Fusion Plasma Theory I

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Homework 5

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1. [Z-Pinch Stability] Consider a straight cylinder Z-pinch configuration with $B_z = 0$, $J_\theta = 0$ but with the axial "uniform" current density $\vec{j} = j\hat{z}$ that confines plasma within $0 < r < a$. Review the stability criteria for Z-pinch (due to Kadomtsev as discussed in the class) and find all the poloidal mode numbers m that can make this pinch unstable internally.

2. [Resistive Tearing Modes in Slab] Consider a slab representation near a resonant surface where the magnetic field can be modeled by $\vec{B} = \vec{\nabla}\psi \times \hat{z} + B_0\hat{z}$. In equilibrium, there is no flow but a finite current $j_z = j_0$ for $x < |a|$ that generates a flux $\psi_0(x)$. Now add a magnetic perturbation with $\psi(\vec{x}, t) = \psi_0(x) + \psi_1(x)e^{iky+\gamma t}$ and assume a perturbed incompressible flow $\vec{u} = \vec{\nabla}\phi_1(\vec{x}, t) \times \hat{z}$. Ignore a variation along the strong constant magnetic field B_0 (i.e. $k_z = 0$). The time evolutions of the two fields (ψ, ϕ) are then determined by the following Ohm's law and vorticity equation;

$$\frac{\partial\psi}{\partial t} + \vec{u}\cdot\vec{\nabla}\psi = -\eta j, \tag{1}$$

$$\rho\frac{\partial\omega}{\partial t} + \rho\vec{u}\cdot\vec{\nabla}\omega = \vec{B}\cdot\vec{\nabla}j, \tag{2}$$

where $j = -\nabla^2\psi$ and $\omega = -\nabla^2\phi$, assuming the constant mass density ρ.

These equations are the same as the HW#3.4 with just slightly different notations. In fact, the RHS term in Eq. (6) in HW#3.4 should be replaced by the RHS in Eq. (2). Also the last statement including $\nabla_\parallel = \partial/\partial z$ in HW#3.4 is incorrect as it should really mean the parallel gradient along the total magnetic field including a perturbation.

(a) Derive the coupled differential equations for $\psi_1(x)$ and $\phi_1(x)$ for inner resonant layer, assuming the narrowness of the layer. These equations are essentially the same as the ones shown in the class, or many other literature such as the Rutherford's book.

(b) Using the constant-ψ approximation across the inner-layer and matching condition $\Delta' = d\ln\psi/dx|_-^+$, obtain the expression for the growth rate γ. Note that the appropriate solution for $Y'' - X^2Y = -X$ is given by $Y = (X/2)\int_0^1 dt e^{-X^2t/2}(1-t^2)^{-1/4}$.

(c) Let $\psi_1(\pm a) = \zeta$. Solve the outer-layer solution for $\psi_1(x)$ to calculate Δ', assuming the ideal MHD equilibrium except the layer near $x = 0$. Show that the tearing mode is internally stable, i.e. when $\zeta = 0$, but could be unstable when it is forced by $\zeta \neq 0$.

3. [**Optional Problem for Gravitational Instabilities**] An ideal MHD plasma couples half of the space from $z = 0$ and $z = \infty$, and is supported against gravity by a superconducting and solid boundary at $z = 0$. Mass density ρ, pressure p, and magnetic field strength are functions of z only, and the static equilibrium is given by

$$\frac{d}{dz}\left(p + \frac{B^2}{2\mu_0}\right) + \rho g = 0, \tag{3}$$

where $\vec{B}$ is in the x direction and $\vec{g}$ is a constant gravity pointing in the $-z$ direction.

(a) Assume that the displacement vector is given by $\vec{\xi} = (\xi_x, \xi_y, \xi_z)e^{ik_x x + ik_y y}$. Let $\zeta \equiv i\xi_x$, $\eta \equiv i\xi_y$, and $\xi \equiv \xi_z$ are all real functions of z for simplicity. Note ξ_z is out of the phase of ξ_x and ξ_y. Show the perturbed potential energy is given by

$$\delta W = -\frac{1}{2}\int_0^\infty dz \vec{\xi}^* \cdot \vec{F}[\vec{\xi}] = \frac{1}{2}\int_0^\infty dz \left\{ \frac{B^2}{\mu_0}\left[k_x^2(\xi^2 + \eta^2) + \left(\frac{d\xi}{dz} + k_y \eta\right)^2\right] \right.$$
$$\left. + \gamma p \left(\frac{d\xi}{dz} + k_x\zeta + k_y\eta\right)^2 - 2\rho g \xi \left(\frac{d\xi}{dz} + k_x\zeta + k_y\eta\right) - g\xi^2 \frac{d\rho}{dz}\right\}. \tag{4}$$

You may want to start with the intuitive form for δW (Friedberg Ch. 8) and add the gravity contribution, and treat (ζ, η, ξ) as real variables.

(b) When $k_x = 0$, only interchange instabilities are possible and ζ drops out from δW. First show in this case that the energy integral reduces to

$$\delta W = \frac{1}{2}\int_0^\infty dz \left[-g\xi^2\left(\frac{d\rho}{dz} + \frac{\rho^2 g}{\gamma p + B^2/\mu_0}\right) + \left(\gamma p + \frac{B^2}{\mu_0}\right)\left(\frac{d\xi}{dz} + k_y\eta - \frac{\rho g \xi}{\gamma p + B^2/\mu_0}\right)^2\right]. \tag{5}$$

What is the necessary and sufficient condition for interchange stability? Show also that the condition reduces to $\vec{g} \cdot \vec{\nabla} \ln(\rho/B) \geq 0$ in the cold plasma limit.

(c) When $k_x \neq 0$, show the energy integral can be expressed as

$$\delta W = \frac{1}{2}\int_0^\infty dz \left[\left(k_x^2 B^2 - \frac{\rho^2 g^2}{\gamma p} - g\frac{d\rho}{dz}\right)\xi^2 + \frac{k_x^2 B^2}{k_x^2 + k_y^2}\left(\frac{d\xi}{dz}\right)^2 \right.$$
$$\left. + \gamma p\left(\frac{d\xi}{dz} + k_x\zeta + k_y\eta - \frac{\rho g \xi}{\gamma p}\right)^2 + (k_x^2 + k_y^2)B^2\left(\eta + \frac{k_y(d\xi/dz)}{k_x^2 + k_y^2}\right)^2\right]. \tag{6}$$

Then, show that plasma can be unstable even if the interchange stability condition in (b) is satisfied. It is called Parker instability (1966) as well known in astrophysical science.