

Engineering Economic Analysis

Spring 2019

Problem set 3 Solution

Due: 2019.05.28

1.

Let σ be the elasticity of substitution

$$\begin{aligned}\sigma &= \frac{TRS}{(x_2/x_1)} \frac{d(x_2/x_1)}{dTRS} = \frac{d \ln(x_2/x_1)}{d \ln|TRS|} \\ TRS &= -\frac{\partial f / \partial x_1}{\partial f / \partial x_2} = -\frac{1/\rho(a_1 x_1^\rho + a_2 x_2^\rho)^{1/\rho-1}(\rho a_1 x_1^{\rho-1})}{1/\rho(a_1 x_1^\rho + a_2 x_2^\rho)^{1/\rho-1}(\rho a_2 x_2^{\rho-1})} = -\frac{a_1}{a_2} \left(\frac{x_1}{x_2}\right)^{1-\rho} \\ \therefore \ln\left(\frac{x_1}{x_2}\right) &= \frac{1}{1-\rho} \left\{ \ln\left(\frac{a_2}{a_1}\right) + \ln|TRS| \right\} \\ \therefore \sigma &= \frac{d \ln(x_2/x_1)}{d \ln|TRS|} = \frac{1}{1-\rho}\end{aligned}$$

2.

(a)

$$c(q) = 0.04q^3 - 0.8q^2 + 10q + 5$$

$$AVC = \frac{VC}{q} = 0.04q^2 - 0.8q + 10$$

$$\frac{dAVC}{dq} = 0.08q - 0.8 = 0$$

Hence its minimum point is at $q = 10$, and $AVC = 6$.

$$MC = 0.12q^2 - 1.6q + 10 - p = 0$$

$$\text{Using quadratic formula, } q = \frac{20 \pm 5\sqrt{3p-14}}{3}$$

But, $q = \frac{20-5\sqrt{3p-14}}{3}$ cannot be a solution.

$$\text{Thus, } S(p) = \begin{cases} \frac{20+5\sqrt{3p-14}}{3} & \text{if } p \geq 6 \\ 0 & \text{if } p < 6 \end{cases}$$

(b)

$$c(q) = q^3 - 4q^2 + 8q$$

$$AC(q) = q^2 - 4q + 8$$

$$\frac{\partial AC}{\partial q} = 2q - 4 = 0$$

$$\therefore q = 2$$

$$\text{Minimum of } AC(q) = 4.$$

3.

(a) We want to maximize $20x - x^2 - wx$. The F.O.C. is $20 - 2x - w = 0$

(b) For the optimal x to be zero, the derivative of profit with respect to x must be non-positive at $x=0$: $20 - 2x - w < 0$ when $x=0$, or $w \geq 20$.

(c) The optimal x will be 10 when $w=0$.

(d) The factor demand function is $x=10-w/2$, or, to be more precise, $x=\max\{10-w/2, 0\}$

(e) Profits as a function of output are $20x - x^2 - wx$. Substitute $x=10-w/2$ to find $\pi(w) = \left[10 - \frac{w}{2}\right]^2$

(f) The derivative of profit with respect to w is $-(10-w/2)$, which is, of course, the negative of the factor demand.

4.

The F.O.C. is $p/x=w$, which gives us the demand function $x=p/w$ and the supply function $y=\ln(p/w)$. The profits from operating at this point are $p\ln(p/w)-p$. Since the firm can always choose $x=0$ and make zero profits, the profit function becomes $\pi(p, w) = \max\{p \ln(p/w) - p, 0\}$.

5.

$$c'(y_1) = y_1, c'(y_2) = 1$$

$$\text{Let } y_1 + y_2 = Y$$

$$\text{Then, when } y_1 < 1, MC_1 < MC_2$$

$$\text{Plant 1 supply only when } y_1 > 1, \text{ or } MC_1 > MC_2$$

Plant 2 supply only when plant 1 supply $\frac{1}{2}$ and plant 2 make others.

$$\therefore c(y) = \begin{cases} \frac{1}{2}y^2 & \text{if } y \leq 1 \\ y - \frac{1}{2} & \text{if } y > 1 \end{cases}$$

6.

