Solution of homework III

Problem 0.1. (Text, 4.1) Skip

Problem 0.2. (Text, 4.2)

(a) $\Leftarrow$: Suppose that there exist $v \neq 0$ with $Av \leq 0$ and $\bar{x}$ with $A\bar{x} < b$. Let $x_\alpha := \bar{x} + \alpha v$ for $\alpha > 0$. Then,

$$Ax_\alpha = A\bar{x} + \alpha Av < b.$$

Thus, $x_\alpha \in \mathrm{dom} f_0$ for all $\alpha > 0$. As $\alpha \to \infty$ we can increase or decrease x_α arbitrarily large. Thus, $\mathrm{dom} f_0$ is unbounded.
$\Rightarrow$: Suppose f_0 is unbounded, and for all $v \neq 0$ $Av > 0$. For any $v \neq 0$, consider the solution $x_v := \bar{x} + \alpha \frac{v}{\|v\|}$ where $\alpha > 0$. To ensure the feasibility of x_v, it must satisfy

$$Ax_v = A\bar{x} + \alpha Av < b.$$

Thus, we cannot increase α arbitrarily large for any direction v.This implies that $\mathrm{dom} f_0$ is bounded, a contradiction.

(b)

$$\begin{array}{cc} & \text{There exists a } v \text{ with } Av \leq 0,\ Av \neq 0. \\ \Leftrightarrow & \text{There exists no } z \succ 0 \text{ such that } A^T z = 0. \\ \Leftrightarrow & \text{There exists no } z \succ 0 \text{ such that } \sum_{i=1}^m z_i a_i = 0. \\ \Leftrightarrow & \text{There exists no } z \prec 0 \text{ such that } \sum_{i=1}^m z_i a_i = 0. \\ \Leftrightarrow & \sum_{i=1}^m \frac{a_i}{b_i - a_i^T x} \neq 0,\ \forall x \in \mathrm{dom} f_0. \end{array}$$

Thus, since our problem is feasible, but doesn't have any optimal solution, it is unbounded below.

(c) From (b).

(d) Skip.

Problem 0.3. (Text, 4.3) Note that a given problem is a convex optimization problem. Thus, we will use the optimality condition that "For convex minimization with differentiable f_0, feasible x is optimal iff $\nabla f_0(x)^T(y - x) =\geq 0$ for any feasible y". For a given point x^*, $\nabla f(x^*) = (-1, 0, 2)^T$. Thus,

$$\nabla f_0(x)^T(y - x) = -(y_1 - 1) + 2(y_3 + 1).$$

For any feasible y such that $y_i \in [-1, 1]$, $-(y_1 - 1) \geq 0$ and $(y_3 + 1) \geq 0$, and hence $\nabla f_0(x)^T(y - x) \geq 0$ for any feasible y. Thus, x^* is optimal.

Problem 0.4. (Text, 4.5) Skip

Problem 0.5. (Text, 4.8)

(a) Skip

(b) Skip

(c) Set $x_i = l_i$ if $c_i > 0$, and $x_i = u_i$ if $c_i \le 0$.

(d) Choose the index i such that $c_i \le c_j$ for all $j = 1, \ldots, n$. Then, set $x_i = 1$ and $x_j = 0$ for all $j \neq i$. If the equality constraint is replaced by an inequality, set $x_i = 1$ and $x_j = 0$ for all $j \neq i$, if $c_i \le 0$, and set $x_j = 0$ for all j if $c_j \ge 0$ for all j.

(e) We can assume that $c_1 \le c_2 \le \cdots \le c_n$ without loss of generality. Now we set $x_i = 1$ from $i = 1$ to $i = k$ where $\sum_{i=1}^{k} x_i = \alpha$. If α is not an integer, for the last index, say k, $x_k = \alpha - (k-1)$. If the equality constraint is replaced by an inequality, it suffices to consider the x_i such that $c_i \le 0$.

