1. a) <u>방법 1</u>

$$m\frac{d\vec{v}_\perp}{dt} = q\left(\vec{E}_\perp + \vec{v}_\perp \times \vec{B}\right)$$

$$m\frac{d^2\vec{v}_\perp}{dt^2} = q\left(\frac{d\vec{v}_\perp}{dt}\times\vec{B} + \vec{v}_\perp \times \frac{d\vec{B}}{dt}\right)$$

$$= \frac{q^2}{m}\left[\vec{E} + \vec{v}_\perp\times\vec{B}\right]\times\vec{B} + q\,\vec{v}_\perp\times\frac{d\vec{B}}{dt}$$

$$\rightarrow \frac{d^2\vec{v}_\perp}{dt^2} - \frac{q}{m}\vec{v}_\perp\times\frac{d\vec{B}}{dt} + \omega_c^2\,\vec{v}_\perp = \frac{q^2}{m^2}\vec{E}\times\vec{B}$$

$$\rightarrow \text{let } \vec{v}_\perp = \begin{pmatrix} v_x \\ v_y \end{pmatrix}. \qquad \ddot{\vec{v}}_\perp + \begin{pmatrix} \omega_c^2 & -\frac{q}{|q|}\omega_c\alpha \\ +\frac{q}{|q|}\omega_c\alpha & \omega_c^2 \end{pmatrix}\vec{v}_\perp = \omega_c^2\,\vec{v}_E$$

i) Fundamental Solution →

정답과 무관하지만 큰 Insight를 주는 부분입니다.
Lorentz Equation의 Homogeneous part는
Gyro-motion을 나타내고 Particular part는
Drift-motion을 나타냅니다. / <u>스킵해도 무관</u>

$$A := \begin{pmatrix} \omega_c^2 & -\frac{q}{|q|}\omega_c\alpha \\ \frac{q}{|q|}\omega_c\alpha & \omega_c^2 \end{pmatrix}$$

$$\det\left(A - \lambda I\right) = 0 \rightarrow \lambda = \omega_c^2 \pm i\frac{q}{|q|}\omega_c\alpha$$

<u>Case I</u>

$$\lambda_1 = \omega_c^2 + i\frac{q}{|q|}\omega_c\alpha \rightarrow A - \lambda_1 I = \begin{pmatrix} -i\frac{q}{|q|}\omega_c\alpha & -\frac{q}{|q|}\omega_c\alpha \\ \frac{q}{|q|}\omega_c\alpha & -i\frac{q}{|q|}\omega_c\alpha \end{pmatrix} \rightarrow v_1 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

<u>Case II</u>

$$\lambda_2 = \omega_c^2 - i\frac{q}{|q|}\omega_c\alpha \rightarrow A - \lambda_2 I = \begin{pmatrix} i\frac{q}{|q|}\omega_c\alpha & -\frac{q}{|q|}\omega_c\alpha \\ \frac{q}{|q|}\omega_c\alpha & i\frac{q}{|q|}\omega_c\alpha \end{pmatrix} \rightarrow v_2 = \begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$\rightarrow \vec{v}_{\perp,h} = \mathcal{Re}\left[\begin{pmatrix} 1 \\ -i \end{pmatrix}\left(c_1\exp(i\sqrt{\lambda_1}t) + c_2\exp(-i\sqrt{\lambda_1}t)\right) + \begin{pmatrix} 1 \\ i \end{pmatrix}\left(c_3\exp(i\sqrt{\lambda_2}t) + c_4\exp(-i\sqrt{\lambda_2}t)\right)\right]$$

$$\sqrt{\lambda_1} = w_c \sqrt{1 + i \frac{q}{|q|} \frac{\alpha}{w_c}} \sim w_c + i \frac{1}{2} \frac{q}{|q|} \alpha$$

$$\sqrt{\lambda_2} = w_c \sqrt{1 - i \frac{q}{|q|} \frac{\alpha}{w_c}} \sim w_c - i \frac{1}{2} \frac{q}{|q|} \alpha$$

$$\vec{V}_{\perp,h} = \mathrm{Re}\Bigg[\begin{pmatrix} 1 \\ -i \end{pmatrix} \exp\left(-\frac{1}{2} \frac{q}{|q|} \alpha t\right) \left(c_1 \cos \sqrt{w_c}\, t + c_2 \sin \sqrt{w_c}\, t\right)$$

$$+ \begin{pmatrix} 1 \\ i \end{pmatrix} \exp\left(\frac{1}{2} \frac{q}{|q|} \alpha t\right) \left(c_3 \cos \sqrt{w_c}\, t + c_4 \sin \sqrt{w_c}\, t\right) \Bigg]$$

In here, only $\exp\left(\frac{1}{2}\alpha t\right)$ part is physially valid.

If $\alpha > 0$, adiabatic invariant increases perpendicular energy. In this sense, $\exp\left(-\frac{1}{2}\alpha t\right)$ is invalid.

Thus, $\underline{\vec{V}_{\perp,h} \sim \exp\left(\frac{1}{2}\alpha t\right) \vec{V}_{cyc,0}}$

Corresponding to Gyro-motion

(Not the answer)

ii) Particular solution.

$$\ddot{\vec{V}}_{\perp,p} + \begin{pmatrix} w_c^2 & -\frac{q}{|q|} w_c \alpha \\ \frac{q}{|q|} w_c \alpha & w_c^2 \end{pmatrix} \vec{V}_{\perp,p} = w_c^2 \vec{V}_E$$

Try to substitute $\vec{V}_{\perp,p} = \begin{pmatrix} w_c^2 & -\frac{q}{|q|} w_c \alpha \\ \frac{q}{|q|} w_c \alpha & w_c^2 \end{pmatrix}^{-1} w_c^2 \vec{V}_E$.

$\ddot{\vec{V}}_E = 0$ and it's satisfied well.

$$\vec{V}_{\perp,p} = \frac{1}{w_c^2 + \alpha^2} \begin{pmatrix} w_c^2 & \frac{q}{|q|} w_c \alpha \\ -\frac{q}{|q|} w_c \alpha & w_c^2 \end{pmatrix} \vec{V}_E$$

3-D Extension

$$\vec{V}_{\perp,p} = \frac{w_c^2}{w_c^2 + \alpha^2} \vec{V}_E + \frac{q}{|q|} \frac{w_c \alpha}{w_c^2 + \alpha^2} \vec{V}_E \times \hat{b}$$

$$= \frac{w_c^2}{w_c^2 + \alpha^2} \vec{V}_E - \frac{q}{|q|} \frac{w_c \alpha}{w_c^2 + \alpha^2} \frac{1}{B} \vec{E}_\perp$$

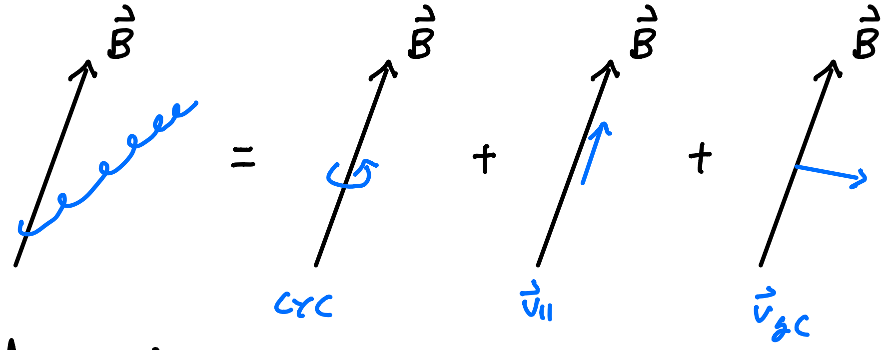
ExB drift

Drift by time-varying $\vec{B}$

1. a) 방법 2

$$m\frac{d\vec{v}}{dt} = q\left(\vec{E} + \vec{v}\times\vec{B}\right)$$

$$\underline{\vec{v} = \vec{v}_{cyc} + \vec{v}_{\parallel} + \vec{v}_{gc}}$$



Here, Our interest is drift motion.

