Seoul National University — Optimization Lab.

Nonlinear programming

2009 — Yun-Hong Min

Solution of homework I

Note that solutions for Exercise 2.19 (c) and (d) are incomplete.

Problem 0.1. (Text, 2.1) When $k = 2$, it holds by definition of convexity. Suppose that when $k < n$, by induction hypothesis, $\lambda_1 x_1 + \cdots + \lambda_k x_k \in C$. Then,

$$\lambda_1 x_1 + \cdots + \lambda_k x_k + \lambda_{k+1} x_{k+1} \quad = (\lambda_1 + \cdots + \lambda_k)(\tfrac{1}{\lambda_1+\cdots+\lambda_k}(\lambda_1 x_1 + \cdots + \lambda_k x_k)) + \lambda_{k+1} x_{k+1}$$

By the induction hypothesis, $(\frac{1}{\lambda_1+\cdots+\lambda_k}(\lambda_1 x_1 + \cdots + \lambda_k x_k)) \in C$, and the definition of convexity, $(\lambda_1 + \cdots + \lambda_k)(\frac{1}{\lambda_1+\cdots+\lambda_k}(\lambda_1 x_1 + \cdots + \lambda_k x_k)) + \lambda_{k+1} x_{k+1} \in C$.

Problem 0.2. (Text, 2.2)

(a) C is convex set $\Rightarrow$ Since any line is convex, the intersection of C and a line is convex.

(b) The intersection of any line and C is convex $\Rightarrow$ C is convex set: Suppose that C is not convex. Then, there exist $x, y \in C$ and $\lambda \in [0,1]$ such that $(1-\lambda)x + \lambda y \notin C$. Consider the line L through x and y. Since $x, y \in C \cap L$, by convexity of $C \cap L$, $(1-\lambda)x + \lambda y \in C \cap L$, but $(1-\lambda)x + \lambda y \notin C$. A contradiction.

(c) A set is affine iff its intersection with any line is affine: Similar to (a) and (b).

Problem 0.3. (Text, 2.9)

(a) For any x_0 and x_i,

$$\begin{aligned} \|x - x_0\|_2 \le \|x - x_i\|_2 \quad &\Longleftrightarrow \quad (x-x_0)^T(x-x_0) \le (x-x_i)^T(x-x_i) \\ &\Longleftrightarrow \quad 2(x_i - x_0)^T x \le x_i^T x_i - x_0^T x_0 \end{aligned}$$

Let us denote $2(x_i - x_0)$ by a_i and $x_i^T x_i - x_0^T x_0$ by b_i. Then, $V = \{x \in \mathbb{R}^n | a_i^x \le b_i, i = 1, \ldots, K\}$, which is a polyhedron. If we express V in the form $V = \{x | Ax \preceq b\}$, then

$$A = \begin{bmatrix} 2(x_1 - x_0)^T \\ \vdots \\ 2(x_K - x_0)^T \end{bmatrix} \text{ and } b = \begin{bmatrix} x_1^T x_1 - x_0^T x_0 \\ \vdots \\ x_K^T x_K - x_0^T x_0 \end{bmatrix}.$$

(b) Suppose that x_0 is an any interior point in P. Since P is an intersection of finite number, say K, of halfspaces, $P = \{x | (a^i)^T x \le b^i, i = 1, \ldots, K\}$ we can define $x_1, \ldots, x_K$ as follows: for $i = 1, \ldots, K$, x_i is plane-symmetry point of x_0 w.r.t the ith hypeplane $\{x | (a^i)^T x = b^i\}$. Then, ith halfspace of P can be represented as $\{x | \|x - x_0\|_2 \le \|x - x_i\|_2\}$.

(c) Consider the polyhedral decomposition of $\mathbb{R}$ such that

$$V_0 = \{x | x \leq -6\}, V_1 = \{x | -6 \leq x \leq -4\}, V_2 = \{x | -4 \leq x \leq 4\},$$
$$V_3 = \{x | 4 \leq x \leq 6\}, V_4 = \{x | x \geq 6\}.$$

Then, we must have

(i) $-6 < x_1 < -4, 4 < x_3 < 6$.

(ii) $-4 - x_1 = x_2 - (-4) \Rightarrow x_2 = -8 - x_1$.

(iii) $x_3 - 4 = 4 - x_2 \Rightarrow x_2 = 8 - x_3$.

However, (i) and (ii) implies $-4 < x_2 < -2$ and (i) and (iii) implies $2 < x_2 < 4$, a contradiction. Thus, we cannot determine x_3 for a given partition of $\mathbb{R}$.

