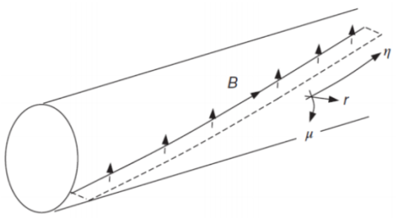
1. 수업자료 참고.

2. (a) Near a resonant surface → choose $\psi_0 = -\frac{x^2}{2}$.



Helical coordinates.

$$\hat{\mathbf{e}}_r = \hat{\mathbf{e}}_r$$

$$\hat{\mathbf{e}}_\mu = \frac{1}{\sqrt{1+(\frac{r}{R_0}\frac{1}{q_s})^2}}\left(\hat{\mathbf{e}}_\theta - \frac{r}{R_0}\frac{1}{q_s}\hat{\mathbf{e}}_z\right)$$

$$\hat{\mathbf{e}}_\eta = \frac{1}{\sqrt{1+(\frac{r}{R_0}\frac{1}{q_s})^2}}\left(\hat{\mathbf{e}}_z + \frac{r}{R_0}\frac{1}{q_s}\hat{\mathbf{e}}_\theta\right)$$

$$B_\mu \approx -B_\theta(r_s)\frac{q'(r_s)}{q(r_s)}(r-r_s)$$

$B_\mu \sim 0$ means $\psi_0'(r_s+x)\Big|_{x=0} \sim 0$

$\underline{\psi_0 = -\frac{x^2}{2}}$ choose ψ_0. with a proper normalization.

$$\partial_t \psi + \vec{u}\cdot\vec{\nabla}\psi = -\eta j$$

$$\rightarrow \gamma\psi_1 + ik\phi_1\,\partial_x\psi_0 = \eta\left(\partial_x^2 - k^2\right)\psi_1$$

$$\rightarrow \gamma\psi_1 - ikx\phi_1 = \eta\,\partial_x^2\psi_1 \quad \cdots (1)$$

$\partial_x\psi_0 = -x$

$\partial_x^2 - k^2 \sim \frac{1}{\varepsilon^2}\partial_X^2 - k^2 \sim \frac{1}{\varepsilon^2}\partial_X^2$

with $X = x/\varepsilon$, $O(\partial_x) \sim O(k)\sim 1$

(the narrowness of the layer)

$$\rho\,\partial_t w + \rho\,\vec{u}\cdot\vec{\nabla}w = \vec{B}\cdot\vec{\nabla}j$$

$$\rightarrow -\rho\gamma\left(\partial_x^2 - k^2\right)\phi_1 = \partial_x\psi_0\, ik\left(\partial_x^2-k^2\right)\psi_1 - ik\psi_1\,\partial_x^3\psi_0$$

$$\rightarrow \rho\gamma\,\partial_x^2\phi_1 = ikx\,\partial_x^2\psi_1 \quad \cdots (2)$$

b) $\gamma \chi_1 - i k x \phi_1 = \eta \, \partial_x^2 \chi_1$ ··· (1)

$\rho \gamma \, \partial_x^2 \phi_1 = i k x \, \partial_x^2 \chi_1$ ··· (2)

$\rightarrow \partial_x^2 \phi_1 - \left(\frac{k^2}{\gamma \eta \rho}\right) x^2 \phi_1 = - \left(- \frac{ik}{\eta \rho}\right) x \chi_1$

$\rightarrow \frac{d^2 y}{dx^2} - \left(\frac{k^2}{\gamma \eta \rho}\right) x^2 y = - \left(- \frac{ik}{\eta \rho}\right) x$ where $y = \phi_1 / \chi_1$

Let $X = x/\delta$, $Y = y/\beta$,

$$\frac{d^2 Y}{dX^2} - X^2 Y = - X \quad \text{with} \quad \delta = \left(\frac{\gamma \eta \rho}{k^2}\right)^{1/4}$$

$$\beta = - \frac{ik}{\eta \rho} \delta^3 = - i \frac{\gamma^{3/4}}{\eta^{1/4} \rho^{1/4} k^{1/2}}$$

$$Y = \frac{X}{2} \int_0^1 dt \, e^{-X^2 t/2} (1 - t^2)^{-1/4}$$

(1) $\rightarrow \frac{1}{\chi_1} \partial_x^2 \chi_1 = \frac{\delta^2 \gamma}{\eta} (1 - XY)$ $\quad \epsilon \sim \mathcal{O}(\delta)$

$$\Delta' = \int_{-\epsilon}^{\epsilon} \frac{1}{\chi_1} \partial_x^2 \chi_1 \, dx = \frac{\gamma \delta}{\eta} \int_{-\infty}^{\infty} (1 - XY) \, dX$$

$$= \frac{\gamma \delta}{\eta} \left(\frac{\pi \Gamma(3/4)}{\Gamma(1/4)} \right)$$

Note δ is real length quantity.
So, $\mathrm{sign}(\Delta') = \mathrm{sign}(\gamma)$

$$\gamma = \left(\frac{\Gamma(1/4)}{\pi \Gamma(3/4)} \right)^{4/5} \Delta'^{4/5} \eta^{3/5} k^{2/5} \rho^{-1/5}$$

$B_0 = 1$, $\mu_0 = 1$
$\tau_A = \rho^{1/2} r_s$
$\tau_R = r_s^2 / \eta$

$$= \left(\frac{\Gamma(1/4) r_s^2}{\pi \Gamma(3/4)} \right)^{4/5} k^{2/5} \Delta'^{4/5} \tau_R^{-3/5} \tau_A^{-2/5}$$

c) Outer Region $\rightarrow \vec{B}\cdot\vec{\nabla} j \sim 0$

$$\rightarrow \partial_x \psi_0\, ik \left(\partial_x^2 - k^2\right)\psi_1 \sim 0.$$

$$\rightarrow \psi_1 = \psi_{10}\left(\cosh kx - \frac{\sinh k|x|}{\tanh ka}\right) + \xi \frac{\sinh k|x|}{\sinh ka}$$

$$\Delta' = \frac{1}{\psi_0}\left(\psi_1'\big|_{+\varepsilon} - \psi_1'\big|_{-\varepsilon}\right)$$

$$= -\frac{2k}{\tanh ka}\left(1 - \xi \cosh ka\right)$$

If $\xi = 0 \rightarrow \Delta' < 0$ & $\gamma < 0$

Stable

If $\xi \neq 0$

$\xi > \operatorname{sech} ka \rightarrow \Delta' > 0$ & $\gamma > 0$

Unstable

$\xi < \operatorname{sech} ka \rightarrow \Delta' < 0$ & $\gamma < 0$

Stable.