1. a)

Assume isotropic p_e to simplify it.

Electron momentum conservation equation.

$$m_e n_e \frac{d}{dt}\vec{u}_e = -en_e\left(\vec{E}+\vec{u}_e\times\vec{B}\right) - \vec{\nabla}p_e + en_e\eta\vec{j}$$

$$-\frac{m_e}{e}\left(\partial_t \vec{u}_e + \underbrace{(\vec{u}_e\cdot\vec{\nabla})\vec{u}_e}_{\Leftrightarrow\, =\, \frac{1}{2}\vec{\nabla}|\vec{u}_e|^2 - \vec{u}_e\times\vec{\nabla}\times\vec{u}_e}\right) = \vec{E} + \vec{u}_e\times\vec{B} + \frac{1}{en_e}\vec{\nabla}p_e - \eta\vec{j}$$

Substitue $\vec{u}_e = -\frac{1}{en_e}\vec{j} = -\frac{1}{en_e\mu_0}\vec{\nabla}\times\vec{B}$, $\vec{j} = \frac{1}{\mu_0}(\vec{\nabla}\times\vec{B})$

$$d_e^2\left(\partial_t(\vec{\nabla}\times\vec{B}) - \vec{u}_e\times(\vec{\nabla}\times(\vec{\nabla}\times\vec{B}))\right) - \cancel{\frac{m_e}{2e}\vec{\nabla}|\vec{u}_e|^2} \quad (\vec{\nabla}\times\vec{\nabla})=0$$

$$= \vec{E} + \vec{u}_e\times\vec{B} + \cancel{\frac{1}{en_e}\vec{\nabla}p_e} - \frac{\eta}{\mu_0}(\vec{\nabla}\times\vec{B}) \quad (\vec{\nabla}\times\vec{\nabla})=0$$

where $d_e^2 = \frac{m_e}{e^2 n_e \mu_0}$

Take a curl $\vec{\nabla}\times$ and use $\vec{\nabla}\times\vec{\nabla}\times\vec{B} = -\vec{\nabla}^2\vec{B}$

$$-\partial_t\left(d_e^2\vec{\nabla}^2\vec{B}\right) + \vec{\nabla}\times\left(\vec{u}_e\times d_e^2\vec{\nabla}^2\vec{B}\right)$$

$$= -\partial_t\vec{B} + \vec{\nabla}\times(\vec{u}_e\times\vec{B}) + \frac{\eta}{\mu_0}\vec{\nabla}^2\vec{B}$$

$$\therefore\ \partial_t\left(\partial_t\vec{B} - d_e^2\vec{\nabla}^2\vec{B}\right) = \vec{\nabla}\times\left[\vec{u}_e\times\left(\vec{B} - d_e^2\vec{\nabla}^2\vec{B}\right)\right] + \frac{\eta}{\mu_0}\vec{\nabla}^2\vec{B}$$

b) A. Intrinsic length scale

Ideal EMHD $\rightarrow$ $\underline{\eta \rightarrow 0}$

$$\partial_t \left(\vec{B} - d_e^2 \vec{\nabla}^2 \vec{B}\right) = \vec{\nabla} \times \left[\vec{u_e} \times \left(\vec{B} - d_e^2 \vec{\nabla}^2 \vec{B}\right)\right]$$

For $\vec{B} - d_e^2 \vec{\nabla}^2 \vec{B} = 0$, it's always satisfied.

Thus, there is the intrinsic scale length d_e for ideal EMHD.

B. Dispersion relation.

$$\vec{u}_{eo} = -\frac{1}{e n_e \mu_0} \vec{\nabla} \times \vec{B} = 0$$

To simplify the system, choose uniform $\vec{B}_0$ and linearize ideal EMHD with $\delta \vec{A} = \tilde{A} e^{i(\vec{k}\cdot\vec{x} - \omega t)}$, $\delta f = \tilde{f} e^{i(\vec{k}\cdot\vec{x} - \omega t)}$

