Fusion Plasma Theory I

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Homework 2

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1. [Drift under Time-varying Magnetic Field] Suppose a particle with the charge q and mass m gyrating along with a spatially homogenous magnetic field $\vec{B} = B\hat{z}$ and drifting under a constant electric field $\vec{E}$.

(a) Let the magnetic field varying in time linearly $B(t) = (1 + \alpha t)B_0$ where $\alpha \ll \omega_c$ and $\omega_c = |q|B_0/m$ is the gyrofrequency with respect to the magnetic field B_0. Estimate the total drift including the $\vec{E} \times \vec{B}$ drift. Ignore the effect by induced electric field by $\vec{\nabla} \times \vec{E} = -\partial\vec{B}/\partial t$.

(b) Take the leading order correction in α/ω_c. Could this drift be deduced from the so-called inertia drift, $(m/qB^2)(\vec{B} \times d\vec{v}_0/dt)$? Here, $\vec{v}_0$ and d/dt includes the drift by the parallel motion and $\vec{E} \times \vec{B}$.

2. [μ-invariance up to the First Order] The small m/q asymptotic expansion on the equation of a single particle motion with the charge q, mass m, and its gyro-frequency ω_c, and its weighted gyro-average, leads to

$$2\omega_c\left\langle \frac{\partial\vec{\rho}}{\partial\gamma} \cdot \frac{\partial^2\vec{\rho}}{\partial t\partial\gamma} \right\rangle + \frac{d\omega_c}{dt}\left\langle \frac{\partial\vec{\rho}}{\partial\gamma} \cdot \frac{\partial\vec{\rho}}{\partial\gamma} \right\rangle + \frac{1}{2}\omega_c(\vec{v}_0 \cdot \vec{\nabla})\left\langle \frac{\partial\vec{\rho}}{\partial\gamma} \cdot \frac{\partial\vec{\rho}}{\partial\gamma} \right\rangle \tag{1}$$

$$= \frac{q}{m}\left(\left\langle \frac{\partial\vec{\rho}}{\partial\gamma} \cdot (\vec{\rho} \cdot \vec{\nabla})\vec{E}_\perp \right\rangle + \left\langle \frac{\partial\vec{\rho}}{\partial\gamma} \cdot \frac{\partial\vec{\rho}}{\partial t} \times \vec{B} \right\rangle + \left\langle \frac{\partial\vec{\rho}}{\partial\gamma} \cdot \vec{v}_0 \times (\vec{\rho} \cdot \vec{\nabla})\vec{B} \right\rangle \right). \tag{2}$$

Here $\langle A \rangle = (1/2\pi)\oint A d\gamma$, where the γ is the phase of the gyro-motion, i.e. $\vec{\rho} = \rho(t)(\hat{x}\sin\gamma + \hat{y}\cos\gamma)$ where the gyro-radius $\rho = mv_\perp/qB$, with respect to the magnetic field $\vec{B} = B\hat{z}$. Show that this implies the invariance of $\mu = mv_\perp^2/2B$ up to the first-order convective drift $\vec{v}_0$, i.e. $d\mu/dt = 0$.

3. [Drift from Guiding-Center Lagrangian] Show that the $\vec{E} \times \vec{B}$, curvature, and $\vec{\nabla}B$ drifts are all included in the compact expression

$$\vec{v}_d = \frac{v_\parallel}{B}\left(\vec{B} + \vec{\nabla} \times (\rho_\parallel\vec{B})\right) \tag{3}$$

under the fixed $U = mv_\parallel^2/2 + \mu B + q\phi$, where $\rho_\parallel = mv_\parallel/qB$ and ϕ is the electric potential. There is in fact a remainder which is however often ignored.

4. [Magneto-Electric Mirror][1] Consider a magneto-electric particle trap in the region $-L < z < L$. To accomplish this trap, suppose a magnetic field in the z direction such that

$$B = \begin{cases} B_0 \left(1 + (R-1)(\frac{z}{L_m})^2\right), & \text{if } -L_m < z < 0; \\ B_0, & \text{if } z \geq 0. \end{cases}$$

Suppose also an electric potential

$$\phi = \begin{cases} 0, & \text{if } z < 0; \\ \phi_0 \left(\frac{z}{L_e}\right)^2, & \text{if } 0 \leq z < L_e; \\ \phi_0, & \text{if } z > L_e. \end{cases}$$

(a) Describe how ions might be trapped in this configuration of magnetic and electric fields. Would electrons also be trapped in the same fields?

(b) Derive a trapping condition for confined particles in terms of the particle midplane perpendicular energy $W_{\perp 0}$ and midplane parallel energy $W_{\parallel 0}$, where these energies are defined at the axial location $z = 0$.

(c) If trapped ions of charge state q were scattered in pitch-angle, but not in energy, through collisions, from what end of the device would they leave? How does this answer depend on the midplane energy coordinates $W_{\perp 0}$ and $W_{\parallel 0}$?

(d) Suppose now that the electric potential is a varying function of time. Show that the second adiabatic invariant can be put in the form

$$W_{\parallel 0}^{1/2} \left(z_M + z_E\right) = \text{const.}$$

Here z_M and z_E are the turning points in the regions $z < 0$ and $z > 0$ respectively. What are z_M and z_E in terms of the parameters L_e, L_m, R, $W_{\perp 0}$, and $W_{\parallel 0}$. Define $W_c \equiv q\phi_0/(R-1)$. Show that, if $W_{\perp 0}/W_c \sim O(1)$, then $L_e \gg L_m$ implies $z_e \gg z_m$.

(e) Suppose that the length $L_e(t)$ slowly changes in time, but assume that $L_e(t) \gg L_m$ for all t. Show that, if $L_e(t)$ is slowly shortened from $t = 0$ to $t = t_0$, such that $L_e(0)/L_e(t_0) = \alpha > 1$, then there is a region in $W_{\perp 0} - W_{\parallel 0}$ space (where coordinates are given at $t = 0$), such that any ions in that region will escape on a different side of the trap by the time $t = t_0$, than they otherwise would have eventually escaped by rare but finite pitch angle scattering had the trap potential not been altered ($\alpha = 1$). Show that this region is triangular in shape with area

$$A \simeq \frac{1}{2}(q\Phi_0)^2 \left(1 - \alpha^{-1}\right)^2.$$

[1] One of the problems issued for the 2020 General examinations in the Princeton plasma physics program.