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AE545-Problem Set #3 – Solutions

Prob #1

$$f(\lambda) = \lambda - \mu \tan\alpha - \frac{C_T}{2\sqrt{\mu^2+\lambda^2}} = 0 \qquad (1)$$

$$f'(\lambda_n) = \lim_{n\to\infty} \frac{f(\lambda_{n+1}) - f(\lambda_n)}{\lambda_{n+1} - \lambda_n}$$

Solve for λ_{n+1}

$$\lambda_{n+1} - \lambda_n = \frac{f(\lambda_{n+1}) - f(\lambda_n)}{f'(\lambda_n)}$$

As $n \to \infty$, λ_{n+1} approaches the exact value λ and $f(\lambda) = 0$, thus

$$\lim f(\lambda_{n+1}) = f(\lambda) = 0$$

$$\lambda_{n+1} = \lambda_n - \frac{f(\lambda_n)}{f'(\lambda_n)} \qquad (2)$$

Evaluate λ_{n+1} for a given $f(\lambda)$, using Eq(1)

$$f'(\lambda) = 1 + \frac{1}{2}\frac{\lambda C_T}{(\mu^2+\lambda^2)^{3/2}} \quad 2\left(\frac{1}{2}\right) = 1 + \frac{\lambda C_T}{2(\mu^2+\lambda^2)^{3/2}} \qquad (3)$$

Using Eq(2)

$$\lambda_{n+1} = \frac{f'(\lambda_n)\lambda_n - f(\lambda_n)}{f'(\lambda_n)} =$$

$$= \frac{\lambda_n + \dfrac{C_T \lambda_n^2}{2(\mu^2+\lambda^2)^{3/2}} - \left(\lambda_n - \mu\tan\alpha - \dfrac{C_T}{2(\mu^2+\lambda^2)^{1/2}}\right)}{1 + \dfrac{\lambda_n C_T}{2(\mu^2+\lambda_n^2)^{3/2}}} \qquad (4)$$

$$\lambda_{n+1} = \frac{\mu \tan\alpha + \frac{C_T}{2(\mu^2+\lambda_n^2)^{1/2}}\left(1+\frac{\lambda_n^2}{(\mu^2+\lambda_n^2)}\right)}{1+\frac{\lambda_n C_T}{2(\mu^2+\lambda_n^2)^{3/2}}}$$

$$= \frac{\mu \tan\alpha + \frac{C_T}{2(\mu^2+\lambda_n^2)^{1/2}}\frac{\mu^2+2\lambda_n^2}{(\mu^2+\lambda_n^2)}}{1+\frac{\lambda_n C_T}{2(\mu^2+\lambda_n^2)^{3/2}}}$$

$$\lambda_{n+1} = \frac{\mu \tan\alpha + \frac{C_T}{2}\frac{\mu^2+2\lambda_n^2}{(\mu^2+\lambda_n^2)^{3/2}}}{1+\frac{\lambda_n C_T}{2(\mu^2+\lambda_n^2)^{3/2}}} \qquad (4)$$

Equation (4) represents the relation that we wanted to prove.

Implementing this iterative scheme for the 3 cases required yields the iteration history provided below

Case \ Iteration #	#1	#2	#3	Exact Solution
Case 1	0.03266	0.032523	0.032523	0.0325231
Case 2	0.033385	0.033351	0.033351	0.0333506
Case 3	0.045077	0.045076	0.045076	0.0450762

The iterative method converges in two steps (or iterations)

Prob #2

The problem can be approached from the concept of minimizing power per unit thrust, or maximizing the power loading. The ratio of power required to hover to the thrust produced is

$$\frac{P}{T} = \frac{P}{W} = \frac{\Omega R\, C_P}{C_T}$$

This quantity should be as close as possible to the ideal value for best hovering efficiency. Because T is proportional to $(\Omega R)^2$ but P is proportional to $(\Omega R)^3$, this requires low tip speed (ΩR). Also, using simple momentum theory the ratio

$$\frac{P}{T} = v \qquad \text{(see page 22 of the notes)}$$

$$\frac{P}{T} = v = \sqrt{\frac{T}{2\rho A}} = \frac{T^{3/2}}{T\sqrt{2\rho A}} = \sqrt{\frac{1}{2\rho}\left(\frac{T}{A}\right)}$$

$$\left(T/A\right) = \text{disk loading } DL$$

Thus (P/T) is proportional to the square of the disk loading, therefore to minimize power per thrust the disk loading should be low, i.e. disk area should be large (large blade radius) for a given gross weight to minimize the induced velocity <u>and</u> the tip speed should be low.

