < b+s_0$. This is the ROC of $Y(s)$.

(c) Consider the signal $y(t) = x(at)$. Now,

$$Y(s) = \int_{-\infty}^{\infty} x(at)e^{-st} dt.$$

Replacing at by τ and assuming that $a > 1$, we get

$$\begin{aligned} Y(s) &= (1/a) \int_{-\infty}^{\infty} x(\tau)e^{-s(\tau/a)} d\tau \\ &= (1/a) X(s/a). \end{aligned}$$

If $a < 0$, then

$$\begin{aligned} Y(s) &= -(1/a) \int_{-\infty}^{\infty} x(\tau)e^{-s(\tau/a)} d\tau \\ &= -(1/a) X(s/a). \end{aligned}$$

Therefore,

$$Y(s) = \frac{1}{|a|} X\left(\frac{s}{a}\right).$$

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If $X(s)$ converges in the range $\alpha < \mathcal{R}\{s\} < \beta$, then $X(s/a)$ converges in the range $\alpha/a < s < \beta/a$ when $a > 0$. When $a < 0$, then $X(s/a)$ converges in the range $\beta/a < s < \alpha/a$.

(d) Consider the signal $y(t) = x(t) * h(t)$. Now,

$$\begin{aligned} Y(s) &= \int_{-\infty}^{\infty} [x(t) * h(t)]e^{-st} dt \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau e^{-st} dt \\ &= \int_{-\infty}^{\infty} x(\tau) \left[\int_{-\infty}^{\infty} h(t-\tau)e^{-st} dt \right] d\tau \end{aligned}$$

Using the time-shifting property, we get

$$\begin{aligned} Y(s) &= \int_{-\infty}^{\infty} x(\tau)H(s)e^{-s\tau} d\tau \\ &= H(s) \int_{-\infty}^{\infty} x(\tau)e^{-s\tau} d\tau \\ &= H(s)X(s) \end{aligned}$$

Clearly, $Y(s)$ converges at least in the region where both $X(s)$ and $H(s)$ converge. Its ROC may be larger depending on whether some of the poles of either $H(s)$ or $X(s)$ get cancelled out by the zeros of $X(s)$ or $H(s)$, respectively.

9.53. (a) From the example worked out in the text we have

$$e^{-at} \left(\frac{t^n}{n!} \right) u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{(s+a)^{n+1}}, \quad \mathcal{R}\{s\} > a.$$

With $a = 0$, we get

$$x^n(0+) \left(\frac{t^n}{n!} \right) u(t) \xleftrightarrow{\mathcal{L}} \frac{x^n(0+)}{s^{n+1}}, \quad \mathcal{R}\{s\} > 0.$$

(b) We may rewrite eq. (P9.53-1) as

$$x(t) = \sum_{n=0}^{\infty} x^n(0+) \left(\frac{t^n}{n!} \right).$$

Taking the Laplace transform of both sides of this equation and using the result of part (a), we get

$$X(s) = \sum_{n=0}^{\infty} \frac{x^n(0+)}{s^{n+1}}. \quad (\text{S9.53-1})$$

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(c) From the result of part (b), we have

$$sX(s) = x^{(0)}(0+) + x^{(1)}(0+)/s + \dots$$

Therefore,

$$\lim_{s \rightarrow \infty} sX(s) = x^{(0)}(0+) = x(0+).$$

(d) (1) Assuming that the ROC is $s > -2$, we get

$$x(t) = e^{-2t}u(t).$$

Therefore, $x(0+) = 1$. Now,

$$\lim_{s \rightarrow \infty} sX(s) = \lim_{s \rightarrow \infty} \frac{s}{s+2} = 1.$$

(2) The partial fraction expansion of $X(s)$ is

$$X(s) = \frac{2}{(s+3)} - \frac{1}{s+2}.$$

Assuming that the ROC is $s > -2$, we get

$$x(t) = 2e^{-3t}u(t) - e^{-2t}u(t).$$

Therefore, $x(0+) = 1$. Now,

$$\lim_{s \rightarrow \infty} sX(s) = \lim_{s \rightarrow \infty} \frac{s^2 + s}{s^2 + 5s + 6} = 1.$$

