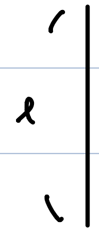
Prob 1.  & charge Q. $\rightarrow$ $\rho_\ell := Q/\ell$.

Assume) enough long wire for z-dir symmetry. / $n_{i0} \sim n_{e0} \sim n_0$

Adiabatic response for electrons.

수업 때 크게 다뤄지지 않았는데 헷갈려 하시는 분들이 있어서...

왜 Boltzmann Response를 electron에 대해서만 적용하는지

간단히 소개 드립니다.

Intuitively 전자가 빨라서 가능하다고 이해하면 됩니다. 이온은 거의 fix

electrons mtm eqns

$$m_e n_e \frac{d}{dt} \vec{u}_e = - n_e e \left(\vec{E} + \vec{u} \times \vec{B} \right) - \vec{\nabla} p_e$$

linearizing with $\vec{u}_0 \sim 0$

$$m_e n_{e,0} \partial_t \delta \vec{u}_e = - n_{e,0} e \left(\delta \vec{E} + \delta \vec{u} \times \vec{B}_0 \right) - T_e \vec{\nabla} \delta n_e$$

Take $\cdot \hat{b}$ and ignore electrons Inertia (Invalid for ions)

$$\rightarrow \quad n_e e \vec{\nabla}_{\parallel} \delta\phi - T_e \vec{\nabla}_{\parallel} \delta n_e = 0$$

$$\therefore \quad \delta n_e / n_{e,0} = e \, \delta\phi / T_e$$

Poisson's equation

$$\varepsilon_0 \vec{\nabla}^2 \phi = e \left(n_i - n_e \right)$$

$$= e \left(n_0 - n_0 \exp\left(-\frac{e\phi}{T_e} \right) \right)$$

$$\sim \frac{e^2 n_0}{T_e} \phi$$

$$\rightarrow \quad \vec{\nabla}^2 \phi - \frac{1}{\lambda_D^2} \phi = 0 \qquad \text{where } \lambda_D^2 = \frac{\varepsilon_0 T_e}{n_0 e^2}$$

$$\rightarrow \quad \frac{1}{r} \partial_r r \partial_r \phi - \frac{1}{\lambda_D^2} \phi = 0$$

$$\rightarrow \quad \frac{1}{x} \partial_x x \partial_x \phi - \phi = 0 \qquad \text{where } x = r/\lambda_D$$

with $\alpha = 0$.

$x^2 \ddot{\phi} + x\dot{\phi} - x^2\phi = 0$: the modified Bessel's equation.

$\phi(x) = C_1 I_0(x) + C_2 K_0(x)$ where I_0 is 1st kind of ~

K_0 " 2nd "

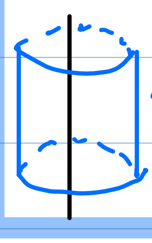
Far from the wire

→ No electric force !?

Boundary Condition.

① $\partial_r \phi \big|_{r\to\infty} = 0$

② $2\pi r E_r \big|_{r\to 0} = \frac{1}{\varepsilon_0} \rho_\ell$



Gauss' law

→ $2\pi r l E_r = \frac{1}{\varepsilon_0} \rho_\ell \cdot l$

Limiting forms of $I_0(x)$ & $K_0(x)$

$0 < \lvert x\rvert \ll 1$	$\lvert x\rvert \gg 1$
$I_0(x) \to 1$	$I_0(x) \to \frac{1}{\sqrt{2\pi x}} \exp(x)$
$K_0(x) \to -\ln\frac{x}{2} - \gamma$	$K_0(x) \to \sqrt{\frac{\pi}{2x}} \exp(-x)$

From B. C ①, $C_1 = 0$

From B. C. ②, $C_2 = \frac{\rho_\ell}{2\pi\varepsilon_0}$

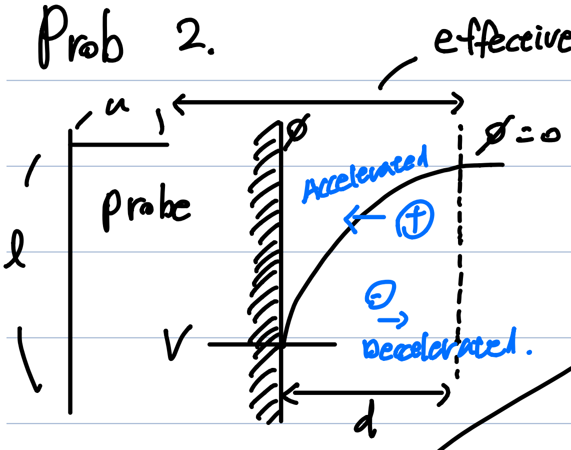
$$\therefore \quad \phi(r) = \frac{Q}{2\pi l \varepsilon_0} K_0\left(r/\lambda_0\right)$$

Note that $K_0(z) \sim \sqrt{\frac{\pi}{2z}}\, e^{-z} \left(1 - \frac{1}{8z} + \frac{a}{2!(8z)^2} + \cdots\right)$.

asymptotic large argument expansions.

e^{-z} factor makes ϕ exponentially small after a few λ_0.

