Solution of homework VI

Problem 0.1. (a) **Solution.** Since P is not PSD, there exists $\bar{x}$ such that $\bar{x}^T P\bar{x} < 0$. For any $t > 0$,

$$f(tx) = (1/2)t^2\bar{x}^T P\bar{x} + tq^T x + r$$

Thus, as $t \to \infty$, $f(tx) \to -\infty$. $\square$

(b) **Solution.** Since $Px = -q$ does not have a solution, there exists v such that $v^T P = 0$ and $-q^T v > 0$. Then, for any $\bar{x}$,

$$\begin{aligned} f(\hat{x} + tv) &= f(\hat{x}) + t(v^T P\hat{x} + (1/2)q^T v) + tv^T q \\ &= f(\hat{x}) + tv^T q \end{aligned}$$

Since $v^T q < 0$, as $t \to \infty$, $f(\hat{x} + tv) \to -\infty$. $\square$

Problem 0.2. (a) **Solution.** $p^* = 1$. $\square$

(b) **Solution.** $S = \{x | x_1^2 + x_2^2 \leq 8,\ x_1 > 1\}$.

$$\nabla^2 f(x) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \succ 0.$$

for all $x \in S$. Thus, f is strongly convex on S. $\square$

(c) skip.

Problem 0.3. ...

Problem 0.4. (a)

$$\begin{aligned} \nabla f(x)^T \Delta x_{nsd} &= \min\{\nabla f(x)^T v | \|v\| = 1\} \\ &= -\max\{-\nabla f(x)^T v | \|v\| = 1\} \\ &= -\|\nabla f(x)\|_*. \end{aligned}$$

(b)

$$\begin{aligned} \nabla f(x)^T \Delta x_{sd} &= \|\nabla f(x)\|_* \nabla f(x)^T \Delta x_{nsd} \\ &= -\|\nabla f(x)\|_*^2. \end{aligned}$$

(c)

Problem 0.5. Skip.

1 Additional problems

Problem 1.1. Skip.

Problem 1.2. (i) $\|z\|_{2*} = \sup\{z^T x | \|x\|_2 \le 1\} = \|z\|_2$.

(ii)

$$\begin{aligned} \|z\|_{P*} &= \sup\{z^T x | x^T P x \le 1\} \\ &= \sup\{z^T P^{-1/2} y | \|y\|_2 \le 1\} \\ &= \|z^T P^{-1/2}\|_2 \\ &= z^T P^{-1} z = \|z\|_{P^{-1}}. \end{aligned}$$

(iii) $\|z\|_{\infty *} = \sup\{z^T x | \|x\|_\infty \le 1\} = \sum_{i=1}^n |z_i| = \|z\|_1$

Problem 1.3. Solution. By (9), $\Delta x_{nsd} = -(\nabla f(x)^T (\nabla^2 f(x)^{-1}) \nabla f(x))^{-1/2} (\nabla^2 f(x)^{-1}) \nabla f(x)$ when we use the quadratic norm defined by $\nabla^2 f(x)$. Then, by $\Delta x_{sd} = \|\nabla f(x)\|_* \Delta x_{nsd} = -\nabla^2 f(x)^{-1} \nabla f(x) = \Delta x_{nt}$. $\square$