Ultimately the tip speed is set on the basis of various performance requirements for a given rotor

size, and may include autorotational requirements as well as noise

The use of a low solidity rotor will help minimize profile power and maximize the figure of merit. This can be seen from the expression for figure of merit. Recall from pg 24 of the notes that

$$FM = \frac{C_{P_{ideal}}}{C_{P_i} + C_{P_o}} \qquad (1)$$

where $C_{P_{ideal}} = \frac{C_T^{3/2}}{\sqrt{2}} = \lambda C_T$

and C_{P_i} can be written in terms of the induced power factor κ as $C_{P_i} = \kappa \frac{C_T^{3/2}}{\sqrt{2}}$

Assuming uniform inflow and $C_d = C_{do} = \text{const}$, the profile power for a rectangular blade is

$$C_{P_o} = \frac{1}{8} \sigma C_{do} \qquad (2)$$

From Eqs (1) and (2) it is clear that to maximize FM for a given C_T, the profile power must be kept to a minimum by minimizing the solidity. However, minimizing the solidity must be done carefully so as to avoid stall. Also propulsive force requirements in forward flight will determine the lowest allowable solidity.

The use of blade twist will help produce a more uniform inflow velocity over the rotor disk

in hovering flight. This helps minimize induced power and maximize the figure of merit. The use of blade taper helps to minimize profile power by more optimally distributing the chord (solidity) over the disk. Removing blade area from the tip and redistributing it further inboard on the blade helps the profile power (lowers power consumption at the tip region). Taper also helps to give a more uniform lift coefficient over the blade, helping to ensure all blade elements operate closer to their best L/D ratio, and also helping to delay the onset of stall.

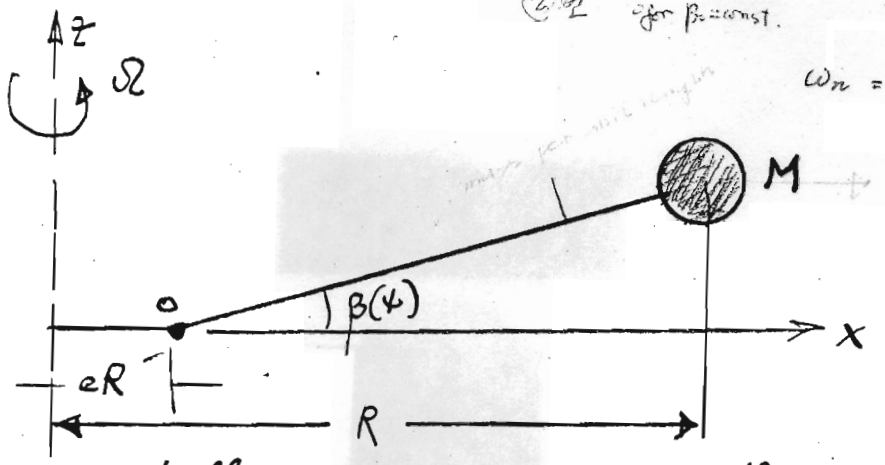
Problem # 3

Part (a)

$$\underbrace{\int_{eR}^{R} L(x-eR)\,dx}_{②\ \text{of}\ L} = I_b\ddot{\beta} + I_b\Omega\beta + M\Omega^2 R(R-eR)\beta + \int_{eR}^{R} m\Omega^2 x(x-eR)\beta\,dx = 0$$

for $\beta = \text{const.}$

$$\omega_n = \frac{\beta\Omega^2(\ )}{I}$$



According to the material provided in the notes there are two principal contributions to the inertia moment about the hinge O, one is associated with inertia loads (proportional to acceleratoon) and the second is due to centrifugal effects. [Note β small]

$$M_{I_0} = M_C + M_I = M\Omega^2 R(R-eR)\sin\beta + \int_{eR}^{R} x\, m\Omega^2(x-eR)\sin\beta\,dx + M(R-eR)^2\ddot{\beta} + \int_{eR}^{R} m(x-eR)^2\ddot{\beta}\,dx \qquad (1)$$