Prob 2.



$\because$ Ions have const J.

Ion saturation current

$$J_{is} = 0.61\, e n_\infty \left(T_e/m\right)^{1/2}$$

Child-Langmuir space limited current

$$J_0 = 4/9 \left(\frac{2e}{m}\right)^{1/2} \frac{\varepsilon_0 |\phi_0|^{3/2}}{d^2}$$

$$d = \left(\frac{1}{0.61\, e n_\infty} \cdot \left(\frac{m}{T_e}\right)^{1/2} \cdot \frac{4}{9} \cdot \left(\frac{2e}{m}\right)^{1/2} \cdot \varepsilon_0 |\phi_0|^{3/2}\right)^{1/2}$$

$$I = \underbrace{A_p}_{\text{probe surface}} J_e - \underbrace{A_s}_{\text{sheath surface}} J_i$$

[J_e : Considered near probe boundary

J_i : considered near sheath boundary] due to negative V ??

$$= 2\pi a l \left(T_e/m_i\right)^{1/2} \left(\frac{1}{2\sqrt{\pi}} \left(\frac{2m_i}{m_e}\right)^{1/2} \exp(eV/T_e) - \frac{A_s}{A_p} \exp\left(-\frac{1}{2}\right)\right)$$

$$= 2\pi a l \left(T_e/m_i\right)^{1/2} \Bigg[\frac{1}{2\sqrt{\pi}} \left(\frac{2m_i}{m_e}\right)^{1/2} \exp(eV/T_e)$$

$$- \left(1 + \frac{1}{a}\left(\frac{1}{0.61\, e n_\infty} \left(\frac{m}{T_e}\right)^{1/2} \cdot \frac{4}{9} \cdot \left(\frac{2e}{m}\right)^{1/2} \varepsilon_0 |V|^{3/2}\right)^{1/2}\right) \exp\left(-\frac{1}{2}\right) \Bigg]$$

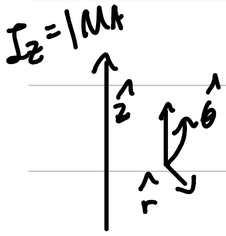
$$I_{is} = \lim_{\frac{eV}{T_e} \to -\infty} I$$

$$\sim -2\pi a l \left(\frac{T_e}{m_i}\right)^{1/2} \exp\left(-\frac{1}{2}\right) \left(1 + \frac{1}{a}\left(\frac{1}{0.61\, e n_\infty} \left(\frac{m}{T_e}\right)^{1/2} \frac{4}{9} \left(\frac{2e}{m}\right)^{1/2} \varepsilon_0 |V|^{3/2}\right)^{1/2}\right)$$

The reason $I_{is} = I_{is}(V)$ //

Prob 3.

a) $$\vec{V}_{\nabla B} = \frac{1}{qB}\frac{\hat{R}_c\times\hat{b}}{R_c}\,W_\perp$$



$$\vec{V}_{curv} = \frac{1}{qB}\frac{\hat{R}_c\times\hat{b}}{R_c}\,2W_\parallel$$

① $$\langle\vec{V}_{\nabla B}\rangle = \frac{1}{qB}\frac{\hat{R}_c\times\hat{b}}{R_c}\langle W_\perp\rangle$$

$$= \frac{1}{qB}\frac{\hat{R}_c\times\hat{b}}{R_c}\int\left(\frac{m}{2\pi T_\perp}\right)\left(\frac{m}{2\pi T_\parallel}\right)^{1/2}\exp\left(-\frac{mV_\perp^2}{2T_\perp}-\frac{mV_\parallel^2}{2T_\parallel}\right)\frac{1}{2}mV_\perp^2\,\underbrace{d^3V}_{2\pi V_\perp dV_\perp dV_\parallel}$$

$$= \frac{1}{qB}\frac{\hat{R}_c\times\hat{b}}{R_c}\underbrace{\left[\int_0^\infty\frac{m}{2\pi T_\perp}\cdot\exp\left(-\frac{mV_\perp^2}{2T_\perp}\right)\frac{1}{2}mV_\perp^2\,2\pi V_\perp\,dV_\perp\right]}_{①-①}$$

$$\times\underbrace{\left[\int_{-\infty}^{\infty}\left(\frac{m}{2\pi T_\parallel}\right)^{1/2}\exp\left(-\frac{mV_\parallel^2}{2T_\parallel}\right)dV_\parallel\right]}_{①-②}$$