$$\rightarrow \underbrace{m\frac{d}{dt}\vec{v}_{gc}}_{①} = q\left(\vec{E}_{\perp} + \underbrace{\vec{v}_{gc}\times\vec{B}}_{②}\right)$$

Check the order of terms ①, ②. system's time varying frequency $\sim \alpha$

$$\frac{\left|m\frac{d}{dt}\vec{v}_{gc}\right|}{\left|q(\vec{v}_{gc}\times\vec{B})\right|} \sim \frac{m}{qB}w \sim \frac{w}{w_c} \sim \frac{\alpha}{w_c} \ll 1$$

This justifies small $\frac{\alpha}{w_c}$ expansion in $\vec{v}_{gc} = \vec{v}_{gc}^{(0)} + \vec{v}_{gc}^{(1)} + \cdots$.

$$m\frac{d}{dt}\left[\vec{v}_{gc}^{(0)} + \vec{v}_{gc}^{(1)} + \cdots\right] = q\left(\vec{E}_{\perp} + \left[\vec{v}_{gc}^{(0)} + \vec{v}_{gc}^{(1)} + \cdots\right]\times\vec{B}\right)$$

0-th order $\sim \mathcal{O}(1)$

$$0 = \vec{E}_{\perp} + \vec{v}_{gc}^{(0)}\times\vec{B} \Rightarrow \vec{v}_{gc}^{(0)} = \frac{\vec{E}\times\vec{B}}{B^2} = \vec{v}_E$$

1-st order $\sim \mathcal{O}(\alpha/w_c)$

$$\frac{d}{dt}\vec{v}_{gc}^{(0)} = \frac{|q|}{q}\frac{w_c}{B_0}\vec{v}_{gc}^{(1)}\times\vec{B}$$

Take a $\times\vec{B}$

$$\frac{d}{dt}\vec{v}_{gc}^{(0)}\times\vec{B} = \frac{|q|}{q}\frac{w_c}{B_0}\left(\vec{v}_{gc}^{(1)}\times\vec{B}\right)\times\vec{B}$$

$$LHS = \left[\left(\frac{d}{dt} + \cancel{\vec{v}_{gc}^{(0)} \cdot \vec{\nabla}}\right) \frac{\vec{E}\times\vec{B}}{B^2}\right] \times \vec{B}$$

$$= \frac{d}{dt}\frac{1}{B}\left(\vec{E}\times\hat{b}\right)\times\vec{B}$$

$$= -\frac{dB_0}{B^2}\left(\vec{E}\times\hat{b}\right)\times\vec{B}$$

$$= \frac{dB_0}{B}\vec{E}_\perp$$

Note

plasma physics 과제에서 $(\vec{A}\times\hat{b})\times\hat{b}$ 형태의 계산은 굉장히 빈번하게 나오니 잘 숙지해두면 많은 도움이 됩니다. $(\vec{A}\times\hat{b})\times\hat{b} = -\vec{A}_\perp$

$$RHS = -\frac{|q|}{q}\frac{\omega_c}{B_0}B^2\,\vec{v}_{gc}^{(1)}$$

$$\Rightarrow \vec{v}_{gc}^{(1)} = -\frac{|q|}{q}\frac{\alpha}{\omega_c}\frac{B_0^2}{B^3}\vec{E}_\perp$$

$$\therefore \vec{v}_{gc} = \frac{\vec{E}\times\vec{B}}{B^2} - \frac{|q|}{q}\frac{\alpha}{\omega_c}\frac{B_0^2}{B^3}\vec{E}_\perp$$

1\. (a) 방법 3.

수업 때 했던 Asymptotic Expansion Method는

(b) 에서 물어보고 있으니 생략하겠습니다.

1. (b)

$$\vec{V}_{\text{방법1}} = \frac{\omega_c^2}{\omega_c^2+\alpha^2}\vec{V}_E - \frac{q}{|q|}\frac{\omega_c\alpha}{\omega_c^2+\alpha^2}\frac{1}{B}\vec{E}_\perp$$

$$= \frac{1}{1+\left(\frac{\alpha}{\omega_c}\right)^2}\frac{\vec{E}\times\hat{b}}{B_0(1+\alpha t)} - \frac{q}{|q|}\frac{\frac{\alpha}{\omega_c}}{1+\frac{\alpha^2}{\omega_c^2}}\frac{\vec{E}_\perp}{B_0(1+\alpha t)}$$

$$\rightarrow \frac{\vec{E}\times\hat{b}}{B_0} - \frac{q}{|q|}\frac{\alpha}{\omega_c}\frac{\vec{E}_\perp}{B_0} = \vec{V}_{E,0} - \frac{q}{|q|}\frac{\alpha}{\omega_c}\frac{\vec{E}_\perp}{B_0}$$

Gyro motion이 계속 ω_c로 지속되기 위해서는 $\alpha t \ll 1$ 이라는 가정이 추가적으로 필요함

$$\vec{V}_{\text{방법2}} = \frac{\vec{E}\times\vec{B}}{B^2} - \frac{q}{|q|}\frac{\alpha}{\omega_c}\frac{B_0^2}{B^3}\vec{E}_\perp$$

$$\rightarrow \vec{V}_{E,0} - \frac{q}{|q|}\frac{\alpha}{\omega_c}\frac{\vec{E}_\perp}{B_0}$$

$$\therefore \vec{V}_{gc} \rightarrow \vec{V}_{E,0} - \frac{q}{|q|}\frac{\alpha}{\omega_c}\frac{\vec{E}_\perp}{B_0} \qquad \text{where } \vec{V}_{E,0} = \frac{\vec{E}\times\vec{B}_0}{B_0^2}$$

Check $\quad \frac{m}{qB^2}\left(\vec{B}\times\frac{d\vec{V}_0}{dt}\right) \rightarrow \vec{V}_{E,0} - \frac{q}{|q|}\frac{\alpha}{\omega_c}\frac{\vec{E}_\perp}{B_0}$

$$\frac{m}{qB^2}\vec{B}\times\left(\frac{d}{dt} + (\vec{V}_\parallel + \vec{V}_E)\cdot\cancel{\nabla}\right)(\vec{V}_\parallel + \vec{V}_E)$$

$$= \frac{q}{|q|}\frac{1}{\omega_c}\frac{B_0}{B^2}\vec{B}\times\underline{\frac{d}{dt}\vec{V}_E}$$

$$\frac{d}{dt}\frac{\vec{E}\times\hat{b}}{B} = -\frac{B_0\alpha}{B^2}\vec{E}\times\hat{b}$$

$$= \frac{q}{|q|}\frac{1}{\omega_c}\frac{B_0}{B^2}\vec{B}\times\left(-\frac{B_0\alpha}{B^2}\vec{E}_\perp\times\hat{b}\right)$$

$$= -\frac{q}{|q|}\frac{\alpha}{\omega_c}\frac{B_0^2}{B^3}\vec{E}_\perp \rightarrow -\frac{q}{|q|}\frac{\alpha}{\omega_c}\frac{\vec{E}_\perp}{B_0} \quad //$$

