Fusion Plasma Theory I

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Homework 4

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1. [Z-Pinch Equilibrium] Consider an straight cylinder Z-pinch configuration with $B_z = 0$, $J_\theta = 0$ but with the axial current $J_z = (4I/2\pi a^2)(1 - r^2/a^2)$ that confines plasma within $0 < r < a$. Here I is the total axial current and a is the minor radius.

(a) Obtain the peak pressure at the center that can be achieved in this pinch.

(b) Show that the $\beta_p \equiv 2\mu_0 \langle p \rangle / B_{\theta a}^2 = 1$, where $\langle . \rangle$ is the volume average and $B_{\theta a}$ is the poloidal field at the plasma boundary.

2. [Beta-Poloidal in Screw Pinch] Show

$$\beta_p \equiv \frac{\langle p \rangle}{B_{\theta a}^2 / 2\mu_0} = 1 + \frac{B_{\varphi a}^2 - \langle B_\varphi^2 \rangle}{B_{\theta a}^2}, \tag{1}$$

in a straight cylinder plasma equilibrium, where $B_{\varphi a}$ and $B_{\varphi 0}$ are the toroidal field at the plasma boundary and the center, respectively. This relation in fact indicates that any Z-pinch equilibrium will always hold $\beta_p = 1$ independent of the current profiles. Also, argue based on the relation that any force-free cylindrical equilibrium (i.e. $p = 0$) implies a paramagnetism of the plasma.

3. [MHD Equilibrium under Gravity] Derive the Grad-Shafranov equation describing the magnetostatic equilibrium of a plasma embedded in a gravitational field in Cartesian coordinates (x, y, z), assuming that the coordinate z is ignorable. Note that the force balance is now given by

$$\vec{j} \times \vec{B} - \vec{\nabla} p - \rho \vec{\nabla} \Phi_g = 0, \tag{2}$$

where ρ is the mass density and $\Phi_g = \Phi_g(x, y)$ is a prescribed gravitational potential function.

4. [Toroidal Effects in Equilibrium - Freidberg 6.7] Consider an ohmically heated tokamak plasma held in equilibrium by a vertical field B_v, with the total toroidal current I, the major radius R_0 and minor raduis a.

(a) Show that the toroidal correction to the flux function $\psi_1(r, \theta)$, evaluated at a minor radius $r > a$ in the vicinity of the vacuum chamber, is given by

$$\psi_1(r, \theta) = \frac{I}{2\pi} \left[\frac{r^2 - a^2}{2r} \left(\beta_p + \frac{l_i - 1}{2} \right) + \frac{r}{2} \ln \frac{r}{a} - \Delta(a) \frac{R_0}{r} \right] \cos\theta, \tag{3}$$

where the beta-poloidal β_p and the internal inductance l_i are defined as usual and normalized by the poloidal field at the plasma boundary (i.e. $B_{\theta a}$), and the $\Delta(a)$ is the radial (Shafranov) shift of the circular plasma boundary.

(b) Calculate the difference in signals from two "B_θ" probes located at $r = c$, $\theta = 0$ and $r = c, \theta = \pi$.

(c) Calculate the signal on a "B_r" probe located at $r = c$, $\theta = \pi/2$.

(d) Assuming $c = a(1 + \delta)$ with $\delta \ll 1$, show how the "B_r" and "B_θ" signals can be combined to give separate measurements of $\Delta(a)$ and $\beta_p + l_i/2$.

(e) Consider now a "B_ϕ" probe located at $r = c$, $\theta = \pi/2$, and then a large "diamagnetic" loop of radius $r = c$, surrounding the vacuum chamber. Discuss how these two sensors can be used to determine β_p. This means you can estimate l_i as well by combining the technique discussed in (d).

5. [Shear Alfven Wave Damping] Obtain the changes in the dispersion relation for shear Alfven waves due to a finite resistivity η. The waves propagate through an infinite homogeneous medium with the mass density ρ_0, the pressure p_0, the magnetic field $\vec{B}_0$, but without flow in equilibrium. Assume $\vec{k} \cdot \vec{B}_0 = 0$ and incompressibility in the first place for simplicity. Include the finite resistive effects only through the Ohm's law and so resulting magnetic induction equation, i.e.

$$\frac{\partial \vec{B}}{\partial t} = \vec{\nabla} \times (\vec{u} \times \vec{B}) + \frac{\eta}{\mu_0} \nabla^2 \vec{B}. \tag{4}$$