최적화근사해법
제 7주 과제

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A supplementary lecture note for the book *Approximation algorithms* by V. Vazirani

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12.5 Is the following a theorem : An approximation algorithm designed using an LP-relaxation cannot achieve a better approximation guarantee than the integrality gap of the relaxation. **Hint**: In principle it may be possible to show,, using additional structural properties, that whenever an instance has a bad gap, the cost of the solution found by the algorithm is much less that αOPT, where α is the integrality gap of the relaxation.(Observe that if the instance has a bad gap, the cost of the solution found cannot be much less than αOPT_f

12.6Use the max-flow min-cut theorem to derive Menger's theorem: **Theorem**. Let $G = (V, E)$ be a directed graph with $s, t \in V$. Then, the maximum number of edge-disjoint (vertex-disjoint) s-t paths is equal to the minimum number of edges (vertices) whose removal disconnects s from t.

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12.9

1. Let $G = (V, E)$ be an undirected graph, with weights w_e on edges. The following is an exact LP-relaxation for the problem of finding a maximum weight matching in G.(By $e : e \in S$ we men edges $e$that have both endpoints in S.)

$$\begin{array}{lcl} \max & \sum_e w_e x_e & \\ s.t & \sum_e x_e \leq 1 & e : \ incident \ at \ v \in V \\ & \sum_e x_e \leq \frac{(|S|-1)}{2} & e \in S \subset V, |S| : odd \\ & x_e \geq 0 & e \in E \end{array}$$

Obtain the dual of this LP. If the weight function is integral, the dual is also exact. Observe that Theorem 1.7 follows form these facts.

2. Assume that $|V|$ is even. The following is an exact LP-relaxation for the minimum weight perfect matching problem in G (a matching is perfect if it matches all vertices). Obtain the dual of this LP. Use complementary slackness conditions to give conditions satisfied by a pair of optimal primal (integral) and dual solutions for both formulations.

$$
\begin{array}{lcl}
\min & \Sigma_e w_e x_e & \\
s.t & \Sigma_e x_e = 1 & e: \ incident \ at \ v \in V \\
 & \Sigma_e x_e \leq \frac{(|S|-1)}{2} & e \in S \subset V, |S| : odd \\
 & x_e \geq 0 & e \in E
\end{array}
$$

13.2 Give an example in which the dual solution, price(e), for each element e, computed by Algorithm 2.2 overpacks some sets, S, by a factor of essentially $H_{|S|}$

13.3 Give examples to show that the lower bound used by Algorithm 2.2, y, can be smaller that OPT by a factor of $O(\log n)$

14.1 Modify Algorithm 14.1 so that is picks all sets that are nonzero in the fractional solution. Show that the algorithm also achieves a factor of f.
Hint: Use the primal complementary slackness conditions to prove this.

14.2 consider the collection of sets, C, picked by the randomized rounding algorithm. Show that with some constant probability, C covers at least half the elements at a cost of at most $O(OPT$.

14.4 Give a (non-bipartite) tight example for the half-integrality-based algorithm for weighted vertex cover

14.5 Give a polynomial time algorithm for the following problem. Given a graph G with nonnegative vertex weights and a valid, though not necessarily optimal, coloring of G, find a vertex cover of weight $\leq (2 - \frac{2}{k})OPT$, where k is the number of colors used.

15.1 This extends Exercise 12.4. Show that if x and y are primal and dual feasible solutions and satisfy the conditions of Section 15.2 with $\alpha = 1$ and $\beta = f$, then y pays for x at a rate of f, *i.e.*

$$\sum_S c(S)x_S \leq f \cdot \sum_e y_e$$

15.4 Give a tight example for Algorithm 15.2 in which f is a fixed constant(for the infinite family constructed in Example 15.4, f is unbounded)