If $p=2$, the firm will produce 1 unit of output.

If $p=1$, the F.O.C. suggests $y=1/2$, but this yields negative profits. The firm can get zero profits by choosing $y=0$.

The profit function is $\pi(p) = \max\left\{\frac{p^2}{4} - 1, 0\right\}$

7.

(a)

$$c(tw, y) = y^{1/2} (tw_1 tw_2)^{3/4} = t^{3/2} \left(y^{1/2} (w_1 tw_2)^{3/4} \right) = t^{3/2} c(\mathbf{w}, y)$$

→ Not homogeneous.

$$\frac{\partial c}{\partial w_1} = \frac{3}{4} y^{1/2} w_1^{-1/4} w_2^{3/4} > 0, \frac{\partial c}{\partial w_2} = \frac{3}{4} y^{1/2} w_1^{3/4} w_2^{-1/4} > 0$$

→ Monotone.

$$H = \begin{bmatrix} -\frac{3}{16} y^{1/2} w_1^{-5/4} w_2^{3/4} & \frac{9}{16} y^{1/2} w_1^{-1/4} w_2^{-1/4} \\ \frac{9}{16} y^{1/2} w_1^{-1/4} w_2^{-1/4} & -\frac{3}{16} y^{1/2} w_1^{3/4} w_2^{-5/4} \end{bmatrix}$$

$$|H_1| < 0, |H_2| = -\frac{72}{256} \frac{y}{\sqrt{w_1 w_2}} < 0$$

→ Not concave.

Obviously continuous.

(b)

$$c(tw, y) = y \left(tw_1 + \sqrt{tw_1 tw_2} + tw_2 \right) = ty \left(w_1 + \sqrt{w_1 w_2} + w_2 \right) = tc(\mathbf{w}, y)$$

→ Homogeneous.

$$\frac{\partial c}{\partial w_1} = y \left(1 + \frac{1}{2} \sqrt{\frac{w_2}{w_1}} \right) > 0, \frac{\partial c}{\partial w_2} = y \left(1 + \frac{1}{2} \sqrt{\frac{w_1}{w_2}} \right) > 0$$

→ Monotone.

$$H = \begin{bmatrix} -\frac{1}{4} y w_2^{1/2} w_1^{-3/2} & \frac{1}{4} y w_2^{-1/2} w_1^{-1/2} \\ \frac{1}{4} y w_2^{-1/2} w_1^{-1/2} & -\frac{1}{4} y w_2^{-3/2} w_1^{1/2} \end{bmatrix}$$

$$|H_1| < 0, |H_2| = 0$$

→ Concave

Obviously continuous.

Production functions:

$$x_1(\mathbf{w}, y) = y \left(1 + \frac{1}{2} \sqrt{\frac{w_2}{w_1}} \right), x_2(\mathbf{w}, y) = y \left(1 + \frac{1}{2} \sqrt{\frac{w_1}{w_2}} \right)$$

Rearranging these equations:

$$x_1 - y = \frac{y}{2} \sqrt{\frac{w_2}{w_1}}, x_2 - y = \frac{y}{2} \sqrt{\frac{w_1}{w_2}}$$

$$\text{Multiply these two equations: } (x_1 - y)(x_2 - y) = \frac{y^2}{4}.$$

$$\text{Thus, } y = \frac{2}{3}(x_1 + x_2) \pm \frac{2}{3} \sqrt{x_1^2 + x_2^2 - x_1 x_2}$$

(c)

$$c(t\mathbf{w}, y) = y(tw_1 e^{-tw_1} + tw_2) = ty(w_1 e^{-tw_1} + w_2) \neq tc(\mathbf{w}, y)$$

\rightarrow Not homogeneous.

$$\frac{\partial c}{\partial w_1} = y(-w_1 e^{-w_1} + e^{-w_1}) = ye^{-w_1}(-w_1 + 1) > 0 \text{ only if } w_1 < 1$$

$$\frac{\partial c}{\partial w_2} = y > 0$$

\rightarrow Not monotone.

$$H = \begin{bmatrix} y(w_1 - 2)e^{-w_1} & 0 \\ 0 & 0 \end{bmatrix}$$

$$|H_1| = y(w_1 - 2)e^{-w_1} < 0, \text{ only if } w_1 < 2$$

$$|H_2| = 0$$

\rightarrow Not concave.