(e) Assuming that $x^{(n)}(0+) = 0$ for $n < N$, eq. (S9.53-1) may be written as

$$X(s) = \sum_{n=N}^{\infty} \frac{x^{(n)}(0+)}{s^{n+1}}.$$

Now,

$$s^{N+1}X(s) = x^{(N)}(0+) + \frac{x^{(N+1)}(0+)}{s} + \frac{x^{(N+2)}(0+)}{s^2} + \dots$$

Therefore,

$$\lim_{s \rightarrow \infty} s^{N+1}X(s) = x^{(N)}(0+).$$

9.54. (a) We have

$$x(t) = \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{-st}ds.$$

Conjugating both sides, we get

$$x^*(t) = -\frac{1}{2\pi j} \int_{\sigma+j\infty}^{\sigma-j\infty} X^*(s)e^{s^*t}ds.$$

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Pair 13: Using the shifting in the s -domain property on pair 11, we get

$$e^{-at} \cos(\omega_0 t)u(t) \xleftrightarrow{\mathcal{L}} \frac{s+a}{(s+a)^2 + \omega_0^2}, \quad \mathcal{R}\{s\} > -a.$$

Pair 14: Using the shifting in the s -domain property on pair 12, we get

$$e^{-at} \sin(\omega_0 t)u(t) \xleftrightarrow{\mathcal{L}} \frac{\omega_0}{(s+a)^2 + \omega_0^2}, \quad \mathcal{R}\{s\} > -a.$$

Pair 15: From pair 1 of Table 9.2, we have

$$u_0(t) = \delta(t) \xleftrightarrow{\mathcal{L}} 1, \quad \text{All } s.$$

Using the differentiation in time-domain property on this signal, we get

$$u_1(t) = \frac{d\delta(t)}{dt} \xleftrightarrow{\mathcal{L}} s, \quad \text{All } s.$$

Continuing along these lines and differentiating $\delta(t)$ n times, we get

$$u_n(t) = \frac{d^n \delta(t)}{dt^n} \xleftrightarrow{\mathcal{L}} s^n, \quad \text{All } s.$$

Pair 16: From pair 2 of Table 9.2, we have

$$u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s}, \quad \mathcal{R}\{s\} > 0.$$

By applying the convolution property, we get

$$u_{-2}(t) = u(t) * u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s^2}, \quad \mathcal{R}\{s\} > 0.$$

Continuing along these lines and convolving $u(t)$ with itself n times, we get

$$u_{-n}(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s^n}, \quad \mathcal{R}\{s\} > 0.$$

9.56. Given that

$$\int_{-\infty}^{\infty} |x(t)|e^{-\sigma_0 t}dt < \infty,$$

we need to prove that $|X(s_0)| = 0$, where $s_0 = \sigma_0 + j\omega_0$. We have

$$|X(s_0)| = \left| \int_{-\infty}^{\infty} x(t)e^{-s_0 t}dt \right| = \left| \int_{-\infty}^{\infty} x(t)e^{-\sigma_0 t}e^{-j\omega_0 t}dt \right|.$$

Using eq. (P9.56-1), we get

$$\begin{aligned} |X(s_0)| &\leq \int_{-\infty}^{\infty} |x(t)|e^{-\sigma_0 t}e^{-j\omega_0 t}dt \\ &\leq \int_{-\infty}^{\infty} |x(t)|e^{-\sigma_0 t}dt \\ &\leq \infty \end{aligned}$$

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For a real signal $x(t) = x^*(t)$. Therefore,

$$x(t) = -\frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} X^*(s)e^{s^*t}ds.$$

Replacing s^* by p and noting that $dp = -ds$ for a fixed σ , we get

$$\begin{aligned} x(t) &= -\frac{1}{2\pi j} \int_{\sigma+j\infty}^{\sigma-j\infty} X^*(p^*)e^{p^*t}dp \\ &= \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} X^*(p^*)e^{p^*t}dp \end{aligned}$$

Therefore, $\mathcal{L}\{x(t)\} = X^*(s^*)$. This implies that $X(s) = X^*(s^*)$.

(b) Let $X(s)$ have a zero at $s = s_1$. Then $X(s_1) = 0$. From the result of part (a), we know that $X^*(s_1^*) = 0$. This implies that $X(s_1^*) = 0$, which in turn implies that $X(s)$ has a zero at s_1^* . The same approach may be used to show that poles occur in conjugate pairs.