①-①: $$\int_0^\infty\frac{m}{2\pi T_\perp}\cdot\exp\left(-\frac{mV_\perp^2}{2T_\perp}\right)\cdot\frac{1}{2}mV_\perp^2\cdot 2\pi V_\perp\,dV_\perp$$

$$= T_\perp\int_0^\infty\exp(-x)\,x\,dx \qquad \text{where} \quad x = \frac{mV_\perp^2}{2T_\perp}$$

$$= T_\perp$$

$$A := \int_{-\infty}^{\infty}\exp(-x^2)\,dx$$

$$A^2 = \int_{-\infty}^{\infty}\exp(-x^2)\,dx\int_{-\infty}^{\infty}\exp(-y^2)\,dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp(-(x^2+y^2))\,dx\,dy$$

$$= \int_0^\infty\exp(-r^2)\,2\pi r\,dr = \pi$$

①-②: $$\int_{-\infty}^{\infty}\left(\frac{m}{2\pi T_\parallel}\right)^{1/2}\exp\left(-\frac{mV_\parallel^2}{2T_\parallel}\right)dV_\parallel$$

$$= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\exp(-x^2)\,dx \qquad \text{where} \quad x = \sqrt{\frac{m}{2T_\parallel}}V_\parallel \quad \& \quad \underline{\int_{-\infty}^{\infty}\exp(-x^2)\,dx = \sqrt{\pi}}$$

$$= 1$$

$$\therefore \langle \vec{V}_{\nabla B} \rangle = \frac{T_\perp}{qB} \frac{\hat{R}_c \times \hat{b}}{R_c}$$

$$② \quad \langle \vec{V}_{curv} \rangle = \frac{1}{qB} \frac{\hat{R}_c \times \hat{b}}{R_c} \langle 2W_\parallel \rangle$$

$$= \frac{1}{qB} \frac{\hat{R}_c \times \hat{b}}{R_c} \int \left(\frac{m}{2\pi T_\perp}\right)\left(\frac{m}{2\pi T_\parallel}\right)^{1/2} \exp\left(-\frac{mV_\perp^2}{2T_\perp}\right) \exp\left(-\frac{mV_\parallel^2}{2T_\parallel}\right) mV_\parallel^2 \underbrace{d^3V}_{2\pi V_\perp dV_\perp dV_\parallel}$$

$$= \frac{1}{qB} \frac{\hat{R}_c \times \hat{b}}{R_c} \underbrace{\left[\int_0^\infty \frac{m}{2\pi T_\perp} \exp\left(-\frac{mV_\perp^2}{2T_\perp}\right) 2\pi V_\perp dV_\perp\right]}_{②-①}$$

$$\times \underbrace{\left[\int_{-\infty}^{\infty} \left(\frac{m}{2\pi T_\parallel}\right)^{1/2} \exp\left(-\frac{mV_\parallel^2}{2T_\parallel}\right) mV_\parallel^2 dV_\parallel\right]}_{②-②}$$

②-①: $\displaystyle\int_0^\infty \frac{m}{2\pi T_\perp} \exp\left(-\frac{mV_\perp^2}{2T_\perp}\right) 2\pi V_\perp dV_\perp$

$$= \int_0^\infty \exp(-x)\, dx = 1 \qquad \text{where} \quad x = \frac{mV_\perp^2}{2T_\perp}$$

②-②: $\displaystyle\int_{-\infty}^{\infty} \left(\frac{m}{2\pi T_\parallel}\right)^{1/2} \exp\left(-\frac{mV_\parallel^2}{2T_\parallel}\right) mV_\parallel^2 dV_\parallel$

$$= 2\frac{T_\parallel}{\sqrt{\pi}} \underline{\int_{-\infty}^{\infty} \exp(-x^2)\, x^2\, dx}$$

$$= T_\parallel$$

$$\int_{-\infty}^{\infty} \exp(-\alpha x^2)\, dx = \frac{\sqrt{\pi}}{\sqrt{\alpha}}$$

Take a derivative w.r.t α.

