1. a) $J_z = \frac{\mu_0 I}{2\pi a} \cdot \frac{4}{\mu_0 a}\left(1 - r^2/a^2\right)$

$= B_{\theta a} \cdot \frac{4}{\mu_0 a}\left(1 - r^2/a^2\right)$

$B_{\theta a} = \frac{\mu_0 I}{2\pi a}$

$\hat{z} \cdot \left(\mu_0 \vec{J} = \vec{\nabla} \times \vec{B}\right) \rightarrow B_\theta = B_{\theta a}\left(2r - \frac{r^3}{a^2}\right)$

$\hat{r} \cdot \left(\vec{J} \times \vec{B} = \vec{\nabla} p\right) \rightarrow p = \frac{B_{\theta a}^2}{\mu_0}\left(\frac{5}{3} - 4\frac{r^2}{a^2} + 3\frac{r^4}{a^4} - \frac{2}{3}\frac{r^6}{a^6}\right)$

$\therefore p(0) = \frac{5}{3}\frac{B_{\theta a}^2}{\mu_0} = \frac{5\mu_0 I^2}{12\pi^2 a^2}$

b) ⓑ에서 묻고 있는 질문은 parabolic current profile이 아니더라도 성립하는 성질입니다. Z-pinch 구조에서는 general한 성질인 것으로 이해 할 수 있습니다.

$\hat{r} \cdot \left(\vec{J} \times \vec{B} = \vec{\nabla} p\right) \rightarrow \partial_r p = -\partial_r\left(\frac{B_\theta^2}{2\mu_0}\right) - \frac{B_\theta^2}{\mu_0 r}$

$\langle p \rangle = \frac{2}{a^2}\int_0^a p\, r\, dr$

$= \frac{2}{a^2}\left[\cancel{\frac{r^2}{2} p \Big|_0^a} - \int_0^a \frac{r^2}{2}\partial_r p\, dr\right]$

$= \frac{2}{a^2}\left[\int_0^a \frac{r^2}{2}\partial_r\left(\frac{B_\theta^2}{2\mu_0}\right) + r\frac{B_\theta^2}{2\mu_0}\, dr\right]$

$= \frac{2}{a^2}\left[\cancel{\int_0^a \frac{r^2}{2}\partial_r\left(\frac{B_\theta^2}{2\mu_0}\right) dr} + \frac{r^2}{2}\left(\frac{B_\theta^2}{2\mu_0}\right)\Big|_0^a - \cancel{\int_0^a \frac{r^2}{2}\partial_r\left(\frac{B_\theta^2}{2\mu_0}\right) dr}\right]$

$= \frac{B_{\theta a}^2}{2\mu_0}$ $\therefore \beta_p = 1$

2. $\hat{r} \cdot \left(\vec{J} \times \vec{B} = \vec{\nabla} p \right) \rightarrow \partial_r \left(p + \frac{B_\theta^2 + B_\varphi^2}{2\mu_0} \right) = - \frac{B_\theta^2}{\mu_0 r}$

$$\langle p \rangle = \frac{2}{a^2} \int_0^a r p \, dr$$

$$= \frac{2}{a^2} \left[\cancel{\frac{r^2}{2} p \Big|_0^a} - \int_0^a \frac{r^2}{2} \partial_r p \, dr \right]$$

$$= \frac{2}{a^2} \int_0^a \frac{r^2}{2} \left[\partial_r \left(\frac{B_\theta^2 + B_\varphi^2}{2\mu_0} \right) + \frac{B_\theta^2}{\mu_0 r} \right] dr$$

1번 참고.

$$= \frac{B_{\theta a}^2}{2\mu_0} + \frac{2}{a^2} \int_0^a \frac{r^2}{2} \partial_r \left(\frac{B_\varphi^2}{2\mu_0} \right) dr$$

$$= \frac{B_{\theta a}^2}{2\mu_0} + \frac{2}{a^2} \left[\frac{r^2}{2} \cdot \frac{B_\varphi^2}{2\mu_0} \Big|_0^a - \int_0^a r \frac{B_\varphi^2}{2\mu_0} dr \right]$$

$$= \frac{B_{\theta a}^2}{2\mu_0} + \frac{B_{\varphi a}^2}{2\mu_0} - \frac{\langle B_\varphi^2 \rangle}{2\mu_0}$$

$$\therefore \beta_p = 1 + \frac{B_{\varphi a}^2 - \langle B_\varphi^2 \rangle}{B_{\theta a}^2}$$

대에~충 thermal motion을 무시

Force-free cylindrical equilibrium $\rightarrow$ $p = 0$ $\rightarrow$ $\beta_p = 0$.