9.55: Pair 10: From pair 1 of Table 9.2, we have

$$\delta(t) \xleftrightarrow{\mathcal{L}} 1, \quad \text{All } s.$$

Using the time-shifting property, we get

$$\delta(t-T) \xleftrightarrow{\mathcal{L}} e^{-sT}, \quad \text{All } s.$$

Pair 11: From pair 6 of Table 9.2, we have

$$e^{j\omega_0 t}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s-j\omega_0}, \quad \mathcal{R}\{s\} > 0 \quad (\text{S9.55-1})$$

and

$$e^{-j\omega_0 t}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+j\omega_0}, \quad \mathcal{R}\{s\} > 0. \quad (\text{S9.55-2})$$

Note that $\cos(\omega_0 t) = (1/2)e^{j\omega_0 t} + (1/2)e^{-j\omega_0 t}$. Now using eqs. (S9.55-1) and (S9.55-2) with the linearity property, we get

$$\cos(\omega_0 t)u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{2} \left[\frac{1}{s-j\omega_0} \right] + \frac{1}{2} \left[\frac{1}{s+j\omega_0} \right] = \frac{s}{s^2 + \omega_0^2}.$$

The ROC will be $\mathcal{R}\{s\} > 0$.

Pair 12: Note that $\sin(\omega_0 t) = (1/2j)e^{j\omega_0 t} - (1/2j)e^{-j\omega_0 t}$. Now using eqs. (S9.55-1) and (S9.55-2) with the linearity property, we get

$$\sin(\omega_0 t)u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{2j} \left[\frac{1}{s-j\omega_0} \right] - \frac{1}{2j} \left[\frac{1}{s+j\omega_0} \right] = \frac{\omega_0}{s^2 + \omega_0^2}.$$

The ROC will be $\mathcal{R}\{s\} > 0$.

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Plausibility of eq. (P9.56-1): Integration is akin to the addition of an infinite number of complex numbers. For any two complex numbers A and B , we know that $|A+B| \leq |A|+|B|$. Using this, we may argue that the same should hold for an infinite sum of complex numbers or the integration of a complex function.

9.57. Since $x(t)$ has an impulse at $t = 0$, the numerator polynomial of $X(s)$ must be of the same/larger degree than the denominator polynomial of $X(s)$. This implies that $X(s)$ has at least 4 zeros.

9.58. Since $g(t) = \mathcal{R}\{h(t)\}$,

$$g(t) = \frac{h(t) + h^*(t)}{2}.$$

Using the linearity and conjugation properties, we get

$$G(s) = \frac{H(s) + H^*(s^*)}{2}.$$

The ROC of $G(s)$ will be at least the intersection of the ROCs of $H(s)$ and $H^*(s^*)$. This means that the ROC of $G(s)$ will be at least as much as the ROC of $H(s)$. Therefore, if $H(s)$ is causal and stable, then $G(s)$ also has to be causal and stable.

9.59. (a) Let $y(t) = x(t-1)$. Then,

$$\mathcal{Y}(s) = e^{-s}X(s) + e^{-s} \int_{-1}^0 x(t)e^{-st}dt.$$

(b) Let $y(t) = x(t+1)$. Then,

$$\mathcal{Y}(s) = e^s X(s) - e^s \int_0^1 x(t)e^{-st}dt.$$

(c) Let $y(t) = \int_{-\infty}^t x(\tau)d\tau$. Then,

$$\mathcal{Y}(s) = \frac{X(s) + \int_{-\infty}^0 x(t)e^{-st}dt}{s}.$$

(d) Let $y(t) = d^3x(t)/dt^3$. Then,

$$\mathcal{Y}(s) = s^3X(s) - s^2x(0^-) - sx'(0^-) - x''(0^-).$$

9.60. (a) We have

$$h(t) = \alpha\delta(t-T) + \alpha^3\delta(t-3T).$$

From Tables 9.1 and 9.2,

$$H(s) = \alpha e^{-sT} + \alpha^3 e^{-3sT}, \quad \text{All } s.$$

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(b) To determine the zeros of $H(s)$, note that we require

$$\alpha e^{-sT} + \alpha^3 e^{-3sT} = \alpha e^{-sT} [1 + \alpha^2 e^{-2sT}] = 0.$$

Therefore, at the zeros

$$1 + \alpha^2 e^{-2sT} = 0 \Rightarrow \alpha e^{-sT} = \pm j.$$

This implies that the zeros occur at

$$s = \frac{1}{T} \log_e \alpha \pm \left[\frac{\pi}{2T} \pm \frac{2k\pi}{T} \right], \quad k = 0, \pm 1, \pm 2, \dots$$

At the poles, $H(s) = \infty$. Therefore, at the poles we require that

$$\alpha e^{-sT} + \alpha^3 e^{-3sT} = \alpha e^{-sT} [1 + \alpha^2 e^{-2sT}] = \infty.$$

This is not possible at any finite s . Therefore, there are no poles in the finite s -plane.