Define $\displaystyle\int_{eR}^{R} m(x-eR)^2\,dx = I_b \qquad (2)$

Combining Eqs (1) and (2)

$$\int_{eR}^{R} m x\Omega^2(x-eR)\beta\,dx + M\Omega^2 R^2(1-e)\beta + I_b\ddot{\beta} + M(1-e)^2R^2\ddot{\beta} = 0 \qquad (3)$$

The first term in Eq (3) can be manipulated

$$\int_{eR}^{R} m\Omega^2(x-eR)(x-eR+eR)\,dx\,\beta = I_1$$

$$I_1 = \Omega^2\beta \int_{eR}^{R} m(x-eR)^2 dx + \Omega^2 \beta eR \int_{eR}^{R} m(x-eR)\,dx$$

$$= \Omega^2 \beta I_b + \Omega^2 \beta eR \int_{eR}^{R} m(x-eR)\,dx \qquad (4)$$

Combining Eqs (3) and (4)

$$I_b \ddot{\beta} + M(1-e)^2 R^2 \ddot{\beta} + \Omega^2 I_b \beta + \Omega^2 \beta eR \int_{eR}^{R} m(x-eR)\,dx$$

$$+ M\Omega^2 R^2 (1-e)\beta = 0$$

$$\left[I_b + M(1-e)^2 R^2\right]\ddot{\beta} + \Omega^2\beta\left[I_b + eR\int_{eR}^{R} m(x-eR)\,dx\right.$$

$$\left. + MR^2(1-e)\right] = 0 \qquad (*)$$

The natural frequency is given by

$$\omega_n^2 = \frac{\Omega^2\left[I_b + eR\int_{eR}^{R} m(x-eR)\,dx + MR^2(1-e)\right]}{I_b + M(1-e)^2R^2}$$

$$\left(\frac{\omega_n}{\Omega}\right)^2 = \frac{I_b + eR\int_{eR}^{R} m(x-eR)\,dx + MR^2(1-e)}{I_b + M(1-e)^2R^2} \qquad (5)$$

The second term can be evaluated for convenience

$$I_2 = eR\int_{eR}^{R} m(x-eR)\,dx = eR\, m\left[\frac{x^2}{2} - eRx\right]\Big|_{eR}^{R}$$

$$= meR\left[\frac{1}{2}(R^2 - e^2R^2) - eR(R-eR)\right]$$

$$= meR\left[\frac{1}{2}R^2(1-e^2) - R^2(1-e)\,e\right]$$

$$= meR\left[\frac{1}{2}R^2(1-e^2) - R^2(e-e^2)\right] = mR^3\left[\frac{e^3}{2} - e^2 + \frac{1}{2}e\right]$$

Part B

The aerodynamic moment is given by (about hinge O)

$$M_{AO} = \int_{eR}^{R} L\,(x-eR)\,dx = \frac{1}{2}\rho a c \Omega^2 \int_{eR}^{R} x^2\left(\theta_0 - \frac{\lambda \Omega R}{\Omega x}\right)(x-eR)\,dx$$

$$= \frac{1}{2}\rho a c \Omega^2 \int_{eR}^{R} (\theta_0 x^2 - \lambda R x)(x-eR)\,dx =$$

$$= \frac{1}{2}\rho a c \Omega^2 \int_{eR}^{R} (\theta_0 x^3 - \lambda R x^2 - \theta_0 e R x^2 + \lambda R e R x)\,dx$$

$$= \frac{1}{2}\rho a c \Omega^2 R^4 \left[\frac{\theta_0}{4}(1-e^4) - \frac{\lambda}{3}(1-e^3) - \frac{\theta_0 e}{3}(1-e^3) + \lambda e (1-e^2)\right]$$