$$-i\omega \left(1 + k^2 d_e^2\right) \delta \vec{B} = \vec{\nabla} \times \left[\cancel{\vec{u}_{eo}} \times \left(\delta \vec{B} - d_e^2 \vec{\nabla}^2 \delta \vec{B}\right)\right] + \vec{\nabla} \times \left[\delta \vec{u}_{eo} \times \vec{B}_0\right]$$

$$\delta \vec{u}_{eo} = \frac{\delta n_{eo}}{e n_{eo}^2} \cancel{\vec{j}_0} - \frac{1}{e n_{eo} \mu_0} \vec{\nabla} \times \delta \vec{B}$$

$$-i\omega \left(1 + k^2 d_e^2\right) \delta \vec{B} = \frac{1}{e n_{eo} \mu_0} \vec{k} \times \left[\left(\vec{k} \times \delta \vec{B}\right) \times \vec{B}_0\right]$$

$$= \frac{1}{e n_{eo} \mu_0} \left(\vec{k} \times \delta \vec{B}\right) \left(\vec{B}_0 \cdot \vec{k}\right)$$

$\because \left(\vec{k} \times \delta \vec{B}\right) \times \vec{B}_0 = -\vec{k}\left(\delta \vec{B} \cdot \vec{B}_0\right) + \delta \vec{B}\left(\vec{k} \cdot \vec{B}_0\right)$

$\vec{k} \times \left[\left(\vec{k} \times \delta \vec{B}\right) \times \vec{B}_0\right] = \left(\vec{k} \times \delta \vec{B}\right)\left(\vec{k} \cdot \vec{B}_0\right)$

$$-i\omega(1+k^2 d_e^2)\overleftrightarrow{I}\cdot\delta\vec{B}$$

$$=\frac{\vec{B}_0\cdot\vec{k}}{e n_e \mu_0}\begin{pmatrix} 0 & k_2 & -k_3 \\ k_3 & 0 & -k_1 \\ k_1 & -k_2 & 0 \end{pmatrix}\delta\vec{B}$$

$$-i\omega(1+k^2 d_e^2)$$

2. $$\vec{u}\cdot\Big[\underbrace{\rho\frac{d\vec{u}}{dt}}_{①} = \underbrace{\sigma\vec{E}}_{②} + \underbrace{\vec{j}\times\vec{B}}_{③} - \underbrace{\vec{\nabla}p}_{④}\Big]$$

①. $$\rho\,\vec{u}\cdot\frac{d\vec{u}}{dt} = \frac{1}{2}\rho\Big(\underbrace{\partial_t|\vec{u}|^2}_{①-①} + \underbrace{(\vec{u}\cdot\vec{\nabla})|\vec{u}|^2}_{①-②}\Big)$$

$$①-① = \frac{1}{2}\rho\,\partial_t|\vec{u}|^2 = \partial_t\Big(\frac{1}{2}\rho|\vec{u}|^2\Big) - \frac{1}{2}|\vec{u}|^2\partial_t\rho$$

$$= \partial_t\Big(\frac{1}{2}\rho|\vec{u}|^2\Big) + \frac{1}{2}|\vec{u}|^2\,\vec{\nabla}\cdot(\rho\vec{u})$$

$$①-② = \frac{1}{2}\rho\,(\vec{u}\cdot\vec{\nabla})|\vec{u}|^2 = \vec{\nabla}\cdot\Big(\frac{1}{2}\rho|\vec{u}|^2\vec{u}\Big) - \frac{1}{2}|\vec{u}|^2\,\vec{\nabla}\cdot(\rho\vec{u})$$

$$① = \partial_t\Big(\frac{1}{2}\rho|\vec{u}|^2\Big) + \vec{\nabla}\cdot\Big(\frac{1}{2}\rho|\vec{u}|^2\vec{u}\Big)$$

$$② = \vec{u}\cdot\sigma\vec{E} = -\sigma\vec{u}\cdot(\vec{u}\times\vec{B}) = 0$$

③ $$\vec{u}\cdot\vec{J}\times\vec{B} = \vec{u}\cdot\Big(\frac{1}{\mu_0}\vec{\nabla}\times\vec{B} - \epsilon_0\partial_t\vec{E}\Big)\times\vec{B}$$