(c) The pole-zero plot is as shown in Figure S9.60.

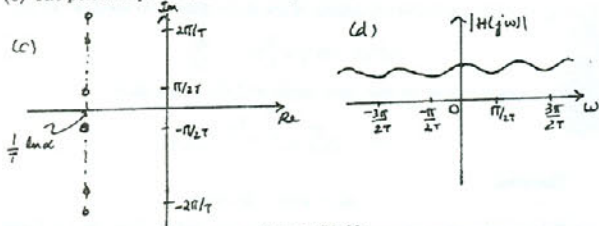


Figure S9.60

(d) From the figure it is clear that $H(j\omega)$ will be periodic and will be as shown in Figure S9.60.

9.61. (a) If we want $\phi_{xx}(t)$ to be the output of the system when $x(t)$ is the input, then

$$\phi_{xx}(t) = \int_{-\infty}^{\infty} x(\tau) h(t - \tau) d\tau.$$

Also we are given that

$$\phi_{xx}(t) = \int_{-\infty}^{\infty} x(\tau) x(t + \tau) d\tau.$$

Therefore,

$$x(t + \tau) = h(t - \tau) \Rightarrow h(t) = x(-t).$$

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But

$$\frac{t^n e^{-t}}{n!} u(t) \xrightarrow{\mathcal{L}} \frac{1}{(s+1)^{n+1}}, \quad s > -1.$$

Therefore,

$$\frac{1}{n!} \frac{d^n}{dt^n} [t^n e^{-t}] u(t) \xrightarrow{\mathcal{L}} \frac{s^n}{(s+1)^{n+1}}, \quad s > -1.$$

It follows that

$$e^{t/2} \frac{1}{n!} \frac{d^n}{dt^n} [t^n e^{-t}] u(t) \xrightarrow{\mathcal{L}} \frac{(s-1/2)^n}{(s+1/2)^{n+1}}, \quad s > -1/2.$$

Therefore,

$$\Phi_n(s) = \frac{(s-1/2)^n}{(s+1/2)^{n+1}}, \quad s > -1/2.$$

(c) Choose

$$H_1(s) = \frac{1}{s+1/2}$$

and

$$H_2(s) = \frac{(s-1/2)}{(s+1/2)}.$$

9.63. (a) We have

$$H(s) = \frac{1}{s+1/2}.$$

The pole-zero plot for $H(s)$ is as shown in the Figure S9.63. Using the geometric method for evaluating the magnitude of the Fourier transform, we may sketch $|H(j\omega)|$ as shown in Figure S9.63.

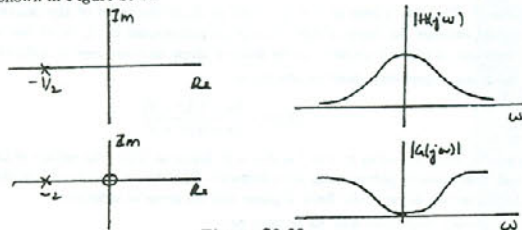


Figure S9.63

Also,

$$G(s) = H(1/s) = \frac{2s}{s+2}.$$

The pole-zero plot for $G(s)$ is as shown in the Figure S9.63. Using the geometric method for evaluating the magnitude of the Fourier transform, we may sketch $|G(j\omega)|$ as shown in Figure S9.63.

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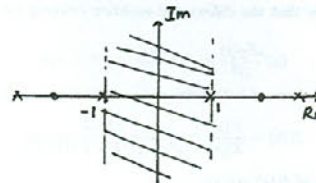


Figure S9.61

(b) Since $\phi_{xx}(t) = x(t) * x(-t)$,

$$\Phi_{xx}(s) = X(s)X(-s)$$

and

$$\Phi_{xx}(j\omega) = X(j\omega)X(-j\omega).$$

If $x(t)$ is real, $X^*(j\omega) = X(-j\omega)$ and

$$\Phi_{xx}(j\omega) = |X(j\omega)|^2.$$

(c) If $X(s)$ has a pole-zero pattern as shown in Figure P9.61, then $X(-s)$ has a pole-zero pattern as shown in Figure S9.61. The corresponding ROC is also shown in Figure S9.61.