$\beta_p = 0 \rightarrow \langle B_\varphi^2 \rangle > B_{\varphi a}^2$

a paramagnetism of the plasma

3. $\vec{J}\times\vec{B} = \vec{\nabla}p + \rho\vec{\nabla}\Phi_g$

cf) $\vec{B} = \vec{B}_z + \vec{\nabla}\psi\times\hat{z}$

$\vec{J} = \frac{1}{\mu_0}\vec{\nabla}\times\vec{B}$

$$\begin{cases} \vec{\nabla}^2\Phi_g = -4\pi G\rho \\ \vec{J}\times\vec{B} = -\frac{1}{\mu_0}\left[\vec{\nabla}^2\psi\,\vec{\nabla}\psi + B_z\vec{\nabla}B_z\right] \end{cases}$$

$$\rightarrow\ -\frac{1}{\mu_0}\left[\vec{\nabla}^2\psi\,\vec{\nabla}\psi + B_z\vec{\nabla}B_z\right] = \vec{\nabla}p - \frac{\vec{\nabla}^2\Phi_g}{4\pi G}\vec{\nabla}\Phi_g$$

Including gravity...

Φ_g의 특성 자체가 전자기력보다는 Order가 낮으니 Leading order를 고려하면 $\vec{\nabla}B_z \sim \partial_\psi B_z\vec{\nabla}\psi$로 나올 것 같지만, 일단 살려서 풀게요.

$B_z(x,y) \rightarrow B_z(\psi,\Phi_g)$

$p(x,y) \rightarrow p(\psi,\Phi_g)$

$$\left[-\frac{1}{\mu_0}\left(\vec{\nabla}^2\psi + B_z\partial_\psi B_z\right) - \partial_\psi p\right]\vec{\nabla}\psi + \left[-\frac{1}{\mu_0}B_z\partial_{\Phi_g}B_z - \partial_{\Phi_g}p + \frac{\vec{\nabla}^2\Phi_g}{4\pi G}\right]\vec{\nabla}\Phi_g = 0$$

$\vec{\nabla}\psi$ & $\vec{\nabla}\Phi_g$ are linearly indep $\rightarrow$ each coeff $=0$.

$$\therefore\ -\frac{1}{\mu_0}\left(\vec{\nabla}^2\psi + B_z\partial_\psi B_z(\psi,\Phi_g)\right) - \partial_\psi p(\psi,\Phi_g) = 0$$

$$-\frac{1}{\mu_0}B_z\partial_{\Phi_g}B_z(\psi,\Phi_g) - \partial_{\Phi_g}p(\psi,\Phi_g) + \frac{\vec{\nabla}^2\Phi_g}{4\pi G} = 0$$

4. a) 프리드버그 & 강의 자료 참고.

b) $\vec{B} = \vec{B}_\phi + \hat{\phi} \times \frac{\vec{\nabla}\psi}{R}$ $\rightarrow$ (R, ϕ, z)

$= \vec{B}_\varphi + \frac{\vec{\nabla}\psi}{R} \times \hat{\varphi}$ $\rightarrow$ (r, θ, φ)

$$\frac{1}{R}\vec{\nabla}\psi = \hat{e}_r \frac{1}{R}\partial_r \psi + \hat{e}_\theta \frac{1}{Rr}\partial_\theta \psi$$

$$\vec{B} = \vec{B}_\varphi + \hat{e}_r \frac{1}{Rr}\partial_\theta \psi - \hat{e}_\theta \frac{1}{R}\partial_r \psi$$

$\rightarrow$ $B_r = \frac{1}{Rr}\partial_\theta \psi$ & $B_\theta = -\frac{1}{R}\partial_r \psi$

$$B_{\theta 1} = -\frac{1}{R}\cdot\frac{\mu_0 I}{2\pi}\left[\frac{r^2+a^2}{2r^2}\left(\beta_p + \frac{l_i - 1}{2}\right) + \frac{1}{2}\ln\frac{r}{a} + \frac{1}{2} + \Delta(a)\frac{R_0}{r^2}\right]\cos\theta$$

o-th order.

