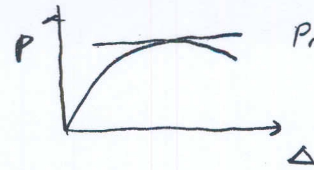
8.1.4 A commentary on stability



Principle of stationary total potential energy

$$\pi = U + V$$

π = total potential energy

U = strain energy

V = changes in potential of the applied load

First theorem

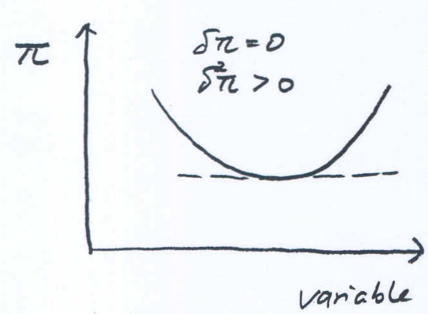
The force-equilibrium that we are seeking can be found under the condition that 1st variation of π becomes zero.

$$\delta\pi = \delta U + \delta V = 0 \quad \Rightarrow \text{ produce equilibrium condition}$$

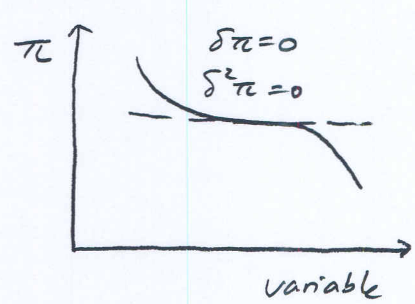
$$\Rightarrow \delta U = -\delta V$$

$$\Rightarrow \delta W_{int} = \delta W_{ext} \quad (= \text{principle of virtual displacement})$$

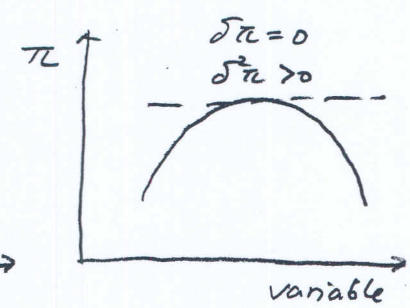
But, characteristic of the equilibrium has not been determined yet.



Stable equilibrium



neutral equilibrium



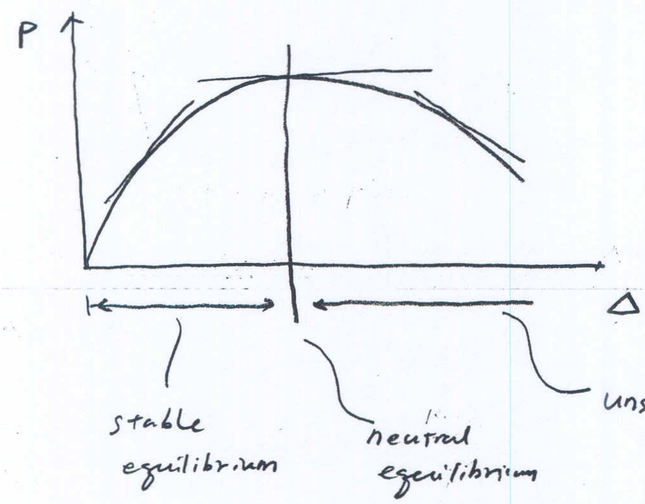
unstable equilibrium

Second theorem

$\delta^2\pi > 0 \Rightarrow$ stable equilibrium

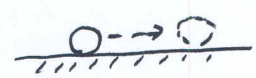
$= 0 \Rightarrow$ neutral equilibrium

$< 0 \Rightarrow$ unstable equilibrium



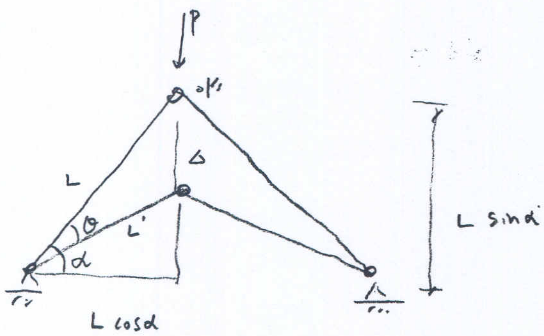
stable equilibrium : with small variations of variables,
the energy level should be increased.

neutral equilibrium : with variations of variables
the energy level is not changed



unstable equilibrium : with small changes of variables
the energy level is quickly dropped

Example 8.8



$$\begin{cases} L' = L\cos\alpha \times \dfrac{1}{\cos(\alpha-\theta)} \\ L - L' = L\left(1 - \dfrac{\cos\alpha}{\cos(\alpha-\theta)}\right) \end{cases}$$

$$\begin{cases} \Delta = L\sin\alpha - L'\sin(\alpha-\theta) \\ \quad = L\sin\alpha - L\cos\alpha\tan(\alpha-\theta) \end{cases}$$

$$\pi = U + V$$

$$= 2\cdot\left(\tfrac{1}{2}k(\Delta L_{ab}^2)\right) - P\Delta \qquad k = \frac{EA}{L}$$

$$= kL^2\left[1 - \frac{\cos\alpha}{\cos(\alpha-\theta)}\right]^2 - PL\left[\sin\alpha - \cos\alpha\tan(\alpha-\theta)\right]$$

$$\delta\pi = \frac{d\pi}{d\theta} = 0$$

$$\Rightarrow P = 2kL\left[\sin(\alpha-\theta) - \cos\alpha\tan(\alpha-\theta)\right]$$