2. $\vec{\rho} = \rho\left(\hat{e}_1 \sin\gamma + \hat{e}_2 \cos\gamma\right)$

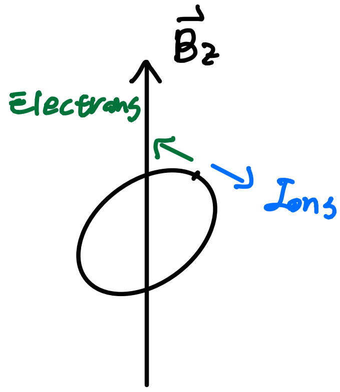
$$2w_c \underbrace{\langle \partial_\gamma \vec{\rho} \cdot \partial_t \partial_\gamma \vec{\rho} \rangle}_{①} + \frac{dw_c}{dt} \underbrace{\langle \partial_\gamma \vec{\rho} \cdot \partial_\gamma \vec{\rho} \rangle}_{②} + \underbrace{\langle \partial_\gamma \vec{\rho} \cdot w_c (\vec{v}_0 \cdot \vec{\nabla}) \partial_\gamma \vec{\rho} \rangle}_{②'}$$

$$= \frac{q}{m}\left(\underbrace{\langle \partial_\gamma \vec{\rho} \cdot (\vec{\rho}\cdot\vec{\nabla})\vec{E}_\perp \rangle}_{③} + \underbrace{\langle \partial_\gamma \vec{\rho} \cdot \partial_t \vec{\rho} \times \vec{B} \rangle}_{④} + \underbrace{\langle \partial_\gamma \vec{\rho} \cdot \vec{v}_0 \times (\vec{\rho}_0 \cdot \vec{\nabla}) \vec{B} \rangle}_{⑤} \right)$$

Set $\vec{\rho} = \rho\left(\hat{e}_1 \sin w_c t + \hat{e}_2 \cos w_c t\right)$

$= \rho\left(\hat{e}_1 \sin\gamma + \hat{e}_2 \cos\gamma\right)$

where $w_c = \frac{Bq}{m}$



Ions $\rightarrow$ $w_c > 0$

Electrons $\rightarrow$ $w_c < 0$

① : $\langle \partial_\gamma \vec{\rho} \cdot \partial_t \partial_\gamma \vec{\rho} \rangle$

$$= \langle \rho\left(\hat{e}_1 \sin\gamma + \hat{e}_2 \cos\gamma\right) \cdot \left(\partial_t \rho \left(\hat{e}_1 \sin\gamma + \hat{e}_2 \cos\gamma\right) + \rho w_c \left(\hat{e}_1 \cos\gamma - \hat{e}_2 \sin\gamma\right)\right) \rangle$$

$$= \rho\, \partial_t \rho$$

Note

$\langle \sin\gamma \cos\gamma \rangle = 0$

$\langle \sin\gamma \sin\gamma \rangle = \frac{1}{2}$

$\langle \cos\gamma \cos\gamma \rangle = \frac{1}{2}$

② : $\langle \rho^2 \rangle = \rho^2$

②' : $\langle \partial_\gamma \vec{\rho} \cdot w_c (\vec{v}_0 \cdot \vec{\nabla}) \partial_\gamma \vec{\rho} \rangle$

$\left(\because \vec{A} \cdot (\vec{v} \cdot \vec{\nabla}) \vec{A} = \frac{1}{2}(\vec{v} \cdot \vec{\nabla}) A^2 \right)$

$$= \frac{1}{2} w_c (\vec{v}_0 \cdot \vec{\nabla}) \langle \partial_\gamma \vec{\rho} \cdot \partial_\gamma \vec{\rho} \rangle$$

$$= \frac{1}{2} w_c (\vec{v}_0 \cdot \vec{\nabla}) \rho^2$$

(이를 살리고 싶다면)

처음 $\frac{d\vec{\rho}}{dt}$ 를 유도할 때 $(\vec{v}_0 \cdot \vec{\nabla})\vec{\rho}$ 를 무시했으므로, 원칙적으로는 처음부터 이를 살려서 다시 계산해야 합니다. (Remind $\frac{d\vec{\rho}}{dt} = \Sigma\, \partial_t \vec{\rho} + w \partial_\gamma \vec{\rho} \neq \Sigma\, \partial_t \vec{\rho} + w \partial_\gamma \vec{\rho} + (\vec{v}_0 \cdot \vec{\nabla})\vec{\rho}$)

그렇지만 adiabatic invariance를 보일 때 $(\vec{v}_0 \cdot \vec{\nabla}) \rho^2 \sim 0$ 으로 두는 것도 괜찮으므로, 이를 인위적으로 살려둡시다. 하지만 이경우 $\frac{1}{2}$이 Typo로 빠져있음을 인지해야 합니다.

③ : $\langle \partial_\gamma \vec{\rho} \cdot (\vec{\rho}\cdot\vec{\nabla}) \vec{E}_\perp \rangle$

$$= \langle \rho(\hat{e}_1\cos\gamma - \hat{e}_2\sin\gamma)\cdot \rho(\sin\gamma\,\partial_x + \cos\gamma\,\partial_y)\vec{E}_\perp \rangle$$

$$= \rho^2 \langle \cos\gamma\,(\cancel{\sin\gamma}\,\partial_x + \cos\gamma\,\partial_y)E_x - \sin\gamma\,(\sin\gamma\,\partial_x + \cancel{\cos\gamma}\,\partial_y)E_y \rangle$$

$$= \frac{1}{2}\rho^2(\partial_y E_x - \partial_x E_y)$$

$$= \frac{1}{2}\rho^2 \partial_t B_z$$

$$\left(\because \vec{\nabla}\times\vec{E}\cdot\hat{z} = -\partial_t\vec{B}\cdot\hat{z} \quad \partial_x E_y - \partial_y E_x = -\partial_t B_z \right)$$

④ : $\langle \partial_\gamma \vec{\rho} \cdot \partial_t \vec{\rho} \times \vec{B} \rangle$

$$= \vec{B}\cdot\langle \partial_\gamma\vec{\rho}\times\partial_t\vec{\rho}\rangle$$

$$\left(\because \vec{A}\cdot\vec{B}\times\vec{C} = \vec{C}\cdot\vec{A}\times\vec{B} \right)$$

$$= \vec{B}\cdot\langle \rho(\cos\gamma\,\hat{e}_1 - \sin\gamma\,\hat{e}_2)\times(\partial_t\rho(\sin\gamma\,\hat{e}_1 + \cos\gamma\,\hat{e}_2) + \rho w_c(\cancel{\cos\gamma\,\hat{e}_1 - \sin\gamma\,\hat{e}_2}))\rangle$$

$$= \vec{B}\cdot\hat{e}_3(\rho\,\partial_t\rho)$$

$$= \left(\partial_t \frac{1}{2}\rho^2\right) B_z$$

⑤ : $\langle \partial_\gamma\vec{\rho}\cdot\vec{v}_0\times(\vec{\rho}_0\cdot\vec{\nabla})\vec{B}\rangle$

$$= \vec{V}_0\cdot\langle(\vec{\rho}\cdot\vec{\nabla})\vec{B}\times\partial_\gamma\vec{\rho}\rangle$$

$$= \vec{V}_0\cdot\langle \rho^2(\sin\gamma\,\partial_x + \cos\gamma\,\partial_y)\cdot(\cos\gamma(B_z\hat{e}_2 - B_y\hat{e}_3) - \sin\gamma(B_x\hat{e}_3 - B_z\hat{e}_1))\rangle$$

$$= \vec{V}_0\cdot\rho^2\langle\cos^2\gamma\,\partial_y(B_z\hat{e}_2 - B_y\hat{e}_3) - \sin^2\gamma\,\partial_x(B_x\hat{e}_3 - B_z\hat{e}_1)\rangle$$

$$= \vec{V}_0\cdot\frac{1}{2}\rho^2\left(\partial_x B_z\,\hat{e}_1 + \partial_y B_z\,\hat{e}_2 - (\partial_x B_x + \partial_y B_y)\hat{e}_3\right)$$

$$= \vec{V}_0\cdot\left(\frac{1}{2}\rho\,\vec{\nabla}B_z\right)$$

$$2w_c\rho\,\partial_t\rho + \frac{dw_c}{dt}\rho^2 + w_c(\vec{V}_0\cdot\vec{\nabla})\frac{1}{2}\rho^2$$
$$= \frac{q}{m}\left(\frac{1}{2}\rho^2\partial_t B_z + \partial_t\left(\frac{1}{2}\rho^2\right)B_z + \vec{V}_0\cdot\frac{1}{2}\rho^2\vec{\nabla}B_z\right)$$