$= \vec{E}$

$$= \Big(\frac{1}{\mu_0}\vec{\nabla}\times\vec{B} - \epsilon_0\partial_t\vec{E}\Big)\cdot\big(\vec{B}\times\vec{u}\big)$$

$\vec{\nabla}\cdot(\vec{B}\times\vec{E}) + \vec{B}\cdot(\vec{\nabla}\times\vec{E})$

$$= -\partial_t\frac{1}{2}\epsilon_0|\vec{E}|^2 + \frac{1}{\mu_0}\big(\vec{E}\cdot\vec{\nabla}\times\vec{B}\big)$$

$$= -\partial_t\Big(\frac{1}{2}\epsilon_0|\vec{E}|^2 + \frac{1}{2\mu_0}|\vec{B}|^2\Big) - \vec{\nabla}\cdot\Big(\frac{1}{\mu_0}\vec{E}\times\vec{B}\Big)$$

④ $-\vec{u}\cdot\vec{\nabla}p = \partial_t p - \gamma p/\rho \frac{d\rho}{dt}$ $\quad \because \frac{d}{dt}(P/\rho^\gamma) = 0$

$= \partial_t p + \gamma p \vec{\nabla}\cdot\vec{u}$ $\quad \because \frac{d\rho}{dt} = -\vec{\nabla}\cdot\vec{u}$

$= \partial_t p + \vec{\nabla}\cdot(\gamma p \vec{u}) - \gamma(\vec{u}\cdot\vec{\nabla})p$

$-\vec{u}\cdot\vec{\nabla}p = -\partial_t \frac{p}{\gamma-1} - \vec{\nabla}\cdot\left(\frac{\gamma}{\gamma-1} p\vec{u}\right)$

3. $\vec{v}_p = \frac{-\vec{\nabla}p\times\vec{B}}{qB^2}$

$\vec{\nabla}\cdot\vec{v}_p = \vec{\nabla}\cdot\left(\cancel{-\vec{\nabla}\times\left(p\frac{\vec{B}}{qB^2}\right)} + p\,\vec{\nabla}\times\frac{\vec{B}}{qB^2}\right)$

$\because$ Vacuum $\vec{B}$

$= \vec{\nabla}\cdot\left(\frac{2nT}{qB^3}\vec{\nabla}B\times\vec{B}\right)$ $\quad \because \vec{\nabla}\times\frac{\vec{B}}{qB^2} = \cancel{\frac{\vec{\nabla}\times\vec{B}}{qB^2}} + 2\frac{\vec{B}}{qB^3}\times\vec{\nabla}B$

$= \vec{\nabla}\cdot\left(n\langle\vec{V}_{\nabla B}\rangle_{ite} + n\langle\vec{V}_{curv}\rangle_{ite}\right)$

See HW 1, Prob 3.

4.

Electron continuity equation

$$\partial_t n + \vec{\nabla}\cdot(n\vec{u}_e) = 0$$

$\vec{u}_e = \vec{u}_i - \frac{\vec{j}}{en}$

$$\partial_t n + \vec{\nabla}\cdot\left(n\vec{u}_i - \frac{\vec{j}}{e}\right) = 0$$

$u_\parallel$: const

$$\partial_t n + \vec{u}_i\cdot\vec{\nabla} n + n(\cancel{\vec{\nabla}\cdot\vec{u}_E}) + n(\cancel{\vec{\nabla}\cdot\vec{u}_\parallel}) - \frac{1}{e}\nabla_\parallel j = 0$$

$\because \vec{u}_E = \vec{\nabla}\phi\times\hat{z} = \vec{\nabla}\times(\phi\hat{z})$

$$\partial_t n + \vec{u}_E\cdot\vec{\nabla} n = \frac{1}{e}\nabla_\parallel j - u_\parallel\nabla_\parallel n$$

Parallel Ohm's Law

$\hat{z}\cdot[\vec{E} + \vec{u}_i\times\vec{B} = \eta\vec{j} - \frac{1}{en}\vec{\nabla}p]$

$\frac{T}{en}\nabla_\parallel n$ $\because$ Const T.

$$\underbrace{E_z}_{①} + \underbrace{\hat{z}\cdot(\vec{u}_i\times\vec{B})}_{②} = \eta j_z - \frac{1}{en}\nabla_\parallel p$$

①. $E_z = \hat{z}\cdot(-\vec{\nabla}V - \partial_t\vec{A})$ $\because$ Well-known Lorenz gauge.