Obviously continuous.

(d)

$$c(t\mathbf{w}, y) = y(tw_1 - \sqrt{tw_1 tw_2} + tw_2) = ty(w_1 - \sqrt{w_1 w_2} + w_2) = tc(\mathbf{w}, y)$$

\rightarrow Homogeneous.

$$\frac{\partial c}{\partial w_1} = y \left(1 - \frac{1}{2} \sqrt{\frac{w_2}{w_1}} \right) > 0 \text{ only if } 1 > \frac{1}{2} \sqrt{\frac{w_2}{w_1}}$$

$$\frac{\partial c}{\partial w_2} = y \left(1 - \frac{1}{2} \sqrt{\frac{w_1}{w_2}} \right) > 0 \text{ only if } 1 > \frac{1}{2} \sqrt{\frac{w_1}{w_2}}$$

$$\text{Thus, monotone only if } \frac{1}{4}w_2 < w_1 < 4w_2$$

\rightarrow Not monotone.

$$H = \begin{bmatrix} \frac{1}{4}yw_2^{1/2}w_1^{-3/2} & -\frac{1}{4}yw_2^{-1/2}w_1^{-1/2} \\ -\frac{1}{4}yw_2^{-1/2}w_1^{-1/2} & \frac{1}{4}yw_2^{-3/2}w_1^{1/2} \end{bmatrix}$$

$$|H_1| > 0, |H_2| = 0$$

→ Not concave. (it is convex)

Obviously continuous.

(e)

$$c(t\mathbf{w}, y) = \left(y + \frac{1}{y} \right) \sqrt{tw_1 tw_2} = tc(\mathbf{w}, y)$$

→ Homogeneous.

$$\frac{\partial c}{\partial w_1} = \frac{1}{2} \left(y + \frac{1}{y} \right) \sqrt{\frac{w_2}{w_1}} > 0, \quad \frac{\partial c}{\partial w_2} = \frac{1}{2} \left(y + \frac{1}{y} \right) \sqrt{\frac{w_1}{w_2}} > 0$$

→ Monotone.

$$H = \begin{bmatrix} -\frac{1}{4} \left(y + \frac{1}{y} \right) w_2^{1/2} w_1^{-3/2} & \frac{1}{4} \left(y + \frac{1}{y} \right) w_2^{-1/2} w_1^{-1/2} \\ \frac{1}{4} \left(y + \frac{1}{y} \right) w_2^{-1/2} w_1^{-1/2} & -\frac{1}{4} \left(y + \frac{1}{y} \right) w_2^{-3/2} w_1^{1/2} \end{bmatrix}$$

$$|H_1| < 0, |H_2| = 0$$

→ Concave.

Not continuous for $y=0$

8.

$$\begin{aligned} & \min w_1 x_1 + w_2 x_2 \\ & s.t. y \leq \sqrt{x_1} + \sqrt{x_2} \\ & L = w_1 x_1 + w_2 x_2 + \lambda(y - \sqrt{x_1} - \sqrt{x_2}) \end{aligned}$$

F.O.C.

$$\frac{\partial L}{\partial x_1} = w_1 - \frac{\lambda}{2\sqrt{x_1}} = 0 \cdots 1$$

$$\frac{\partial L}{\partial x_2} = w_2 - \frac{\lambda}{2\sqrt{x_2}} = 0 \cdots 2$$

$$\lambda(y - \sqrt{x_1} - \sqrt{x_2}) = 0 \cdots 3$$

$$\text{By 1 and 2, } \lambda = 2w_1\sqrt{x_1} = 2w_2\sqrt{x_2}$$

If $\lambda = 0$, $x_1^* = x_2^* = 0 \rightarrow$ trivial sol'n

$$\text{For } \lambda > 0, \text{ since } \sqrt{x_2} = \frac{w_1}{w_2}\sqrt{x_1} \cdots 4$$