LHS : $\frac{dw_c}{dt}\rho^2 + w_c\left(\partial_t\rho^2 + \vec{V}_0\cdot\vec{\nabla}\frac{1}{2}\rho^2\right)$

$$= \frac{dw_c}{dt}\rho^2 + w_c\frac{d\rho^2}{dt} - w_c(\vec{V}_0\cdot\vec{\nabla})\frac{1}{2}\rho^2$$
$$= \frac{d}{dt}(w_c\rho^2) - \underline{w_c(\vec{V}_0\cdot\vec{\nabla})\frac{1}{2}\rho^2}$$

$(\vec{V}_0\cdot\vec{\nabla})\rho^2 \neq 0$ 일때
Typo 수정을 못했으면
이 텀 때문에
문제가 생깁니다.

RHS : $\frac{1}{2}\rho^2\left(\partial_t w_c + \vec{V}_0\cdot\vec{\nabla}w_c\right) + \left(\partial_t\frac{1}{2}\rho^2\right)w_c$

$$= \frac{1}{2}\rho^2\frac{dw_c}{dt} + \left(\partial_t\frac{1}{2}\rho^2\right)w_c$$

$$\frac{d}{dt}w_c\rho^2 - w_c(\vec{V}_0\cdot\vec{\nabla})\frac{1}{2}\rho^2 = \frac{1}{2}\rho^2\frac{dw_c}{dt} + \left(\frac{d}{dt}\frac{1}{2}\rho^2\right)w_c$$

$$\therefore \quad \frac{d}{dt}\frac{1}{2}w_c\rho^2 = \frac{1}{q}\frac{d}{dt}\mu = 0$$

Note

$$\vec{A}\cdot(\vec{V}\cdot\vec{\nabla})\vec{A}$$
$$= A_1(V_1\partial_x + V_2\partial_y + V_3\partial_z)A_1 + \cdots$$
$$= (V_1\partial_x + V_2\partial_y + V_3\partial_z)\frac{1}{2}A_1^2 + \cdots$$
$$= (\vec{V}\cdot\vec{\nabla})\frac{1}{2}\vec{A}\cdot\vec{A}$$

$(\vec{V}_0\cdot\vec{\nabla})\rho^2$ 을 살려서 계산하신 분들도 계셔서 이렇게 풀긴 했는데 $(\vec{V}_0\cdot\vec{\nabla})\rho^2 \sim 0$ 으로 두고 푸는 것이 가장 make sense 해보이긴 합니다.

3. $\vec{V}_d = \frac{V_\parallel}{B}\vec{B} + \frac{V_\parallel}{B}\vec{\nabla}\times(\rho_\parallel\vec{B})$

$\left(\because \vec{\nabla}\times(\rho_\parallel\vec{B}) = \vec{\nabla}\rho_\parallel\times\vec{B} + \rho_\parallel\vec{\nabla}\times\vec{B}\right)$

$= \frac{V_\parallel}{B}\vec{B} + \frac{V_\parallel}{B}\left(\vec{\nabla}\rho_\parallel\times\vec{B} + \rho_\parallel\vec{\nabla}\times\vec{B}\right)$

$\vec{\nabla}\rho_\parallel = \frac{m}{qB}\vec{\nabla}V_\parallel - \frac{mV_\parallel}{qB^2}\vec{\nabla}B$

$$= \frac{V_\parallel}{B}\vec{B} + \underbrace{\frac{mV_\parallel}{qB^2}\vec{\nabla}V_\parallel\times\vec{B}}_{1} - \frac{mV_\parallel^2}{qB^3}\vec{\nabla}B\times\vec{B} + \frac{mV_\parallel^2}{qB^2}\vec{\nabla}\times\vec{B}$$

$\underline{U = \frac{1}{2}mV_\parallel^2 + \mu B + q\phi}$ $\longrightarrow$ $mV_\parallel\vec{\nabla}V_\parallel + (\vec{\nabla}\mu)B + \mu\vec{\nabla}B + q\vec{\nabla}\phi = 0$

Assume it fixed in space

1 : $\frac{1}{qB^2}\left(\cancel{-B\vec{\nabla}\mu} - \mu\vec{\nabla}B - q\vec{\nabla}\phi\right)\times\vec{B}$

$$= \frac{\vec{E}\times\vec{B}}{B^2} + \frac{W_\perp}{qB^3}\vec{B}\times\vec{\nabla}B - \cancel{\frac{1}{qB}\vec{\nabla}\mu\times\vec{B}}$$

$$\vec{V}_d = \frac{V_\parallel}{B}\vec{B} + \frac{\vec{E}\times\vec{B}}{B^2} + \frac{W_\perp}{qB^3}\vec{B}\times\vec{\nabla}B - \underbrace{\frac{2W_\parallel}{qB^3}\vec{\nabla}B\times\vec{B}}_{2}$$

$$- \frac{1}{qB}\vec{\nabla}\mu\times\vec{B} + \underbrace{\frac{2W_\parallel}{qB^2}\vec{\nabla}\times\vec{B}}_{3}$$

2 + 3 : $\frac{2W_\parallel}{qB^2}\left(-\vec{\nabla}B\times\hat{b} + \vec{\nabla}\times\vec{B}\right)$

$$= \underbrace{\frac{2W_\parallel}{qB}\vec{\nabla}\times\hat{b}}_{4}$$

$\left(\because \vec{\nabla}\times\vec{B} = \vec{\nabla}\times(B\hat{b}) = \vec{\nabla}B\times\hat{b} + B\vec{\nabla}\times\hat{b}\right)$

Check ④ $= \hat{b} \times (\hat{b}\cdot\vec{\nabla}\hat{b})$ + "Something" to identify if it is has curvature drift or not

$$\hat{b}\times(\vec{\nabla}\times\hat{b}) = \vec{\nabla}\left(\tfrac{1}{2}|\hat{b}|^2\right) - (\hat{b}\cdot\vec{\nabla})\hat{b}$$
$$= -(\hat{b}\cdot\vec{\nabla})\hat{b}$$

$$\hat{b}\times(\hat{b}\times(\vec{\nabla}\times\hat{b}))$$
$$= \underline{\hat{b}\,(\hat{b}\cdot(\vec{\nabla}\times\hat{b}))} - \vec{\nabla}\times\hat{b}$$