$= -\partial_t A_z$ Note!? $\vec{B} = B_0\hat{z} + \vec{\nabla}\psi\times\hat{z}$

$= -\partial_t\psi$ $= B_0\hat{z} + \vec{\nabla}\times(\psi\hat{z})$ → $\vec{A}_z$

②. $\hat{z}\cdot(\vec{u}_i\times\vec{B}) = -\vec{u}_i\cdot(\hat{z}\times(\cancel{B_0\hat{z}} + \vec{\nabla}\psi\times\hat{z}))$

$\psi = \psi(x,y)$ $= \cancel{\vec{u}_\parallel\cdot\vec{\nabla}_\parallel(\psi)} - \vec{u}_E\cdot\vec{\nabla}(\psi)$

$$\partial_t\psi + \vec{u}_E\cdot\vec{\nabla}\psi = -\eta j + \frac{T}{ne}\nabla_\parallel n$$

Equation of single fluid motion

$$\rho \partial_t \vec{u}_i = \vec{J}\times\vec{B} - \vec{\nabla} p$$

Take $\hat{z}\cdot\vec{\nabla}\times$

$$\vec{\nabla}\times\left(\rho \frac{d}{dt}\vec{u}_i = \vec{J}\times\vec{B} - \cancel{\vec{\nabla} p}\right)$$

$\because (\vec{J}\cdot\vec{\nabla}) \vec{B}$ $\sim j_{\parallel}\nabla_{\parallel}\vec{B}(x,y) = 0$

$$\hat{z}\cdot\left[\underbrace{\vec{\nabla}\rho\times\frac{d}{dt}\vec{u}_i}_{①} + \underbrace{\rho\,\vec{\nabla}\times\frac{d}{dt}\vec{u}_i}_{②} = (\underbrace{(\vec{B}\cdot\vec{\nabla})}_{③} + \cancel{\vec{\nabla}\cdot\vec{B}})\vec{J} - ((\cancel{\vec{J}\cdot\vec{\nabla}}) + \cancel{\vec{\nabla}\cdot\vec{J}})\vec{B}\right]$$

③: $(\vec{B}\cdot\vec{\nabla})\vec{J}\cdot\hat{z} \sim B_0\nabla_{\parallel} j$

See "Extended part"

①. $$\vec{\nabla}\rho\times\frac{d}{dt}\vec{u}_i = \vec{\nabla}\rho\times(\vec{J}\times\vec{B} - \vec{\nabla}p) = 0$$

② $$\rho\,\hat{z}\cdot\vec{\nabla}\times(\underbrace{\partial_t\vec{u}_i}_{②-①} + \underbrace{\vec{u}_i\cdot\vec{\nabla}\vec{u}_i}_{②-②})$$

$u_{\parallel}$ is const

$(\hat{z}\cdot\vec{\nabla})\vec{\nabla}\phi(x,y) = 0.$

②-①. $$\hat{z}\cdot\vec{\nabla}\times\partial_t\vec{u}_i = \partial_t\left(\hat{z}\cdot\vec{\nabla}\times(\cancel{\vec{u}_\parallel} + \vec{\nabla}\phi\times\hat{z})\right)$$
$$= \partial_t\,\hat{z}\cdot\left(\cancel{(\hat{z}\cdot\vec{\nabla})\vec{\nabla}\phi} - \vec{\nabla}^2\phi\,\hat{z}\right)$$
$$= \partial_t w$$

$\vec{\nabla}\times(u_\parallel\hat{z} + \vec{\nabla}\phi\times\hat{z})$ $= -\vec{\nabla}^2\phi\,\hat{z}$ $= w\hat{z}$