Problem 0.4. (Text, 2.10)

(a) C is convex if $\lambda x_1 + (1-\lambda)x_2 \in C$ for any $x_1, x_2 \in C$ and any $\lambda \in [0,1]$. We need to show that

$$(\lambda x_1 + (1-\lambda)x_2)^T A(\lambda x_1 + (1-\lambda)x_2) + b^T(\lambda x_1 + (1-\lambda)x_2) + c \leq 0.$$

From the following calculation, we can conclude that $\lambda x_1 + (1-\lambda)x_2 \in C$ for any $\lambda \in [0,1]$:

$$\begin{aligned}
& (\lambda x_1 + (1-\lambda)x_2)^T A(\lambda x_1 + (1-\lambda)x_2) + b^T(\lambda x_1 + (1-\lambda)x_2) + c \\
= \;& \lambda^2 x_1^T A x_1 + (1-\lambda)^2 x_2^T A x_2 + \lambda(1-\lambda) x_1^T A x_2 + \lambda(1-\lambda) x_2^T A x_1 \\
+ \;& \lambda(b^T x_1 + c) + (1-\lambda)(b^T x_2 + c) \\
= \;& \lambda(x_1^T A x_1 + b^T x_1 + c) + (1-\lambda)(x_2^T A x_2 + b^T x_2 + c) \\
- \;& \lambda(1-\lambda)(x_2 - x_1)^T A(x_2 - x_1) \leq 0.
\end{aligned}$$

since $A \succeq 0$.

(b) Let $K = \{x \in \mathbb{R}^n | x^T A x + bx + c \leq 0\} \cap \{x | g^T x + h = 0\}$ and consider any $x_1, x_2 \in K$ and any $\lambda \in [0,1]$. First, $\lambda x_1 + (1-\lambda)x_2 \in \{x | g^T x + h = 0\}$ is obvious. Second, since $g^T x_1 + h = 0$ and $g^T x_2 + h = 0$, $g^T(x_2 - x_1) = 0$. Moreover, for all $z \in \mathbb{R}^n$, $z^T(A + \mu g g^T)z = z^T A z + (g^T z)^2 \geq 0$, so $-z^T A z \leq (g^T z)^2$. Therefore,

$$\begin{aligned}
& (\lambda x_1 + (1-\lambda)x_2)^T A(\lambda x_1 + (1-\lambda)x_2) + b^T(\lambda x_1 + (1-\lambda)x_2) + c \\
= \;& \lambda(x_1^T A x_1 + b^T x_1 + c) + (1-\lambda)(x_2^T A x_2 + b^T x_2 + c) \\
- \;& \lambda(1-\lambda)(x_2 - x_1)^T A(x_2 - x_1) \\
\leq \;& \lambda(x_1^T A x_1 + b^T x_1 + c) + (1-\lambda)(x_2^T A x_2 + b^T x_2 + c) \\
- \;& \lambda(1-\lambda)(g^T(x_2 - x_1))^2 \\
= \;& \lambda(x_1^T A x_1 + b^T x_1 + c) + (1-\lambda)(x_2^T A x_2 + b^T x_2 + c) \leq 0
\end{aligned}$$

Thus, K is convex.

Problem 0.5. (Text, 2.12)

(a) For $x, y \in C := \{x \in \mathbb{R}^n | \alpha \le a^T x \le \beta\}$ and $0 \le \lambda \le 1$,

$$a^T((1-\lambda)x + \lambda y) = (1-\lambda)a^T x + \lambda a^T y \ge (1-\lambda)\alpha + \lambda\alpha = \alpha$$

and

$$a^T((1-\lambda)x + \lambda y) = (1-\lambda)a^T x + \lambda a^T y \le (1-\lambda)\beta + \lambda\beta = \beta.$$

Thus, $(1-\lambda)x + \lambda y \in C$.

(b) $\{x \in \mathbb{R}^n | \alpha_i \le x_i \le \beta_i, i = 1, \ldots, n\} = \bigcap_i \{x \in \mathbb{R}^n | \alpha_i \le e_i^T x \le \beta_i\}$. Thus, convex.

(c) $\{x \in \mathbb{R}^n | a_1^T x \le b_1, a_2^T x \le b_2\} = \{x \in \mathbb{R}^n | a_1^T x \le b_1\} \cap \{x \in \mathbb{R}^n | a_2^T x \le b_2\}$. Thus, convex.

(d) By definition, we have the following equivalent inequalities:

$$\begin{aligned} \|x - x_0\|_2 \le \|x - y\|_2 \quad &\iff \quad (x - x_0)^T(x - x_0) \le (x - y)^T(x - y) \\ &\iff \quad 2(y - x_0)^T x \le y^T y - x_0^T x_0 \end{aligned}$$

Therefore, $\{x | \|x - x_0\|_2 \le \|x - y\|_2, \forall y \in S\} = \bigcap_{y \in S} \{x | 2(y - x_0)^T x \le y^T y - x_0^T x_0\}$. Thus, convex.