"Not 0" : origin of remainder

$$\cancel{\left(\because \hat{b}\cdot\vec{\nabla}\times\hat{b} = \vec{\nabla}\cdot(\hat{b}\times\hat{b}) - \hat{b}\cdot\vec{\nabla}\times\hat{b} \rightarrow \hat{b}\cdot\vec{\nabla}\times\hat{b} = 0\right)}$$

$$\therefore ④ = \vec{\nabla}\times\hat{b} = -\hat{b}\times(\hat{b}\times(\vec{\nabla}\times\hat{b})) + \hat{b}(\hat{b}\cdot\vec{\nabla}\times\hat{b})$$
$$= \hat{b}\times(\hat{b}\cdot\vec{\nabla})\hat{b} + \hat{b}(\hat{b}\cdot\vec{\nabla}\times\hat{b})$$

②+③: $\dfrac{2W_\parallel}{qB}\left(\hat{b}\times(\hat{b}\cdot\vec{\nabla})\hat{b} + \hat{b}(\hat{b}\cdot\vec{\nabla}\times\hat{b})\right)$

$$\therefore \vec{V}_d = V_\parallel\hat{b} + \underbrace{\frac{\vec{E}\times\vec{B}}{B^2}}_{=\vec{V}_E} + \underbrace{\frac{W_\perp}{qB^3}\vec{B}\times\vec{\nabla}B}_{=\vec{V}_{\nabla B}} + \underbrace{\frac{2W_\parallel}{qB}\hat{b}\times(\hat{b}\cdot\vec{\nabla})\hat{b}}_{=\vec{V}_{curv}}$$

$$+ \underbrace{\frac{2W_\parallel}{qB}\hat{b}(\hat{b}\cdot\vec{\nabla}\times\hat{b})}_{\text{Remainder}} \qquad \cancel{-\frac{1}{qB}\vec{\nabla}\mu\times\vec{B}}$$

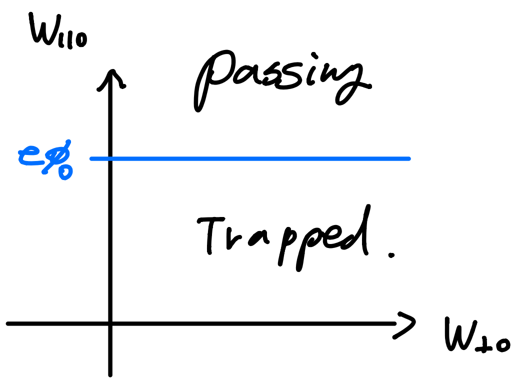
Remainder.

4. a)

Ions

Case 1. $z > 0$. (Electric Mirror)

$E = W_0 = W_\perp + W_\parallel + e\phi(z) = \text{const.}$

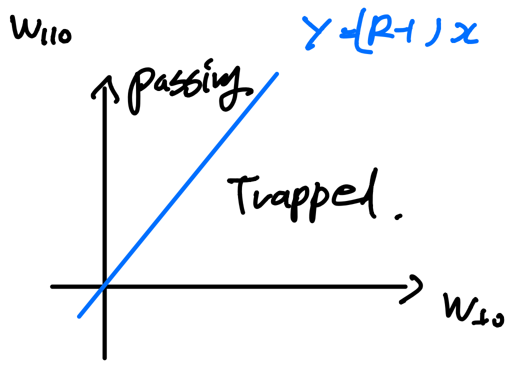


If $W_{\parallel 0} > e\phi_0$, passing.

If $W_{\parallel 0} < e\phi_0$, trapped.

Case 2. $z < 0$ (Magnetic Mirror)

$E = W_0 = W_\parallel + \mu B(z) = \text{Const.}$



$$\left.\begin{aligned} W_{\parallel 0} + \mu B_0 &= \mu R B_0 \\ W_{\parallel 0} &= (R-1) W_{\perp 0} \end{aligned}\right\} \text{ marginally trapped.}$$

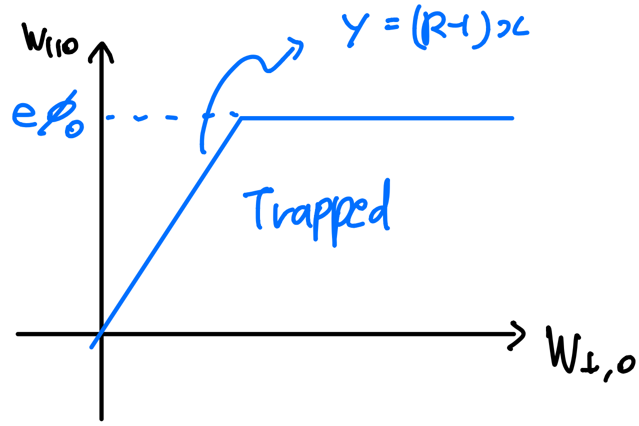
If $W_{\parallel 0} > (R-1) W_{\perp 0}$, passing

If $W_{\parallel 0} < (R-1) W_{\perp 0}$, trapped

Electrons.

Electrons cannot be trapped in electric mirror.

b)

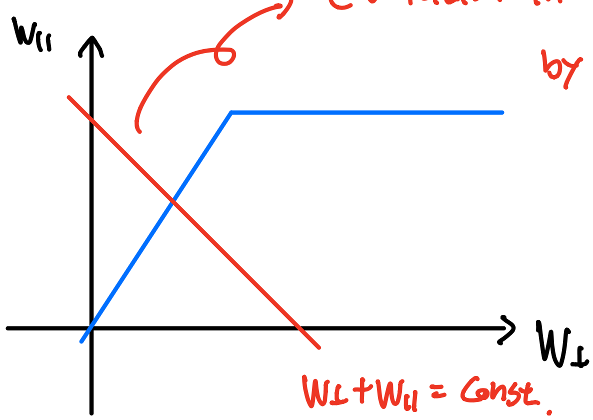


Derivation process

See prob 4 (a)

c)

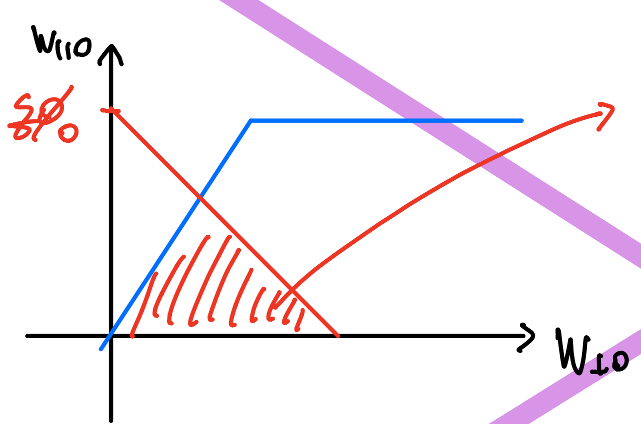
evolution in $W_\perp$ - $W_\parallel$ space by pitch-angle scattering.



Note

pitch-angle scattering 과정을 원형으로 그리신 분들이 많은데... $W_\perp$ - $W_\parallel$ space 에서는 직선 입니다.

