Solution of homework II

Problem 0.1. (Text, 3.1)

(a) By the assumption, $\frac{b-x}{b-a} \geq 0$ and $\frac{x-a}{b-a} \geq 0$. Moreover, $\frac{b-x}{b-a} + \frac{x-a}{b-a} = 1$. Let $\lambda = \frac{b-x}{b-a}$. Then, $x = b - \lambda(b-a) = (1-\lambda)b + \lambda a$. Thus,

$$f(x) = f((1-\lambda)b + \lambda a) \leq \lambda f(a) + (1-\lambda) f(b) = \frac{b-x}{b-a} f(a) + \frac{x-a}{b-a} f(b).$$

(b) $\frac{f(x)-f(a)}{x-a} \leq \frac{f(b)-f(a)}{b-a}$:

$$\begin{aligned} \frac{f(x)-f(a)}{x-a} \leq \frac{f(b)-f(a)}{b-a} \quad &\Leftrightarrow f(x) - f(a) \leq \tfrac{x-a}{b-a} f(b) - \tfrac{x-a}{b-a} f(a) \\ &\Leftrightarrow f(x) \leq \tfrac{x-a}{b-a} f(b) + \tfrac{b-x}{b-a} f(a). \end{aligned}$$

Thus, the first inequality holds by (a). The second inequality can be proved in a similar way.

(c) Since $\frac{f(x)-f(a)}{x-a} \leq \frac{f(b)-f(a)}{b-a}$ for all $x \in (a,b)$,

$$\lim_{x \to a} \frac{f(x)-f(a)}{x-a} = f'(a) \leq \lim_{x \to a} \frac{f(b)-f(a)}{b-a} = \frac{f(b)-f(a)}{b-a}.$$

Similarly, $f'(b) \geq \frac{f(b)-f(a)}{b-a}$.

(d) From (c),

$$f(b) \geq f(a) + f'(a)(b-a), \quad f(a) \geq f(b) + f'(b)(a-b).$$

By summing these two inequalities we have

$$(b-a)(f'(b) - f'(a)) \geq 0 \quad \Rightarrow \frac{f'(b) - f'(a)}{b-a} \geq 0.$$

By limiting $b \to a$, we obtain $f''(a) \geq 0$. $f''(b) \geq 0$ can be proved in a similar way.

Problem 0.2. (Text, 3.2)

Problem 0.3. (Text, 3.3) First, since f is increasing, $f(x_1) = y_1$, $f(x_2) = y_2$, and $y_1 < y_2$ implies $x_1 < x_2$. Thus, g is an increasing function. Moreover,

$$\begin{aligned} g((1-\lambda)y_1 + \lambda y_2) &= g((1-\lambda) f(x_1) + \lambda f(x_2)) \\ &\geq g(f((1-\lambda)x_1 + \lambda x_2)) \\ &= (1-\lambda)x_1 + \lambda x_2 \\ &= (1-\lambda) g(y_1) + \lambda g(y_2). \end{aligned}$$

Thus, g is a concave function.

Problem 0.4. (Text, 3.6)

(i) epi$f$ is a halfspace iff hypo$f$ is a halfspace. Thus, f is both convex and concave, and hence affine function.

(ii) If epi$f$ is a convex cone, for $\alpha, \beta \geq 0$ and $(x, f(x)), (y, f(y)) \in$ epi$f$,

$$f(\alpha x + \beta y) \leq \alpha f(x) + \beta f(y).$$

It implies that f is a sublinear function.

(iii) epi$f$ is a polyhedron iff

$$\text{epi}f = \{(x,t) | \langle s_j, x\rangle + \alpha_j t \leq b_j \ \text{ for } j \in J\}.$$

For this set to be an epigraph each α_j should be nonpositive, so we can assume that $\alpha_j = -1$ for all j. Furthermore, we may denote by $\{1, \ldots, m\} \subseteq J$ such that $\alpha_j = -1$ and by $\{m+1, \ldots, m+p\}$ the rest. With these notations,

$$f(x) := \max\{\langle s_j, x\rangle - b_j : j = 1, \ldots, m\}$$

defines a polyhedron epigraph.

Problem 0.5. (Text, 3.7) Suppose that $f(x) < M$ for all $x \in \mathbb{R}^n$. By the first-order conditions for convexity,

$$f(y) \geq f(x) + \nabla f(x)^T (y - x), \quad \forall x, y \in \mathbb{R}^n.$$

Suppose that f is not constant, which means that there exists a $\bar{x} \in \mathbb{R}^n$ such that $\nabla f(\bar{x}) \neq 0$. For any positive $\alpha > 0$ let $y := \bar{x} + \alpha \nabla f(\bar{x})$. Then, first-order condition implies that

$$f(y) \geq f(\bar{x}) + \alpha \|\nabla f(\bar{x})\|_2^2$$

Let $\alpha = \frac{M - f(\bar{x})}{\|\nabla f(\bar{x}\|_2^2}$. Then,

$$f(\frac{M - f(\bar{x})}{\|\nabla f(\bar{x}\|_2^2} \nabla f(\bar{x})) \geq M,$$

a contradiction.