$\Leftarrow$ $B_\theta^{(0)}$ $\Rightarrow$ Indep of θ.

$$B_\theta\Big|_{r=c,\theta=\pi} - B_\theta\Big|_{r=c,\theta=0}$$

$$= \frac{\mu_0 I}{2\pi}\left[\frac{c^2+a^2}{2c^2}\left(\beta_p + \frac{l_i - 1}{2}\right) + \frac{1}{2}\ln\frac{c}{a} + \frac{1}{2} + \Delta(a)\frac{R_0}{c^2}\right]\left(\frac{1}{R_0 - c} + \frac{1}{R_0 + c}\right)$$

$$= \frac{2R_0}{R_0^2 - c^2}\cdot\frac{\mu_0 I}{2\pi}\left[\frac{c^2+a^2}{2c^2}\left(\beta_p + \frac{l_i - 1}{2}\right) + \frac{1}{2}\ln\frac{c}{a} + \frac{1}{2} + \Delta(a)\frac{R_0}{c^2}\right]$$

c) $$B_r = \frac{1}{Rr}\partial_\theta \psi$$

$$= \frac{1}{Rr}\left(\partial_\theta\left(\psi^{(0)} + \psi^{(1)}\right)\right)$$

$$= \frac{1}{Rr}\partial_\theta \psi^{(1)}$$

$$= -\frac{1}{Rr}\cdot\frac{\mu_0 I}{2\pi}\left[\frac{r^2-a^2}{2r}\left(\beta_p+\frac{l_i-1}{2}\right)+\frac{r}{2}\ln\frac{r}{a}-\Delta(a)\frac{R_0}{r}\right]\sin\theta$$

$$B_r\Big|_{r=c,\theta=\frac{\pi}{2}} = -\frac{1}{R_0 c}\cdot\frac{\mu_0 I}{2\pi}\left[\frac{c^2-a^2}{2c}\left(\beta_p+\frac{l_i-1}{2}\right)+\frac{c}{2}\ln\frac{c}{a}-\Delta(a)\frac{R_0}{c}\right]$$

d) $$\frac{a}{c}=\frac{1}{1+\delta}\sim 1-\delta \qquad \ln(1+x)\sim x$$

$$\Delta B_\theta^* = \frac{2R_0}{R_0^2-c^2}\cdot\frac{\mu_0 I}{2\pi}\left[\frac{c^2+a^2}{2c^2}\left(\beta_p+\frac{l_i-1}{2}\right)+\frac{1}{2}\ln\frac{c}{a}+\frac{1}{2}+\Delta(a)\frac{R_0}{c^2}\right]$$

$$\sim \frac{2}{R_0}\cdot\frac{\mu_0 I}{2\pi}\left[(1-\delta)\left(\beta_p+\frac{l_i-1}{2}\right)+\frac{1}{2}\delta+\frac{1}{2}+\Delta(a)\frac{R_0}{c^2}\right]$$

$$B_r^* = -\frac{1}{R_0 c}\cdot\frac{\mu_0 I}{2\pi}\left[\frac{c^2-a^2}{2c}\left(\beta_p+\frac{l_i-1}{2}\right)+\frac{c}{2}\ln\frac{c}{a}-\Delta(a)\frac{R_0}{c}\right]$$

$$\sim -\frac{1}{R_0}\cdot\frac{\mu_0 I}{2\pi}\left[(1+\delta)\left(\beta_p+\frac{l_i-1}{2}\right)+\frac{1}{2}\delta-\Delta(a)\frac{R_0}{c^2}\right]$$

$$\frac{\Delta B_\theta^*}{2}+B_r^* = \frac{\mu_0 I}{2\pi c}\left[2\frac{\Delta(a)}{c}+\frac{c}{R_0}\left(\frac{1}{2}-2\delta\left(\beta_p+\frac{l_i-1}{2}\right)\right)\right]$$