(e) Consider the following counter example: Let $S = \{x \in \mathbb{R} | x \ge 1\} \cup \{x \in \mathbb{R} | x \le 0\}$ and T be the complement of S. Then,

$$S = \{x | \operatorname{dist}(x, S) \le \operatorname{dist}(x, T)\}.$$

But, S is not convex.

(f) $\{x | x + S_2 \subseteq S_1\} = \bigcap_{y \in S_2} \{x | x \in S_1 - y\}$. Since translation $S_1 - y$ preserves convexity, $\{x | x \in S_1 - y\}$ is convex, and so is $\{x | x + S_2 \subseteq S_1\}$.

(g) When $\theta = 0$ or $\theta = 1$, it is easy to see that $C = \{x | \|x - a\|_2 \le \theta \|x - b\|_2\}$ is convex. For $0 < \theta < 1$,

$$\begin{aligned} \|x - a\|_2 \le \theta \|x - b\|_2 \; &\Leftrightarrow (x - a)^T(x - a) \le \theta^2 (x - b)^T(x - b) \\ &\Leftrightarrow (1 - \theta^2) x^T x - 2(a - \theta^2 b)^T x \le -a^T a + \theta^2 b^T b \\ &\Leftrightarrow \|x - \tfrac{a - \theta^2 b}{1 - \theta^2}\|_2^2 \le \tfrac{(a - \theta^2 b)^T(a - \theta^2 b) + (1 - \theta^2)(-a^T a + \theta^2 b^T b)}{(1 - \theta^2)^2} = \tfrac{\theta^2 \|a - b\|_2^2}{(1 - \theta^2)^2}. \end{aligned}$$

Thus, C is a ball, and hence convex.

Problem 0.6. (Text, 2.19)

(a)

$$\begin{aligned} f^{-1}(C) &= \{x \in \mathrm{dom} f | f(x) \in C\} \\ &= \{x | c^T x + d > 0, \tfrac{Ax+b}{c^T x+d} \in C\} \\ &= \{x | c^T x + d > 0, g^T Ax + g^T b \leq hc^T x + hd\} \\ &= \{x | c^T x + d > 0, (A^T g - hc)^T x \leq hd - g^T b\}. \end{aligned}$$

(b)

$$\begin{aligned} f^{-1}(C) &= \{x \in \mathrm{dom} f | f(x) \in C\} \\ &= \{x | c^T x + d > 0, \tfrac{Ax+b}{c^T x+d} \in C\} \\ &= \{x | c^T x + d > 0, GAx + Gb \leq hc^T x + dh\} \\ &= \{x | c^T x + d > 0, (GA - hc^T) x \leq dh - Gb\}. \end{aligned}$$

(c)

$$\begin{aligned} f^{-1}(C) &= \{x \in \mathrm{dom} f | f(x) \in C\} \\ &= \{x | c^T x + d > 0, \tfrac{Ax+b}{c^T x+d} \in C\} \\ &= \{x | c^T x + d > 0, (Ax+b)^T P^{-1} (Ax+b) \leq (c^T x + d)^2\} \\ &= ?? \end{aligned}$$

(d)

$$\begin{aligned} f^{-1}(C) &= \{x \in \mathrm{dom} f | f(x) \in C\} \\ &= \{x | c^T x + d > 0, \tfrac{Ax+b}{c^T x+d} \in C\} \\ &= \{x | c^T x + d > 0, (Ax+b)_1 A_1 + \cdots + (Ax+b)_n A_n \preceq (c^T x + d) B\} \\ &= ?? \end{aligned}$$

Problem 0.7. (Text, 2.28)

(a) x_1 is PDS iff $x_1 \geq 0$.

(b) $\begin{bmatrix} x_1 & x_2 \\ x_2 & x_3 \end{bmatrix} \succeq 0$ iff $x_1 \geq 0$, $x_3 \geq 0$, and $x_1 x_3 - x_2^2 \geq 0$.

(c) $\begin{bmatrix} x_1 & x_2 & x_3 \\ x_2 & x_4 & x_5 \\ x_3 & x_5 & x_6 \end{bmatrix} \succeq 0$ iff

$$\begin{gathered} x_1 \geq 0, x_4 \geq 0, x_6 \geq 0 \\ x_1 x_4 - x_2^2 \geq 0, x_1 x_6 - x_3^2 \geq 0, x_4 x_6 - x_5^2 \geq 0, \text{ and} \\ x_1(x_4 x_6 - x_5^2) - x_2(x_2 x_6 - x_3 x_5) + x_3(x_2 x_5 - x_3 x_4) \geq 0. \end{gathered}$$