Now, $\Phi_{xx}(s)$ will include the poles of both $X(s)$ and $X(-s)$. Furthermore, its ROC will be the intersection of the ROCs of $X(s)$ and $X(-s)$. (See Figure S9.61)

9.62. (a) We have

$$L_0(t) = e^t e^{-t} = 1,$$

$$L_1(t) = e^t \frac{d}{dt}(te^{-t}) = e^t [e^{-t} - te^{-t}] = 1 - t$$

and

$$\begin{aligned} L_2(t) &= \frac{e^t}{2} \frac{d^2}{dt^2}(t^2 e^{-t}) \\ &= \frac{e^t}{2} [2e^{-t} - 2te^{-t} - 2te^{-t} + t^2 e^{-t}] \\ &= 1 - 2t + \frac{1}{2}t^2. \end{aligned}$$

(b) We have

$$\begin{aligned} \phi_n(t) &= \frac{1}{n!} e^{t/2} \frac{d^n}{dt^n} [t^n e^{-t}] u(t) \\ &= \frac{1}{n!} e^{t/2} \frac{d^n}{dt^n} [t^n e^{-t} u(t) + t^n e^{-t} u(-t)] u(t) \\ &= \frac{1}{n!} e^{t/2} \frac{d^n}{dt^n} [t^n e^{-t} u(t)]. \end{aligned}$$

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(b) LCCDE associated with $H(s)$:

Consider

$$H(s) = \frac{Y(s)}{X(s)} = \frac{1}{s+1/2}.$$

Cross-multiplying and taking the inverse Laplace transform, we obtain

$$\frac{dy(t)}{dt} + \frac{1}{2}y(t) = x(t).$$

LCCDE associated with $G(s)$:

Consider

$$G(s) = \frac{Y(s)}{X(s)} = \frac{2s}{s+2}.$$

Cross-multiplying and taking the inverse Laplace transform, we obtain

$$\frac{dy(t)}{dt} + 2y(t) = 2 \frac{dx(t)}{dt}.$$

(c) Taking the Laplace transform of eq.(P9.63-1), we obtain

$$\sum_{k=0}^N a_k s^k Y(s) = \sum_{k=0}^N b_k s^k X(s).$$

Therefore,

$$H(s) = \frac{Y(s)}{X(s)} = \frac{\sum_{k=0}^N b_k s^k}{\sum_{k=0}^N a_k s^k}.$$

Now,

$$G(s) = H(1/s) = \frac{\sum_{k=0}^N b_k s^{-k}}{\sum_{k=0}^N a_k s^{-k}} = \frac{\sum_{k=0}^N b_k s^{N-k}}{\sum_{k=0}^N a_k s^{N-k}}.$$

(d) Now from the previous part, we have

$$G(s) = \frac{Y(s)}{X(s)} = \frac{\sum_{k=0}^N b_k s^{N-k}}{\sum_{k=0}^N a_k s^{N-k}}.$$

Cross-multiplying and taking the inverse Laplace transform, we obtain

$$\sum_{k=0}^N a_k \frac{d^{N-k} y(t)}{dt^{N-k}} = \sum_{k=0}^N b_k \frac{d^{N-k} x(t)}{dt^{N-k}}.$$

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9.64. For the circuit, we know that the differential equation relating the input $x(t)$ and output $y(t)$ is

$$LC \frac{d^2 y(t)}{dt^2} + RC \frac{dy(t)}{dt} + y(t) = x(t).$$

Taking the Laplace transform of both sides and simplifying, we get

$$H(s) = \frac{Y(s)}{X(s)} = \frac{1/LC}{s^2 + (R/L)s + (1/LC)}.$$

(a) Note that the poles of $H(s)$ are at

$$s = \frac{-RC \pm \sqrt{R^2 C^2 - 4LC}}{2}.$$

If R , L , and C are always positive, then the poles are always in the left half of the s -plane (because the real part of the numerator of the above equation is always negative). Since the system is causal, the ROC is to the right of the right-most pole. Therefore, the ROC includes the $j\omega$ -axis and the system is stable.