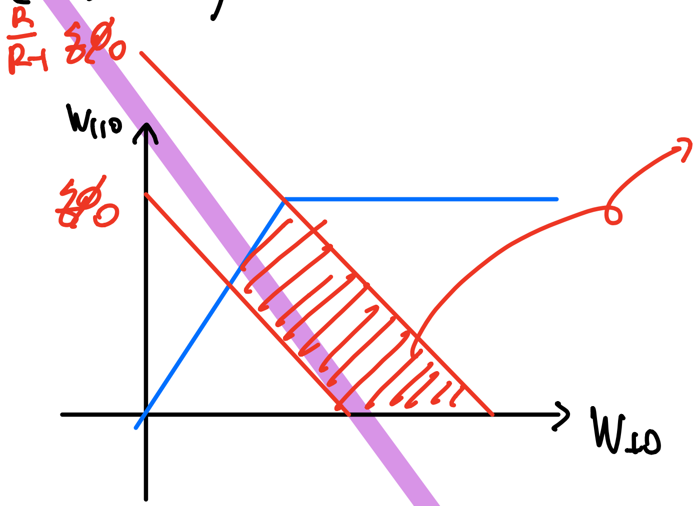
~~Case 1) $W_{\perp 0} + W_{\parallel 0} < e\phi_0$.~~



~~If $W_{\perp 0} + W_{\parallel 0} < e\phi_0$, even if ions gain $W_\parallel$ by pitch-angle scattering, they cannot escape the electric trap.~~

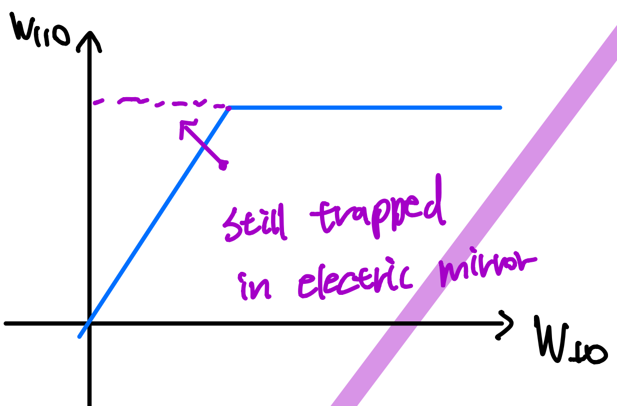
~~Thus, in that case, all ptls pass through magnetic mirror.~~

Case 2) $R\varepsilon\phi_0 > W_{\perp 0} + W_{\parallel 0} > \varepsilon\phi_0$

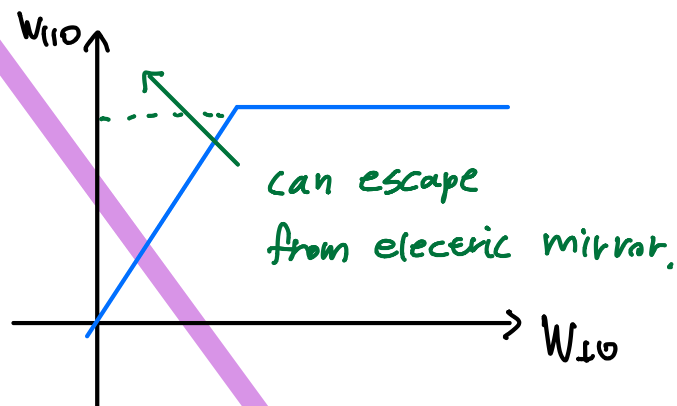


If $\frac{R}{R-1}\varepsilon\phi_0 > W_{\perp 0} + W_{\parallel 0} > \varepsilon\phi_0$, possible scattering mechanisms are "small angle" scattering and "large angle" scattering.

Small angle scattering is likely to change $W_{\parallel}$ a little bit, and in that case, escaping through magnetic mirror is easier. On the other hand, escaping through electric mirror is only possible when large angle scattering occurs.

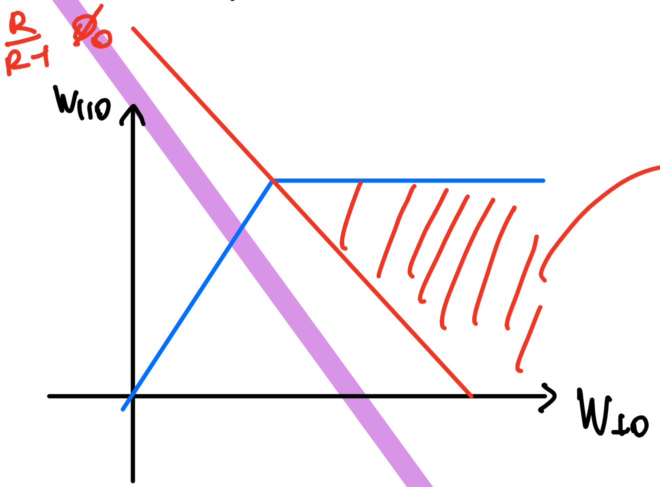


Small angle scattering



Large angle scattering.

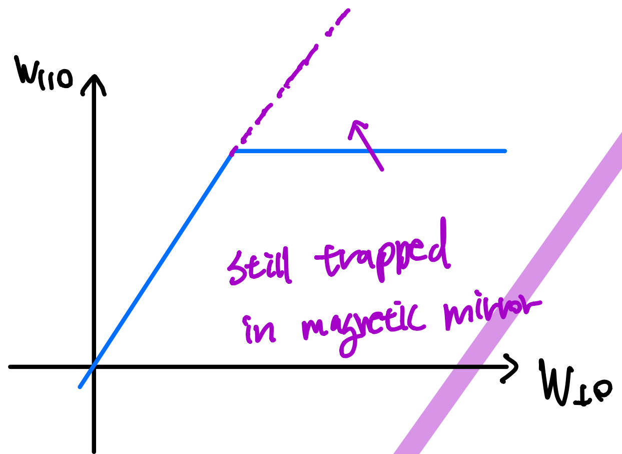
Case 3) $W_{\perp 0} + W_{\parallel 0} > R \xi \phi_0$



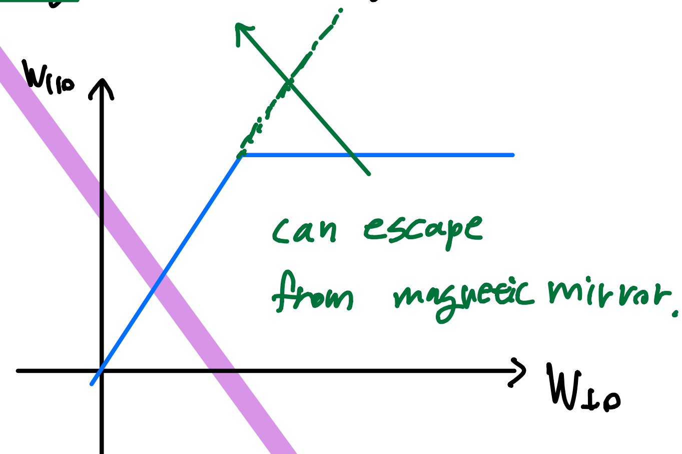
If $W_{\perp 0} + W_{\parallel 0} > \frac{R}{R-1} \xi \phi_0$, the roll of small & large angle scattering changes a little bit

Small angle scattering is likely to change $W_{\parallel}^{r}$, and in that case, escaping through electric mirror is easier.

Escaping through magnetic mirror is only possible when large angle scattering occurs.



Small angle Scattering

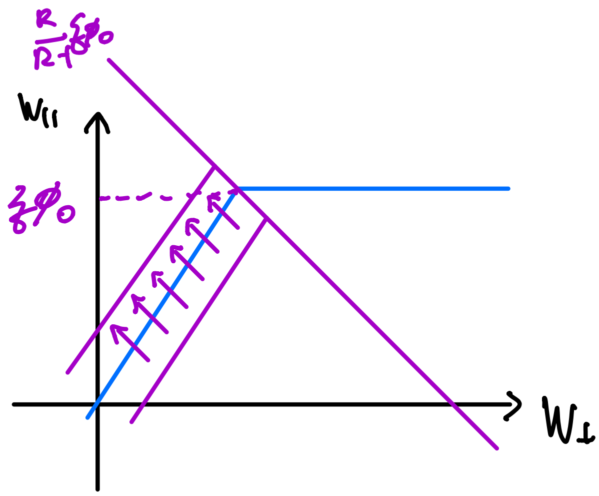


Large angle Scattering.

