Fusion Plasma Theory I

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Homework 3

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1. [**Electron MHD**] Electron MHD (E-MHD) model assumes immobile ions and so that electrons carry all electric current under magnetic field $\vec{B}$.

(a) Assuming uniform density n, isotropic resistivity η, and incompressible electron flow $\vec{u}_e$, derive the magnetic diffusion equation for E-MHD;

$$\frac{\partial}{\partial t}\left(\vec{B} - d_e^2 \nabla^2 \vec{B}\right) = \vec{\nabla} \times \left[\vec{u}_e \times \left(\vec{B} - d_e^2 \nabla^2 \vec{B}\right)\right] + \frac{\eta}{\mu_0} \nabla^2 \vec{B}, \tag{1}$$

where d_e is electron skin depth you can define.

(b) Discuss physical insights that you can develop from these equations for the ideal E-MHD model.

2. [**Energy Conservation in MHD (GR: Problem 8.2)**] Using the single-fluid MHD momentum equation

$$\rho \frac{d\vec{u}}{dt} = \sigma \vec{E} + \vec{j} \times \vec{B} - \vec{\nabla} p \tag{2}$$

and assuming ideal fluid frozen to magnetic fields, i.e. $\vec{E} + \vec{u} \times \vec{B} = 0$, deduce the energy conservation equation;

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho_m u^2 + \frac{p}{\gamma - 1} + \frac{B^2}{2\mu_0} + \frac{E^2}{2\epsilon_0}\right) + \vec{\nabla} \cdot \left(\frac{1}{2}\rho_m u^2 \vec{u} + \frac{\gamma}{\gamma - 1} p \vec{u} + \frac{1}{\mu_0} \vec{E} \times \vec{B}\right) = 0. \tag{3}$$

Note: This problem is closely related to the discussions in the Ch 8.6 and the Problem 8.2 in Goldston&Rutherford's book.

3. [Fluid vs. Particle Drift] Show that the charge accumulation due to the drift motions implied in fluid picture becomes equivalent to one in single particle picture. Specifically, show $\vec{\nabla} \cdot \vec{v}_p = \vec{\nabla} \cdot (\vec{v}_B + \vec{v}_{curv})$, where each drift $\vec{v}$ is the diamagnetic, gradient-B, and curvature drift, respectively, under a vacuum magnetic field $\vec{B}$.

4. [HARD: Three-field Equations] Suppose that the magnetic field $\vec{B} = B_0\hat{z} + \vec{\nabla}\psi \times \hat{z}$ and the ion fluid velocity $\vec{u}_i = u_{\parallel}\hat{z} + \vec{u}_E$ where $\vec{u}_E = \vec{\nabla}\phi \times \hat{z}$ in a slab geometry. Here, $\psi = \psi(x, y)$ and $\phi = \phi(x, y)$, and B_0 is the constant magnetic field and $u_{\parallel}$ is the constant parallel flow in equilibrium. Assume the ion fluid velocity is the same as the single fluid velocity, and the charge neutrality $n_e = n_i = n$. Let $T_e = T$ constant, and $T_i = 0$ (cold ion assumption).

Using the electron continuity equation, the parallel Ohm's law, and the equation of single fluid motion in the perpendicular direction (i.e. taking the operator $\hat{z} \cdot \vec{\nabla}\times$), drive the following three-field $(n, \psi, \omega(\phi))$ closure equations. You will have to use $\vec{E}_z = -\hat{z} \cdot \partial\vec{A}/\partial t$ where $\vec{B} = \vec{\nabla} \times \vec{A}$, and also will need to assume $\vec{\nabla} \cdot \vec{j} = 0$ while having $\vec{j} \approx \hat{z}j$, and then $\vec{\nabla}n \times (\vec{j} \times \vec{B}) \approx 0$. These last two assumptions are ultimately related to a specific ordering which separates these non-ideal evolution from the faster ideal equilibrium response. Ignore electron mass, anisotropic pressure, and also all source terms.

$$\frac{\partial n}{\partial t} + \vec{u}_E \cdot \vec{\nabla}n = \frac{1}{e}\nabla_{\parallel}j - u_{\parallel}\nabla_{\parallel}n, \tag{4}$$

$$\frac{\partial \psi}{\partial t} + \vec{u}_E \cdot \vec{\nabla}\psi = -\eta j + \frac{T}{ne}\nabla_{\parallel}n, \tag{5}$$

$$\rho_m\frac{\partial \omega}{\partial t} + \rho_m\vec{u}_E \cdot \vec{\nabla}\omega = B_0\nabla_{\parallel}j, \tag{6}$$

where the parallel current $j = -\nabla^2\psi$, and the vorticity $\omega \equiv -\nabla^2\phi$. Note that $\nabla_{\parallel} = \partial/\partial z$ simply in our geometry, but generally means the gradient (almost) parallel to the equilibrium magnetic field.