(b) From $H(s)$ we obtain

$$H(s)H(-s) = \frac{1}{L^2 C^2 s^4 + (RLC^2 - RLC^2)s^3 + (2LC - R^2 C^2)s^2 + (RC - RC)s + 1} \\ = \frac{1}{L^2 C^2 s^4 + (2LC - R^2 C^2)s^2 + 1}.$$

For this to represent a second order Butterworth filter, we require

$$2LC - R^2 C^2 = 0 \Rightarrow R = 2\sqrt{\frac{L}{C}}.$$

9.65. (a) The differential equation relating $v_i(t)$ and $v_o(t)$ may be obtained by putting $x(t) = v_i(t)$ and $y(t) = v_o(t)$ in the differential equation given in the previous problem. Therefore,

$$LC \frac{d^2 v_o(t)}{dt^2} + RC \frac{dv_o(t)}{dt} + v_o(t) = v_i(t)$$

or

$$\frac{d^2 v_o(t)}{dt^2} + \frac{R}{L} \frac{dv_o(t)}{dt} + \frac{1}{LC} v_o(t) = \frac{1}{LC} v_i(t)$$

(b) Taking the unilateral Laplace transform of the above differential equation, we get

$$s^2 V_o(s) - sV_o(0^-) - V_o'(0^-) + \frac{R}{L} sV_o(s) - V_o'(0^-) + \frac{1}{LC} V_o(s) = \frac{1}{LC} V_i(s) \quad \text{[S9.65-1]}$$

Now, since $v_i(t) = e^{-3t}u(t)$,

$$V_i(s) = \frac{1}{s+3}, \quad \text{Re}\{s\} > -3.$$

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Substituting this along with the values of R , L , and C in eq. (S9.65-1), we get

$$V_o(s) = \frac{2(s^2 + 5s + 7)}{(s+1)(s+2)(s+3)}.$$

The partial fraction expansion of $V_o(s)$ is

$$V_o(s) = \frac{3}{s+1} - \frac{2}{s+2} + \frac{1}{s+3}.$$

Taking the inverse Laplace transform, we get

$$v_o(t) = 3e^{-t}u(t) - 2e^{-2t}u(t) + e^{-3t}u(t).$$

9.66. (a) The differential equation relating $i(t)$ and v_2 is

$$\frac{di(t)}{dt} + \frac{R}{L}i(t) = \frac{v_2}{L}u(t).$$

Also, $i(0^-) = v_1/R$.

(b) Taking the unilateral Laplace transform of the above differential equation, we get

$$sI(s) - i(0^-) + \frac{R}{L}I(s) = \frac{v_2}{Ls}.$$

(i) This corresponds to the zero state response of the circuit. Here,

$$I(s) = \frac{v_2}{s(s+1)} = v_2 \left[\frac{1}{s} - \frac{1}{s+1} \right].$$

Therefore,

$$i(t) = 2u(t) - 2e^{-t}u(t).$$

(ii) This corresponds to the zero state response of the circuit. Here, $i(0^-) = 4$ and

$$I(s) = \frac{4}{s+1}.$$

Therefore,

$$i(t) = 4e^{-t}u(t).$$

(iii) This corresponds to the total response of the system. It will be the sum of the results of the previous two parts.

$$i(t) = 2u(t) + 2e^{-t}u(t).$$

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Chapter 10 Answers

10.1. (a) The given summation may be written as

$$\sum_{n=-1}^{\infty} \frac{1}{2} \left(\frac{1}{2} r^{-1} \right)^n e^{-j\omega n},$$

by replacing z with $re^{j\omega}$. If $r < \frac{1}{2}$, then $\frac{1}{2}r^{-1} > 1$ and the function within the summation grows towards infinity with increasing n . Also, the summation does not converge. But if $r > \frac{1}{2}$, then the summation converges.

(b) The given summation may be written as

$$\sum_{n=1}^{\infty} \frac{1}{2} (2r)^n e^{j\omega n},$$

by replacing z with $re^{j\omega}$. If $r > (1/2)$, then $2r > 1$ and the function within the summation grows towards infinity with increasing n . Also, the summation does not converge. But if $r < \frac{1}{2}$, then the summation converges.

(c) The summation may be written as

$$\sum_{n=0}^{\infty} \frac{r^{-n} + (-r)^{-n}}{2} e^{-j\omega n}$$

by replacing z with $re^{j\omega}$. If $r > 1$, then the function inside the summation grows towards infinity with increasing n . Also, the summation does not converge. But if $r < 1$, then the summation converges.