Plugging this value into 3 gives

$$y = \sqrt{x_1} + \frac{w_1}{w_2}\sqrt{x_1}$$

$$\text{Thus, } x_1^* = \left(\frac{w_2}{w_1 + w_2} \right)^2 y^2$$

Plugging this value into 4 gives

$$x_2^* = \left(\frac{w_1}{w_1 + w_2} \right)^2 y^2$$

$$\text{Cost fn } c(w_1, w_2, y) = w_1 x_1^* + w_2 x_2^* = \frac{w_1 w_2}{w_1 + w_2} y^2$$

9.

(a)

Technology 1 $f(x_1, x_2)$

If $x_1 < x_2$, $\min\{x_1, x_2\} = x_1$

$$\min\{\alpha x_1, \alpha x_2\} = \alpha x_1 = \alpha f(x_1, x_2)$$

If $x_1 \geq x_2$, similar argument can be applied

Thus, constant returns to scale

Technology 2

$$f(x_1, x_2) = \frac{1}{3}\alpha(x_1 + x_2) = \alpha f(x_1, x_2)$$

Thus, constant returns to scale

(b)

Cost fn of tech.1

$$c(w_1, w_2, y) = (w_1 + w_2)y$$

Cost fn of tech.2

$$c(w_1, w_2, y) = \min\{3w_1y, 3w_2y\}$$

Thus, if $w_1 > w_2$, $c(\tilde{w}, y) = 3w_2y$

If $w_1 < w_2$, $c(\tilde{w}, y) = 3w_1y$

If $w_1 = w_2$, $\alpha 3w_2y + (1-\alpha)3w_1y$ for any $\alpha \in (0, 1)$

If $w_1 > w_2$, $(w_1 + w_2)\bar{y} < 3w_1\bar{y}$, $w_2 < 2w_1 \Rightarrow w_1 > \frac{w_2}{2}$

If $w_1 < w_2$, $(w_1 + w_2)\bar{y} < 3w_1\bar{y}$, $w_1 < 2w_2 \Rightarrow w_2 < w_1 < 2w_2$

If $w_1 = w_2$, if $\alpha 3w_2\bar{y} + (1-\alpha)3w_1\bar{y} > (w_1 + w_2)\bar{y}$, $1 < \frac{3\alpha - 1}{3\alpha - 3}$

Thus, it always holds.

Therefore, $\frac{1}{2}w_2 < w_1 < 2w_2$

In addition, since profits must be nonnegative, $p\bar{y} - (w_1 + w_2)\bar{y} \geq 0$

Thus, $w_1 \leq p - w_2$

$\therefore \frac{1}{2}w_2 < w_1 < 2w_2 \& w_1 \leq p - w_2$

10.

(a)

$$\max p_i q_i - c_i(q_i) = \frac{\theta q^{\beta-1}}{\sum q_j^\beta} q_i - (F + cq_i)$$

F.O.C.

$$\frac{\partial \pi_i}{\partial q_i} = \frac{\theta \left[\beta q_i^{\beta-1} (\sum q_j^\beta) - \beta q_i^{2\beta-1} \right]}{(\sum q_j^\beta)^2} - c = 0$$

By the symmetry such that $q_i^* = q^*$

F.O.C. becomes

$$\frac{\theta \beta (N-1) q_i^{2\beta-1}}{N^2 q^{2\beta}} - c = 0$$

$$\text{Thus, } q^* = \frac{\theta \beta (N-1)}{c N^2}$$

(b)

Note that hand-made food market would be more differentiated than fast-food market.

$$\frac{\partial q^*}{\partial \beta} = \frac{\theta (N-1)}{c N^2} > 0$$

Thus, q^* in hand-made food market will be larger. "True"

(c)

$$\frac{\partial q^*}{\partial N} = \frac{\theta N^2 - \theta(N-1)2cN}{(cN^2)^2} < 0, \text{since } N > 2. \text{ “True”}$$