Anyway... eventually all particles could escape the mirrors by pitch-angle scattering process.

제가 기존에 올렸던 답안 에서는 Small angle scattering 과 Large angle scattering 을 모두 고려했었는데, Small angle scattering 만 고려해야 원하는 답이 나오는 것 같아서 전면적으로 수정하도록 하겠습니다.

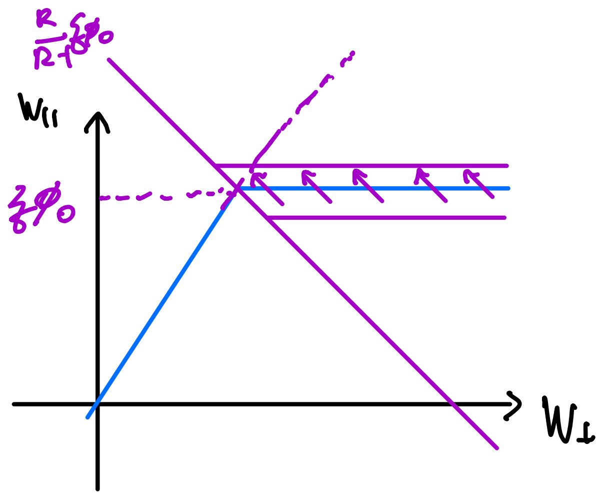
Case 1) $W_\parallel + W_\perp < \frac{R}{R-1} e\phi$



Small angle scattering 만 고려할 경우 $W_\parallel + W_\perp < \frac{R}{R-1} e\phi$ 을 만족하는 ptls 에 대해서는 Magnetic Mirror의 trapping conditions 에서 벗어나는 것을 확인 가능합니다.

→ 빠져나가는 방향 : Magnetic Mirror.

Case 2) $W_\parallel + W_\perp > \frac{R}{R-1} e\phi$



Small angle scattering 만 고려할 경우 $W_\parallel + W_\perp > \frac{R}{R-1} e\phi$ 을 만족하는 ptls 에 대해서는 Electric Mirror의 trapping conditions 에서 벗어나는 것을 확인 가능합니다.

→ 빠져나가는 방향 : Electric Mirror.

✱ 개인적인 Comment...

제가 한참 관심 가지던 영역이 collision part 라서 너무 어렵게 생각해 모두의 시간을 잡아먹은 것 같아 죄송합니다.

다만, 이러한 Large angle scattering 은 분명히 존재하는 Mechanism 이고, 어떠한 물리현상에서는 굉장히 중요할 수 있다는 것 정도만 기억해 주시면 될 것 같습니다.

e.g. Runaway electrons 의 avalanche ~

d) Assume $\left|\frac{dL_e}{dt}\frac{1}{L_e}\right| \ll \omega_b$, $\left|\frac{d\phi_0}{dt}\frac{1}{\phi_0}\right| \ll \omega_b$.

And then, we can regard it as constant during bounce-motion, but time-varying in long time window.

$z>0$ //

$$W_{\parallel 0} = \frac{1}{2} m v_\parallel^2 + \frac{q}{8}\phi(t) \cdot \left(\frac{z}{L_e(t)}\right)^2$$

$$v_\parallel = \pm \sqrt{\frac{2}{m}\left(W_{\parallel 0} - \frac{q}{8}\phi(t)\left(\frac{z}{L_e(t)}\right)^2\right)}$$

Turning point means $v_\parallel = 0 \iff z_E = L_e(t)\sqrt{\frac{W_{\parallel 0}}{\frac{q}{8}\phi_0(t)}}$

$$J_+ = 2\int_0^{z_E} v_\parallel \, dz \quad \searrow \quad x \equiv z/z_E$$

$$= 2\cdot\sqrt{\frac{2}{m}W_{\parallel 0}} \cdot L_e(t)\sqrt{\frac{W_{\parallel 0}}{\frac{q}{8}\phi_0(t)}} \int_0^1 \sqrt{1-x^2}\, dx .$$

$x = \sin\theta$

$$= \int_0^{\frac{\pi}{2}} \cos\theta\cos\theta \, d\theta = \frac{\pi}{4} .$$

$$= \frac{\pi}{\sqrt{2m}} W_{\parallel 0}^{1/2} z_E$$

$z<0$ //

$$W_{\parallel 0} + \mu B_0 = W_\parallel + \mu B_0\left(1 + (R-1)\left(\frac{z}{L_m}\right)^2\right)$$

$$v_\parallel = \pm\sqrt{\frac{2}{m}\left(W_{\parallel 0} - W_{\perp 0}(R-1)\left(\frac{z}{L_m}\right)^2\right)} , \quad z_M = L_M\sqrt{\frac{W_{\parallel 0}}{W_{\perp 0}(R-1)}}$$

$$J_- = 2\int_0^{z_M} v_\parallel \, dz = \frac{\pi}{\sqrt{2m}} W_{\parallel 0}^{1/2} z_M$$ $\rightarrow$ after same process

$$\therefore J = J_+ + J_- = \frac{\pi}{\sqrt{2m}} W_{\parallel 0}^{1/2}\left(z_E + z_M\right)$$

$$z_E / z_M = L_e \sqrt{\frac{W_{\parallel 0}}{\xi \phi_0}} \Big/ L_m \sqrt{\frac{W_\parallel}{W_{\perp,0}(R-1)}}$$

$$= (L_e / L_m) \cdot \sqrt{W_{\perp,0} / W_c} \gg 1$$

e) 방법 1

Equation of motion

$$\frac{1}{2} m V_\parallel^2 = W_{\parallel 0} - \xi \phi_0 \frac{z^2}{L_e^2}$$

$$\rightarrow m\ddot{z} + 2\xi\phi_0 \frac{1}{L_e^2} z = 0$$

$$z_E = \sqrt{\frac{W_{\parallel 0}}{\xi \phi_0}} L_e$$

$$W_E = \sqrt{\frac{2\xi\phi_0}{m}} \frac{1}{L_e}$$

$$V_0 = \sqrt{\frac{2W_{\parallel 0}}{m}}$$

Leading order motion with $\left(\varepsilon = \frac{L_e'}{L_e} t \ll 1 \right)$

$$z \sim z_E \sin W_E t$$

$$v \sim V_0 \cos W_E t$$

Bounce - Averaged Power from slow-varying electric field.

let $0 < t < \frac{2\pi}{W_b}$, then $\frac{L_e'}{L_e} t \ll 1$.

$L_e(t^*)$, t^* is reference time for bounce motion.