$$\delta^2\pi = \frac{dP}{d\theta} = 0$$

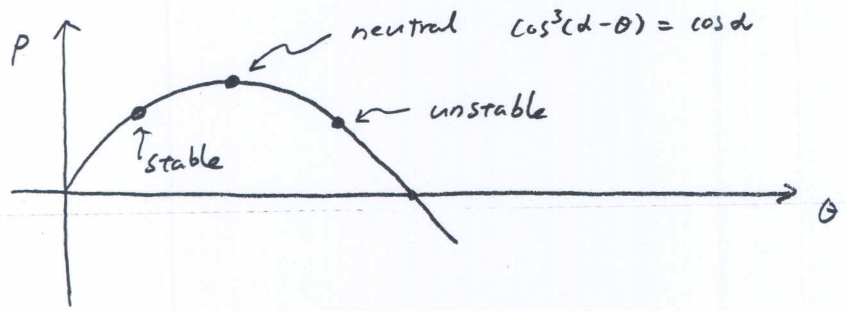
$$\Rightarrow \delta^2\pi = \frac{2kL^2\left[\cos\alpha - \cos^3(\alpha-\theta)\right]}{\cos^4(\alpha-\theta)}$$

for $|\alpha-\theta| < \pi/2$

$$\delta^2\pi = \frac{d^2\pi}{d\theta^2} > 0 \quad \text{for} \quad \cos^3(\alpha-\theta) < \cos\alpha \quad : \text{stable}$$

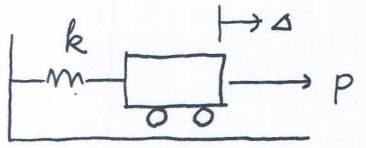
$$\delta^2\pi = \frac{d^2\pi}{d\theta^2} = 0 \quad \text{for} \quad \cos^3(\alpha-\theta) = \cos\alpha \quad : \text{neutral}$$

$$\delta^2\pi = \frac{d^2\pi}{d\theta^2} < 0 \quad \text{for} \quad \cos^3(\alpha-\theta) > \cos\alpha \quad : \text{unstable}$$



Conditions of tangent stiffness ($\underset{\sim}{k}$) regarding to equilibrium

one-degree of freedom system



the principle of stationary total potential energy is assumed to be applicable to incremental force-displacement relationship

$$\pi = \frac{1}{2} k_t \Delta_t^2 - P_t \Delta_t$$

P_t = incremental force

Δ_t = incremental displacement

$$\delta\pi = k_t \Delta_t \delta\Delta_t - P_t \delta\Delta_t = 0$$

$$\Rightarrow \quad k_t \Delta_t = P_t \quad : \text{equilibrium equation}$$

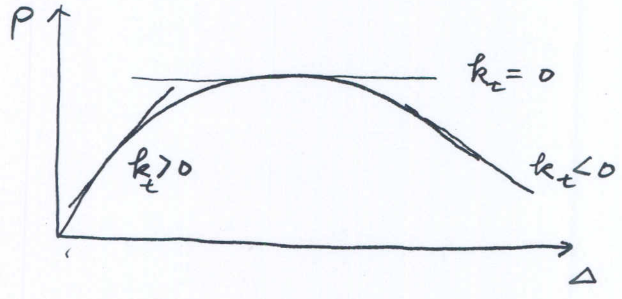
assuming k_t (tangent stiffness) is constant.

$$\delta\pi^2 = k_t \delta\Delta_t \delta\Delta_t$$

$\delta\pi^2 > 0 \Rightarrow k_t > 0$ stable equilibrium

$= 0 \Rightarrow k_t = 0$ neutral equilibrium

$< 0 \Rightarrow k_t < 0$ unstable equilibrium



multi-degree of freedom system

$$\pi = \frac{1}{2} \underset{\sim}{\Delta}_t^T \underset{\sim}{k}_t \underset{\sim}{\Delta}_t - \underset{\sim}{\Delta}_t^T \underset{\sim}{P}_t$$

$$\delta\pi = \delta\underset{\sim}{\Delta}_t^T \underset{\sim}{k}_t \underset{\sim}{\Delta}_t - \delta\underset{\sim}{\Delta}_t^T \underset{\sim}{P}_t$$

$\delta\pi = 0 \Rightarrow \underset{\sim}{k}_t \underset{\sim}{\Delta}_t = \underset{\sim}{P}_t$: equilibrium equation

assuming $\underset{\sim}{k}_t$ = constant

$$\delta^2\pi = \delta\underset{\sim}{\Delta}_t^T \underset{\sim}{k}_t \delta\underset{\sim}{\Delta}_t$$

$\delta^2\pi > 0 \Rightarrow \delta\underset{\sim}{\Delta}_t^T \underset{\sim}{k}_t \delta\underset{\sim}{\Delta}_t > 0$ stable equilibrium

$\underset{\sim}{k}$ = symmetric . positive definite

$\delta^2\pi = 0 \Rightarrow \delta\underset{\sim}{\Delta}_t^T \underset{\sim}{k}_t \delta\underset{\sim}{\Delta}_t = 0$ neutral equilibrium

$\underset{\sim}{k}_t \delta\underset{\sim}{\Delta}_t = \underset{\sim}{0} \Rightarrow |\underset{\sim}{k}_t| = 0$ for any $\delta\underset{\sim}{\Delta}$

eigenvalue problem

$\delta^2\pi < 0 \Rightarrow \delta\underset{\sim}{\Delta}_t^T \underset{\sim}{k}_t \delta\underset{\sim}{\Delta}_t < 0$ unstable equilibrium

When the geometry is not significantly changed,

$\underset{\sim}{k}_t \approx \underset{\sim}{k}$, tangent stiffness = secant stiffness

$\underset{\sim}{k}\underset{\sim}{\Delta} = \underset{\sim}{P}$

$|\underset{\sim}{k}| = 0 \Rightarrow$ critical load analysis