$$\int_{-\infty}^{\infty} \exp(-\alpha x^2)\, x^2\, dx = \frac{1}{2}\frac{\sqrt{\pi}}{(\alpha)^{3/2}}$$

$$\therefore \int_{-\infty}^{\infty} \exp(-x^2)\, x^2\, dx = \frac{1}{2}\sqrt{\pi}$$

$$\therefore \langle \vec{V}_{curv} \rangle = \frac{T_\parallel}{qB} \frac{\hat{R}_c \times \hat{b}}{R_c}$$

$$\frac{1}{qB}\frac{\hat{R}_c \times \hat{b}}{R_c} = \frac{1}{q}\frac{2\pi}{\mu_0 I_z}\hat{z}$$

$$\langle \vec{V}_{\vec{\nabla} B} \rangle_i = \frac{2\pi T_{\parallel}}{e\mu_0 I_z}\hat{z} = \frac{2\pi \cdot 5 \cdot 10^3}{4\pi \cdot 10^{-7} \cdot 10^6} = 25{,}000\,\hat{z}\ [m/s]$$

$$\langle \vec{V}_{curv} \rangle_i = \frac{2\pi T_{\perp}}{e\mu_0 I_z}\hat{z} = 10{,}000\,\hat{z}\ [m/s]$$

$$\langle \vec{V}_{\vec{\nabla} B} \rangle_e = -\frac{2\pi T_{\parallel}}{e\mu_0 I_z}\hat{z} = -25{,}000\,\hat{z}\ [m/s]$$

$$\langle \vec{V}_{curv} \rangle_e = -\frac{2\pi T_{\perp}}{e\mu_0 I_z}\hat{z} = -10{,}000\,\hat{z}\ [m/s]$$

b) $$\vec{J} = en\left(\langle \vec{V}_{\vec{\nabla} B} + \vec{V}_{curv} \rangle_i - \langle \vec{V}_{\vec{\nabla} B} + \vec{V}_{curv} \rangle_e\right)$$

$$= 1.6 \cdot 70{,}000 = 112{,}000\,\hat{z}\ [A/m^2]$$

Prob 4.

system 's frequency

a) $\underbrace{\omega_c \gg \omega}_{①} \gg 1/\tau_{eq}$
② $\omega \gg 1/\tau_{eq}$

Typically $\omega_c \gg 1/\tau_{eq}$ 는 아주 잘 맞는 가정이고, 궁금하신 분들은 Goldston ch 11 참조.

가정 ① : Adiabatic Invariant 가 Valid 하기 위한 조건

②: Parallel & Perpendicular energy 가 충분히 교환되지 않을 조건.

③: B and T are constant in space.

④ : No charge accumulation in space & External $\vec{E}$.

$$\frac{d}{dt}\mu = 0 \rightarrow \mu = \frac{W_\perp}{B} = \text{const} \rightarrow \text{Double } B$$

$\rightarrow$ Double $W_\perp$.

$\rightarrow$ $T_{\perp,1} = 2T_0$

Possible Questions.

Q. Magnetic Field cannot work.

Then why does $T_\perp$ increase?

A. By the induced $\vec{E}$.

$(\partial_t \vec{B} = - \vec{\nabla} \times \vec{E})$

$$m_s \frac{dV_\parallel}{dt} = \cancel{q_s E_\parallel} - \mu \cancel{\vec{\nabla}_\parallel B} \rightarrow \frac{dV_\parallel}{dt} = 0 \rightarrow T_{\parallel,1} = T_0$$

$\because$ No charge accumulation. $\quad \because \nabla_\parallel B = 0$

Energy 보존 때는 가정은 3번과 유사하니 참조.

b) $\frac{1}{2}T_{\parallel,1} + T_{\perp,1} = \frac{3}{2}T_1 \rightarrow T_1 = \frac{5}{3}T_0$

c) After same process with a), $T_{\parallel,2} = \frac{5}{3}T_0$, $T_{\perp,2} = \frac{5}{6}T_0$

d) " " " " c), $T_2 = \frac{10}{9}T_0$

Prob 5.

$$\varepsilon^2 x^8 - \varepsilon x^6 + x - 2 = 0$$

i) $x \sim O(1)$

$\begin{cases} \varepsilon^2 x^8 \to 0 \\ \varepsilon x^6 \to 0 \end{cases} \Rightarrow \underline{x \sim 2}$ // $\quad 1/8$

ii) $x \sim O\left(\sqrt{1/\varepsilon}\right) \gg 1$

$\to \quad \varepsilon^2 x^3 - \varepsilon x + \frac{1}{x^4} - \frac{2}{x^5} = 0$

$\sim \varepsilon^2 x^3 - \varepsilon x = 0 \quad \Rightarrow \quad \underline{x \sim \pm\sqrt{1/\varepsilon}}$ // $\quad 2/8$

iii) $x \sim O\left(\sqrt[5]{1/\varepsilon}\right) \gg 1$

$\to \quad \varepsilon^2 x^7 - \varepsilon x^5 + 1 - \frac{2}{x} = 0$

$\sim \quad -\varepsilon x^5 + 1 = 0 \quad \Rightarrow \quad \underline{x \sim e^{i\frac{2n\pi}{5}} \left(\frac{1}{\varepsilon}\right)^{1/5}}$ // $\quad 5/8.$

$\therefore$ <u>Total 8 solutions mean all solutions found</u>