(d) The summation may be written as

$$\sum_{n=0}^{\infty} \left(\frac{1}{2} r^{-1} \right)^n \cos(\pi n/4) e^{-j\omega n} + \sum_{n=-\infty}^0 \left(\frac{1}{2} r \right)^{-n} \cos(\pi n/4) e^{-j\omega n}$$

by replacing z with $re^{j\omega}$. The first summation converges for $r > \frac{1}{2}$. The second summation converges for $r < 2$. Therefore, the sum of these two summations converges for $\frac{1}{2} < r < 2$.

10.2. Using eq. (10.3),

$$X(z) = \sum_{n=-\infty}^{\infty} \left(\frac{1}{5} \right)^n u[n-3] z^{-n} \\ = \sum_{n=3}^{\infty} \left(\frac{1}{5} \right)^n z^{-n} \\ = \left[\frac{z^{-3}}{125} \right] \sum_{n=0}^{\infty} \left(\frac{1}{5} \right)^n z^{-n} \\ = \left[\frac{z^{-3}}{125} \right] \frac{1}{1 - \frac{1}{5}z^{-1}} \quad |z| > \frac{1}{5}$$

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10.3. By using eq. (9.3), we can easily show that

$$\alpha^n u[-n - n_0] \xleftrightarrow{Z} \frac{-z^{-n_0}}{1 - \alpha z^{-1}}, \quad |z| < |\alpha|.$$

We then obtain

$$X(z) = \frac{1}{1 + z^{-1}} + \frac{-z^{-n_0-1}}{1 - \alpha z^{-1}}, \quad 1 < |z| < |\alpha|.$$

Therefore, $|\alpha|$ has to be 2 . n_0 can take on any value.

10.4. Using eq. (9.3), we have

$$X(z) = \sum_{n=-\infty}^0 \left(\frac{1}{3} \right)^n \cos\left(\frac{\pi}{4}n\right) z^{-n} \\ = (1/2) \sum_{n=-\infty}^0 \left(\frac{1}{3} \right)^n e^{j\pi n/4} z^{-n} + (1/2) \sum_{n=-\infty}^0 \left(\frac{1}{3} \right)^n e^{-j\pi n/4} z^{-n} \\ = (1/2) \sum_{n=0}^{\infty} \left(\frac{1}{3} \right)^{-n} e^{-j\pi n/4} z^n + (1/2) \sum_{n=0}^{\infty} \left(\frac{1}{3} \right)^{-n} e^{j\pi n/4} z^n \\ = (1/2) \frac{1}{1 - 3e^{-j\pi/4}z} + (1/2) \frac{1}{1 - 3e^{j\pi/4}z}, \quad |z| < \frac{1}{3}$$

The poles are at $z = \frac{1}{3}e^{j\pi/4}$ and $z = \frac{1}{3}e^{-j\pi/4}$.

10.5. (a) The given z -transform may be written as

$$X(z) = \frac{z - \frac{1}{2}}{(z - \frac{1}{3})(z - \frac{1}{4})}.$$

Clearly, $X(z)$ has a zero at $z = \frac{1}{2}$. Since in $X(z)$ the order of the denominator polynomial exceeds the order of the numerator polynomial by 1, $X(z)$ has a zero at ∞ . Therefore, $X(z)$ has one zero in the finite z -plane and one zero at infinity.

(b) The given z -transform may be written as

$$X(z) = \frac{(z-1)(z-2)}{(z-3)(z-4)}.$$

Clearly, $X(z)$ has zeros at $z = 1$ and $z = 2$. Since in $X(z)$, the orders of the numerator and denominator polynomials are identical, $X(z)$ has no zeros at infinity. Therefore, $X(z)$ has two zeros in the finite z -plane and no zeros at infinity.

(c) The given z -transform may be written as

$$X(z) = \frac{(z-1)}{z(z - \frac{1}{2})(z + \frac{1}{4})}.$$

Clearly, $X(z)$ has a zero at $z = 1$. Since in $X(z)$ the order of the denominator polynomial exceeds the order of the numerator polynomial by 2, $X(z)$ has two zeros at ∞ . Therefore, $X(z)$ has one zero in the finite z -plane and two zeros at infinity.

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