(f) Skip

Problem 0.6. (Text, 4.13) Consider the simpler robust linear constraint:

$$a_1 x_1 + \cdots + a_n x_n \le b,\ \forall i, \bar{a}_i - v_i \le a_i \le \bar{a}_i + v_i.$$

When we can express it as linear inequalities, we can easily convert the original robust LP to an ordinary LP. Consider the following linear inequality system:

$$\begin{aligned} & t_1 + \cdots + t_n \le b \\ & (\bar{a}_i - v_1) x_i \le t_i \\ & (\bar{a}_i + v_i) x_i \le t_i \end{aligned}$$

Thus, we have total $1 + 2n$ constraints with $2n$ variables. Suppose x is a feasible for our linear inequality system. We need to show that any a_i between $\bar{a}_i - v_i$ and $\bar{a}_i + v_i$, it satisfies $a_1 x_1 + \cdots + a_n x_n \le b$. Since $a_i x_i \le \max\{(\bar{a}_i - v_i)x, (\bar{a}_i + v_i)x\} \le t_i$, the claim holds.

By applying the described conversion to each constraint of robust LP, we can construct $m(1 + 2n)$ constraints with $2nm$ variables which describes the original robust LP.

Problem 0.7. (Text, 4.20) Skip

Problem 0.8. (Text, 4.21)

(a) Consider the problem of minimizing $c^T x$ over the unit ball centered at origin. Let y^* be the optimal solution of this problem. Then, we can construct the optimal solution $x^*$to the problem of minimizing $c^T x$ over the ellipsoid, $x^T A x \le 1$: $x^* = A^{-1/2} y^*$. Since $y^* = -c/\|c\|_2$, $x^* = -A^{-1/2} c/\|c\|_2$.

(b) By the similar argument to (a), $x^* = A^{-1/2}y^* + x_c$.

(c) For any $x \in \mathbb{R}^n$, $x^T Bx \geq 0$. Thus, 0 is a lower bound of the optimal value. Let $x^* := 0$. Then, x^* is feasible, and it achieves 0 for the objective value. Thus, x^* is optimum.

Problem 0.9. (Text, 4.22) First, $(x^*)^T x^* = q^T(P + \lambda I)^{-2}q = I$, and hence x^* is feasible. To prove the optimality of x^* it suffices to show that

$$\nabla f(x^*)^T(y - x^*) \geq 0 \ \forall y, y^T y \leq 1.$$

For $f(x^*) = -P(P + \lambda I)^{-1}q + q$, the minimum value of $\nabla f(x^*)^T(y - x^*)$ over $y^T y \leq 1$ is achieved at $y = -\nabla f(x^*)/\|\nabla f(x^*)\|_2$. Thus, we need to check that

$$(-P(P + \lambda I)^{-1}q + q)^T(y + (P + \lambda I)^{-1}q) \geq 0$$

at $y = -\nabla f(x^*)/\|\nabla f(x^*)\|_2$. By a simple manipulation,

$$\begin{aligned}
\nabla f(x^*)^T(y - x^*) &= \|P(P + \lambda I)^{-1}q + q\| + (-P(P + \lambda I)^{-1}q + q)^T(P + \lambda I)^{-1}q \\
&= \|P(P + \lambda I)^{-1}q + q\| - q^T(P + \lambda I)^{-1}P(P + \lambda I)^{-1}q + q^T(P + \lambda I)^{-1}q
\end{aligned}$$

We will show the Theorem by showing that

$$-q^T(P + \lambda I)^{-1}P(P + \lambda I)^{-1}q + q^T(P + \lambda I)^{-1}q \geq 0.$$

Since $q^T(P + \lambda I)^{-2}q = 1$ and $\lambda \geq 0$,

$$\begin{aligned}
-q^T(P + \lambda I)^{-1}P(P + \lambda I)^{-1}q + q^T(P + \lambda I)^{-1} &= q^T((P + \lambda I)^{-1} - (P + \lambda I)^{-1}P(P + \lambda I)^{-1})q \\
&= (P + \lambda I)^{-1}(P + \lambda I - P)(P + \lambda I)^{-1} \\
&= \lambda q^T(P + \lambda I)^{-1}q = \lambda \geq 0
\end{aligned}$$