1st order correction for electric field.

$$F \sim F_0 + F_1$$

$$= \partial_z \left(-\xi\phi_0 \frac{z^2}{L_e^2} \right) + \underbrace{\partial_z \left(2\xi\phi_0 \frac{z^2}{L_e^3} L_e' t \right)}_{\text{1st order correction.}}$$

$$\frac{\Delta W_{\parallel 0}}{\pi / w_E} \sim \frac{dW_{\parallel 0}}{dt} \sim \frac{1}{\pi / w_E} \oint_b v\left(\cancel{F_0} + F_1\right) dt$$

(F_0 contribution ~ 0)

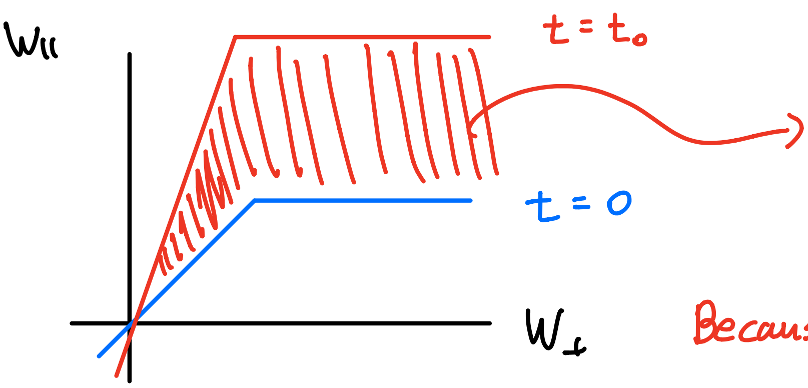
Here $z_E / z_M \gg 1$ used.

$$\frac{dW_{\parallel 0}}{dt} = \frac{w_E}{\pi} \int_0^{\pi / w_E} V_0 \cos w_E t \left(4 \xi \phi_0 \frac{z_E \sin w_E t}{L_e^3} L_e' t \right) dt$$

$$= \frac{w_E}{\pi} \cdot V_0 \cdot 4 \xi \phi_0 \cdot \frac{z_E}{L_e^3} \cdot L_e' \cdot \frac{1}{w_c^2} \underbrace{\int_0^{\pi} \sin\theta \cos\theta \, \theta \, d\theta}_{= -\frac{\pi}{4}} .$$

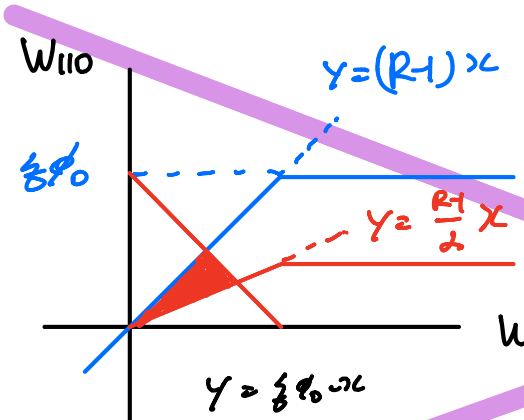
$$= - W_{\parallel 0} \frac{L_e'}{L_e}$$

$$\Rightarrow \frac{d}{dt}\left(W_{\parallel 0}(t) \, L_e(t) \right) = 0 \qquad W_{\parallel f} / W_{\parallel 0} = \alpha$$



They escape during these bounce motion

Because 1st order corrected force increases $W_{\parallel 0}$?? ? ?.



$\triangle$ region : all ions in this region escape through magnetic mirror!. Otherwise, they can escape only by pitch-angle scattering.

$$\text{Area} = \frac{1}{2}(\xi \phi_0)^2 \left(\left(1 + \frac{\alpha}{R-1}\right)^{-1} - \left(1 + \frac{1}{R-1}\right)^{-1} \right) = \frac{1}{2}(\xi \phi_0)^2 \frac{(R-1)(\alpha - 1)}{R(R-1+\alpha)}$$

e) 방법 2.

$$\left\langle \frac{dJ}{dt} \right\rangle_b = \frac{\partial J}{\partial W_{110}} \left\langle \frac{dW_{110}}{dt} \right\rangle_b + \frac{\partial J}{\partial L_e} \left\langle \frac{dL_e}{dt} \right\rangle_b \sim 0.$$

For simplicity, lets omits $\langle\,\rangle_b$ for $\left\langle \frac{dW_{110}}{dt} \right\rangle_b$, $\left\langle \frac{dL_e}{dt} \right\rangle_b$.

$$\rightarrow \left(\frac{d}{dt} W_{110}\right)\left(L_e(t)\sqrt{\frac{1}{\xi\phi_0}} + L_M\sqrt{\frac{1}{w_{s,p}(R-1)}} \right) + W_{110}\left(\frac{d}{dt} L_e(t)\right)\sqrt{\frac{1}{\xi\phi_0}} = 0.$$

$$\rightarrow \ln W_{110} + \ln\left(L_e(t)\sqrt{\frac{1}{\xi\phi_0}} + L_M\sqrt{\frac{1}{w_{s,p}(R-1)}} \right) = \text{const}.$$

If we assume $L_e/L_M \gg 1$, then

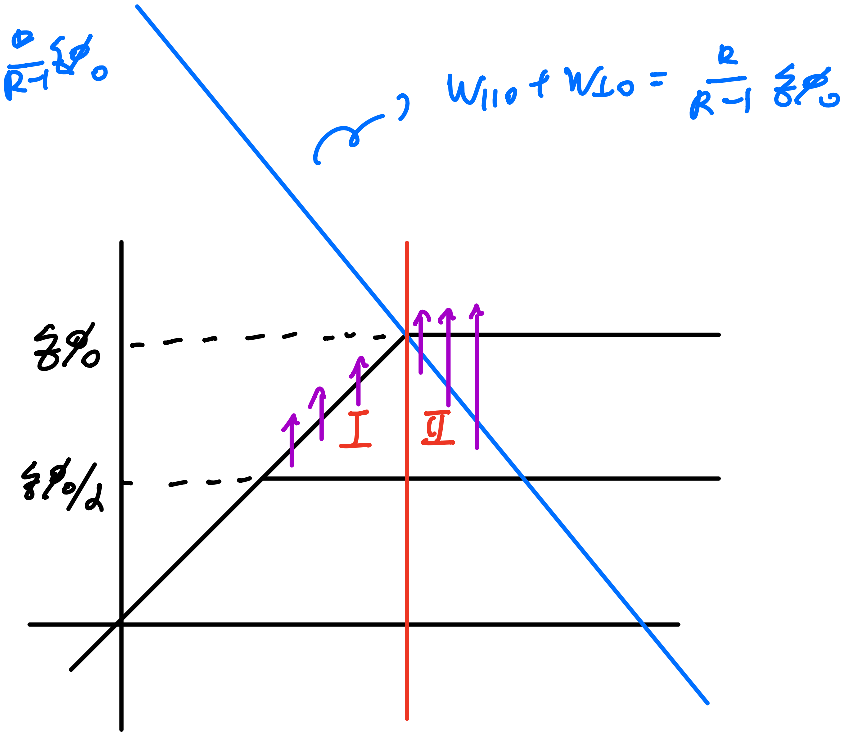
$$\ln(W_{110} L_e) = \text{const}.$$

Other processes are the same.

(e) 번 문항에 approximated area 에 관하여 typo가 있었습니다

$$A \sim \frac{1}{2}(\xi\phi_0)(1-\alpha^{-4/3})^2 \longrightarrow A \sim \frac{1}{2}(\xi\phi_0)(1-\alpha^{-1})^2$$

이것을 뒤늦게 공지드려 죄송합니다...



<u>영역 I.</u> → Bounce-motion에 따른 $W_{\parallel 0}$ 의 증가로
"Magnetic Mirror"로 빠져나감.

<u>영역 II.</u> → Bounce-motion에 따른 $W_{\parallel 0}$ 의 증가로
"Electric Mirror"로 빠져나감.

영역 I & II 모두 원래는 pitch-angle scattering에 의해 "Magnetic Mirror"로 빠져나갈 운명이었지만, $L_c \to L_c/\alpha$ 로 인해 영역 II는 "다른 방향"인 "Electric Mirror"로 빠져나감이 확인 가능합니다.

$$\therefore\ A \sim \frac{1}{2}(\delta\phi_0)^2\left(1-\alpha^{-1}\right)^2$$ //