Inviscid Flows
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Homework #1

1. Using usual rules of index notation (i, j, k = 1,2,3), evaluate the following expressions.

(i) $\delta_{ij}\delta_{ij}$ (ii) $\varepsilon_{ijk}\varepsilon_{kji}$ (iii) $\varepsilon_{ijk}a_i a_k$ (iv) $\varepsilon_{ijk}\dfrac{\partial^2\phi}{\partial x_i \partial x_j}$ (v) $a_i b_j \delta_{ij}$

2. Prove the following vector identities that, among other ideas, extend the chain rule to vector operations. Here, $a(x)$ and $b(x)$ are vector functions of position x; $\phi(x)$ is a scalar function of position.

(i) $\nabla\cdot(\phi a) = \phi\nabla\cdot a + a\cdot\nabla\phi$

(ii) $\nabla\times(\phi a) = \phi\nabla\times a + (\nabla\phi)\times a$

(iii) $\nabla\cdot(\nabla\times a) = 0$

(iv) $\nabla\times(\nabla\phi) = 0$

(v) $\nabla\times(a\times b) = (b\cdot\nabla)a - b(\nabla\cdot a) - (a\cdot\nabla)b + a(\nabla\cdot b)$

(vi) $\nabla\cdot(a\times b) = (\nabla\times a)\cdot b - a\cdot(\nabla\times b)$

(vii) $\nabla\cdot(ab) = (\nabla\cdot a)b + a\cdot(\nabla b)$

(viii) $\nabla\times(\nabla\times a) = \nabla(\nabla\cdot a) - \nabla^2 a$

3. Prove followings.

(i) If C is a second-order tensor and a is an arbitrary vector, prove $a\cdot C = C\cdot a$ if and only if $C = C^T$, i.e., C is symmetric.

(ii) If $C = -C^T$ (C is anti-symmetric), then show $a\cdot C\cdot a = 0$.

(iii) If B is an anti-symmetric second-order tensor, show that $B \cdot B$ is a symmetric second-order tensor.

(iv) Verify $(A \cdot B)^T = B^T \cdot A^T$ using index notation.

4. If x is the position vector $x = (x_1, x_2, x_3) = (x, y, z)$, then evaluate:

(i) $\nabla \cdot x$ (ii) $\nabla \times x$ (iii) ∇x

5. State whether the following equations are correctly written or not. If the index notation is incorrect, briefly state why it is incorrect.

(i)

$$\frac{\partial \phi}{\partial t} = a_k \frac{\partial \phi}{\partial x_k} + \phi \frac{\partial a_k}{\partial x_k} + \nu \frac{\partial^2 a_k}{\partial x_k \partial x_k} + \phi g_k b_k$$

(ii)

$$\frac{\partial a_k}{\partial t} = -u_j \left(\frac{\partial a_k}{\partial x_j} - \frac{\partial a_j}{\partial x_k} \right) + A_{ij} \frac{\partial a_k}{\partial x_i} \frac{\partial a_k}{\partial x_j} + \frac{1}{\rho} \frac{\partial}{\partial x_j} \left(\mu \frac{\partial a_k}{\partial x_j} \right) + g_j b_k$$

(iii)

$$\frac{\partial R_{ij}}{\partial t} = -u_k \frac{\partial R_{ij}}{\partial x_k} - R_{ik} \frac{\partial u_j}{\partial x_k} - R_{jk} \frac{\partial u_i}{\partial x_k} + p \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$
$$+ \phi \left(\frac{\partial u_i}{\partial x_k} - \frac{\partial u_k}{\partial x_i} \right) \left(\frac{\partial u_j}{\partial x_\ell} - \frac{\partial u_\ell}{\partial x_j} \right) + \nu \frac{\partial u_i}{\partial x_k} \frac{\partial u_j}{\partial x_\ell} \left(\frac{\partial a_\ell}{\partial x_k} + \frac{\partial a_k}{\partial x_\ell} \right) - g \delta_{ij}$$

(iv)

$$\frac{\partial}{\partial t}(\rho \phi) + \frac{\partial}{\partial x_k}(\rho \phi u_k) = \nu \frac{\partial \rho}{\partial x_k} \frac{\partial \phi}{\partial x_k} + a \rho \phi \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) + \rho u_k f_k$$

(v)

$$\frac{\partial}{\partial t}(\rho a_j) + \frac{\partial}{\partial x_k}(\rho a_j u_k) = \beta \frac{\partial a_i}{\partial x_k} \frac{\partial u_j}{\partial x_k} + \mu \frac{\partial^2 a_j}{\partial x_k \partial x_k} + \rho f_i$$

(vi)

$$\frac{\partial}{\partial t}(\rho b_{ij}) + \frac{\partial}{\partial x_k}(\rho b_{ij} u_k) = \alpha \left(T_{ij} + T_{ji} \right) + \gamma \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$
$$+ \varepsilon_{ijk} \mu \frac{\partial^2 b_{\ell\ell}}{\partial x_k \partial x_k} + \rho f_i g_j$$