For Equilibrium the aerodynamic moment has to be equal to the centrifugal moment, so that a steady state coning angle given by β_0 is produced. Thus,

$$M_{centrifugal} + M_A = 0 \quad (7)$$

From Eq(*) p.7 and Eq (6) we have

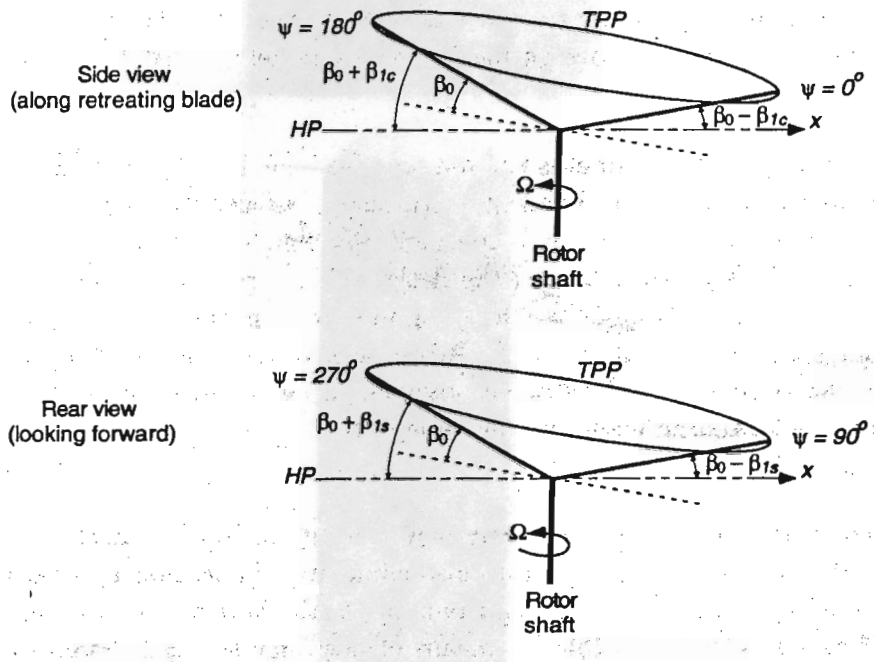
$$\frac{1}{2}\rho a c \Omega^2 R^4 \left[\frac{\theta_0}{4}(1-e^4) - \frac{(1-e^3)}{3}(\lambda + \theta_0 e) + \lambda e (1-e^2)\right] =$$

$$= \Omega^2 \beta_0 \left[I_b + eR\int_{eR}^{R} m(x-eR)\,dx + MR^2(1-e)\right]$$

$$\beta_0 = \frac{\frac{1}{2}\gamma\left[\frac{\theta_0}{4}(1-e^4) - \frac{(1-e^3)}{3}(\lambda+\theta_0 e) + \lambda e(1-e^2)\right]}{\left[1 + \frac{eR}{I_b}\int_{eR}^{R} m(x-eR)\,dx + \frac{MR^2}{I_b}(1-e)\right]} \quad (8)$$

Problem # 4

Part(a) The inclination angles are shown in the figure below



Flapping angles with respect to the hub plane for Question

(b) The coning angle does not affect the inclination of the TPP. Therefore the longitudinal tilt of the TPP is determined by the longitudinal flapping angle β_{1c}. In this case the TPP is tilted back by 4°.

(c) Because in this case the longitudinal and lateral flapping angles are equal, the TPP is also inclined by 4° to the starboard side (retreating blade high)

(d) Based on $\beta(\psi) = 6° - 4° \cos\psi - 4° \sin\psi$ the displacements of the blade are shown on the next page

ψ	β	ψ	β
$0°$	$2°$	$180°$	$10°$
$45°$	$0.343°$	$225°$	$11.657°$
$90°$	$2°$	$270°$	$10°$
$135°$	$6°$	$315°$	$6°$

(e) The maximum and minimum flapping angle can be determined by differentiating the equation for flapping displacement, i.e.

$$\overset{*}{\beta} = -\beta_{1c}\sin\psi + \beta_{1s}\cos\psi \quad (1)$$

where $\overset{*}{(\,)} = \frac{d}{d\psi}$

Maximum and minimum is when Eq (1) $= 0$, which implies

$$\beta_{1c}\sin\psi = \beta_{1s}\cos\psi$$

$$\psi = \tan\left(\frac{\beta_{1c}}{\beta_{1s}}\right) = \tan^{-1}\left(\frac{-4°}{-4°}\right) = 45° \text{ or } 225°$$

Minimum flapping angle is $0.343°$ at $\psi = 45°$ and maximum flapping angle is $11.657°$ at $\psi = 225°$