Problem 0.6. (Text, 3.9)

(a) By the second order condition $\tilde{f}$ is convex if and only if $\nabla^2 \tilde{f}(x) \succeq 0$ for all $x \in$ dom$\tilde{f}$. Thus,

$$\nabla^2 \tilde{f}(x = Fz + \hat{x}) = \nabla^2 f(Fz + \hat{x}) = F^T f(Fz + \hat{x}) F \succeq 0.$$

(b) From (a) $\tilde{f}$ is convex if and only if $F^T\nabla^2\tilde{f}(Fz+\hat{x})F \succeq 0$. This means that

$$y^T F^T \nabla^2 \tilde{f}(Fz+\hat{x})Fy \geq 0, \forall y.$$

Since the null space of A is equal to the range space of F,

$$w^T\nabla^2\tilde{f}(Fz+\hat{x})w \geq 0, \forall w \in \mathcal{N}(A).$$

Thus, by the hint, the proof is completed.

Problem 0.7. (Text, 3.20)

(a)

$$\begin{aligned}\|A((1-\lambda)x+\lambda y)-b\| &= \|(1-\lambda)(Ax-b)+\lambda(Ay-b)\| \\ &\leq (1-\lambda)\|Ax-b\|+\lambda\|Ay-b\|.\end{aligned}$$

where the second inequality is due to the homogeneity and metric property of $\|\cdot\|$.

(b) Not yet...

(c) Not yet...

Problem 0.8. (Text, 3.22) In this problem we will express $f(x)$ as $h(g(x))$, and then apply the composition rules.

(a) $f(x) = h(g(x))$ where $g(x) = -\log(\sum_{i=1}^n e^{a_i^T x+b_i})$ and $h(x) = -\log x$. Then, $g(x)$ is concave, and $h(x)$ is convex and nonincreasing. Thus, $f(x)$ is convex.

(b) $f(x,u,v) = h(u, g(x,u,v))$ where $g(x,u,v) = v - x^T x/u$ and $h(u,y) = -\sqrt{uy}$. Then, $g(x)$ is concave, and $h(u,y)$ is convex and nonincreasing. Thus, $f(x,u,v)$ is convex.

(c) $f(x,u,v) = h(g(x,u,v))$ where $g(x,u,v)$ is a concave function defined as the negative of (b), and $h(x) = -2\log x$ which is a convex and nonincreasing. Thus, $f(x,u,v)$ is convex.

(d) $f(x,t) = h(g(x,t),t)$ where $g(x,t) = t - \frac{\|x\|_p^p}{t^{p-1}}$ and $h(x,y) = -x^{1/p}y^{1-1/p}$. Then, $g(x,t)$ is concave, and $h(x,y)$ is convex and nonincreasing. Thus, $f(x,t)$ is convex.

(e) $f(x,t) = h(g(x,t))$ where $g(x,t)$ is a concave function defined as the negative of (d), and $h(x) = -p\log x$ which is a convex and nonincreasing. Thus, $f(x,t)$ is convex.

Problem 0.9. (Text, 3.32)

(a) $(fg)'' = (f''g + 2f'g' + fg'')$. By assumption, $f, g > 0$ and $f'', g'' \geq 0$ and hence $f''g, fg'' \geq 0$. Since either $f', g' \geq 0$ or $f', g' \leq 0$, $f'g' \geq 0$. Thus, $(fg)'' \geq 0$, and hence fg is convex.

(b) Since $f'', g'' \leq 0$, and $f, g > 0$ and $f'g' \leq 0$, $(fg)'' \leq 0$, and hence fg is concave.

(c) Similar to (a) and (b).

Problem 0.10. (Text, 3.43) It suffices to prove the result for a function on $\mathbb{R}$. Recall that a function f is quasiconvex iff for all $x, y \in \mathrm{dom} f$ and $0 < \lambda < 1$,

$$f((1-\lambda)x + \lambda y) \leq \max\{f(x), f(y)\}.$$

only if part: Suppose that f is quasiconvex and $f(y) \leq f(x)$. By differentiability of f at x, we have

$$f((1-\lambda)x + \lambda y) - f(x) = \lambda f'(x)(y-x) + \lambda|y-x|\alpha(x; \lambda(y-x))$$

for $\lambda \in (0,1)$ where $\alpha(x; \lambda(y-x)) \to 0$ as $\lambda \to 0$. By the quasiconvexity of f, we have $f((1-\lambda)x + \lambda y) - f(x) = \lambda f'(x)(y-x) + \lambda|x-y|\alpha(x; \lambda(y-x)) \leq 0$. Dividing by λ and letting $\lambda \to 0$, we get

$$f'(x)(y-x) \leq 0.$$

if part: Suppose that x and y are any two points such that $f(y) \leq f(x)$. We assume that $y < x$ (A proof for $y > x$ is similar). Suppose f is not quasiconvex, i.e., there exists $z \in (y, x)$ such that $f(z) > \max\{f(x), f(y)\}$. By continuity of f, there exists $z' \in (z, x)$ such that $f'(z') < 0$ and $f(y) \leq f(z')$. Then,

$$f'(z')(y - z') > 0,$$

a contradiction.