②-②. $$\hat{z}\cdot\vec{\nabla}\times(\vec{u}_i\cdot\vec{\nabla}\vec{u}_i) = \hat{z}\cdot\vec{\nabla}\times\left(\cancel{\vec{\nabla}\left(\frac{|\vec{u}_i|^2}{2}\right)} - \vec{u}_i\times\vec{\nabla}\times\vec{u}_i\right)$$
$$= -\hat{z}\cdot\vec{\nabla}\times(\vec{u}_i\times w\hat{z})$$

$\vec{\nabla}\cdot\vec{A}\times\vec{B} = \vec{\nabla}\times\vec{A}\cdot\vec{B} - \vec{\nabla}\times\vec{B}\cdot\vec{A}$

$$= \vec{\nabla}\cdot(\hat{z}\times(\vec{u}_i\times w\hat{z})) - (\vec{u}_i\times w\hat{z})\cdot\cancel{\vec{\nabla}\times\hat{z}}$$

BAC-CAB

$$= \vec{\nabla}\cdot(w\vec{u}_E)$$
$$= \vec{u}_E\cdot\vec{\nabla}w + w\,\cancel{\vec{\nabla}\cdot\vec{u}_E}$$

$\vec{\nabla}\cdot\vec{\nabla}\times(\phi\hat{z}) = 0$

$$\rho_m(\partial_t w + \vec{u}_E\cdot\vec{\nabla}w) = B_0\nabla_{\parallel} j$$

* Extended part... (4번에서 make $B_0\nabla_\parallel j$ 가 살아남는 등 make sense 하지 않아보일 부분에 대한 부연 설명 입니다.)

$$\vec{B} = (B_0 + b)\hat{z} + \vec{\nabla}\psi \times \hat{z} \quad \rightarrow \begin{cases} |\vec{\nabla}\psi|/B_0 \sim \mathcal{O}(\varepsilon) \\ b/B_0 \sim \mathcal{O}(\varepsilon^2) \end{cases}$$

$$\rightarrow \vec{B} \sim B_0\hat{z} + \vec{\nabla}\psi \times \hat{z}$$

$$\vec{J} = \frac{1}{\mu_0}(\vec{\nabla}\times\vec{B}) = \frac{1}{\mu_0}(\vec{\nabla}b \times \hat{z} - \vec{\nabla}^2\psi\,\hat{z})$$

$$\rightarrow \vec{J} \sim j_\parallel \hat{z} \quad (\because |j_\perp|/|j_z| \sim |\vec{\nabla}b|/|\vec{\nabla}^2\psi| \sim \varepsilon)$$

$$\vec{\nabla}\times(\vec{J}\times\vec{B}) = (\vec{B}\cdot\vec{\nabla} + \cancel{\vec{\nabla}\cdot\vec{B}})\vec{J} - (\vec{J}\cdot\vec{\nabla} + \cancel{\vec{\nabla}\cdot\vec{J}})\vec{B}$$

$$= B_0\nabla_\parallel\vec{j} + (\vec{\nabla}\psi\times\hat{z})\cdot\vec{\nabla}\vec{j} - (\vec{j}_\perp + \vec{j}_\parallel)\cdot\vec{\nabla}(b\hat{z} + \vec{\nabla}\psi\times\hat{z})$$

$$\hat{z}\cdot\vec{\nabla}\times(\vec{J}\times\vec{B}) \sim B_0\nabla_\parallel j + (\vec{\nabla}\psi\times\hat{z})\cdot\vec{\nabla}j - (\vec{J}\cdot\vec{\nabla})b$$

$$\frac{|\vec{\nabla}\psi\times\hat{z}\cdot\vec{\nabla}j|}{|B_0\nabla_\parallel j|} \sim \mathcal{O}(\varepsilon), \quad \frac{|\vec{J}\cdot\vec{\nabla}b|}{|B_0\nabla_\parallel j|} \sim \mathcal{O}(\varepsilon^2)$$

$$\therefore \hat{z}\cdot\vec{\nabla}\times(\vec{J}\times\vec{B}) \sim B_0\nabla_\parallel j$$