$$\sim \frac{\mu_0 I}{2\pi c}\cdot 2\cdot\frac{\Delta(a)}{c}$$

$$\frac{\Delta B_\theta^*}{2}-B_r^* = \frac{\mu_0 I}{2\pi R_0}\left[2\cdot\left(\beta_p+\frac{l_i-1}{2}\right)+\delta+\frac{1}{2}\right]$$

$$\sim \frac{\mu_0 I}{2\pi R_0}\cdot 2\cdot\left(\beta_p+\frac{l_i}{2}\right)$$

e) $\beta_p = 1 + \frac{B_{\phi a}^2 - \langle B_\phi^2 \rangle}{B_{\theta a}^2}$

$$B_{\phi a}^2 - \langle B_\phi^2 \rangle \sim 2 B_{\phi a} \left(B_{\phi a} - \langle B_\phi \rangle \right)$$

Assume : B_ϕ varies only weakly across the plasma.

In a large aspect ratio tokamak, $B_\phi = B_{\phi 0} \cdot \frac{R_0}{R} \sim B_{\phi 0}$.

$$\beta_p = 1 + \frac{2 B_{\phi a} \left(B_{\phi a} - \langle B_\phi \rangle \right)}{B_{\theta a}^2}$$

$B_{\phi a} \longleftarrow$ "B_ϕ" probe located at $r = c$, $\theta = \pi/2$

$\langle B_\phi \rangle \longleftarrow$ a large "diamagnetic" loop of $r = c$

5. $\partial_t \vec{B} = \vec{\nabla} \times (\vec{u} \times \vec{B}) + \frac{\eta}{\mu_0} \vec{\nabla}^2 \vec{B}$

$\partial_t A_1 = -iw A_1$, $\vec{\nabla} A_1 = i\vec{k} A_1$)

$\partial_t A_0 \neq -iwA_0$

$\vec{\nabla} A_0 \neq i\vec{k}A_0$

$\rightarrow \partial_t \vec{B}_1 = \left[(\vec{B}_0 \cdot \vec{\nabla}) \vec{u}_1 - \vec{u}_1 (\cancel{\vec{\nabla} \cdot \vec{B}_0}) - (\cancel{\vec{u}_1 \cdot \vec{\nabla}}) \vec{B}_0 + \vec{B}_0 (\cancel{\vec{\nabla} \cdot \vec{u}_1}) \right] + \frac{\eta}{\mu_0} \vec{\nabla}^2 \vec{B}_1$

$\rightarrow \left(-iw + \frac{\eta}{\mu_0} k^2\right) \vec{B}_1 = i(\vec{k} \cdot \vec{B}_0) \vec{u}_1$ ①

$\rho \frac{d}{dt} \vec{u} = -\vec{\nabla} p + \vec{J} \times \vec{B}$ & $\frac{d}{dt}\left(\frac{p}{\rho^\gamma}\right)$ & $\frac{d}{dt}\rho = 0$

$\rightarrow$ 열심히 정리하면

마찬가지

$\partial_t P_1 = 0$.

(Incompressible 하니까 당연)

$\rho_0 \partial_t^2 \vec{u}_1 = -\vec{\nabla} \partial_t P_1 + \frac{1}{\mu_0}\left((\vec{B}_0 \cdot \vec{\nabla}) \partial_t \vec{B}_1 - \vec{\nabla} \partial_t (\cancel{\vec{B}_0 \cdot \vec{B}_1})\right)$

$\rightarrow -\rho_0 w \vec{u}_1 = \frac{1}{\mu_0} (\vec{k} \cdot \vec{B}) \vec{B}_1$ ②

아마 $\vec{k} \cdot \vec{B}_0 = 0$ 이 typo이고 $\vec{B}_1 \cdot \vec{B}_0 = 0$ 인 듯 합니다.

① & ② 연결.

$-\mu_0 \rho_0 w \left(-iw + \frac{\eta}{\mu_0} k^2\right) \vec{u}_1 = i\, k_\parallel^2 B_0^2 \vec{u}_1$

$w^2 + iw \frac{\eta k^2}{\mu_0} - k_\parallel^2 V_A^2 = 0$

$w_r \sim k_\parallel V_A$

$w_i \sim -\frac{\eta k^2}{2\mu_0}$

Damping term by Resistivity sign$(w_i) < 0$