Term Project: Problem Set A

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Due: May 24th
Room: 301-1256

For each problem described below, the following methods in Table 1 are considered.
Class (I) and (II) are all mandatory.

Table1. Numerical methods for SCL

Class (I)	① First-order upwind method ② Lax-Friedrich ③ Lax-Wendroff ④ Beam-Warming
Class (II)	① TVD methods using flux limiters or MUSCL with slope limiters ② WENO interpolation with cell-average values ③ MLP3 and MLP5 methods (for 2-D problem only)

* For class (II), Runge-Kutta time integration is preferred.
The 3rd- order accurate TVD Runge-Kutta method for $q_t = L(q)$ is given by

$$q^{(1)} = q^n + \Delta t L\left(q^n\right),$$
$$q^{(2)} = \frac{3}{4}q^n + \frac{1}{4}q^{(1)} + \frac{1}{4}\Delta t L\left(q^{(1)}\right),$$
$$q^{n+1} = \frac{1}{3}q^n + \frac{2}{3}q^{(2)} + \frac{2}{3}\Delta t L\left(q^{(2)}\right).$$

1. Consider the 1-D linear advection equation

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0, \quad a = 0.5, \quad -1 \le x \le 1.$$

Initial conditions:

1) square wave $u(x,0) = \begin{cases} 1, & -0.5 \le x \le 0.5 \\ 0, & otherwise \end{cases}.$

2) half dome wave $u(x,0) = \begin{cases} \sqrt{1 - \frac{10}{3}x^2}, & -\sqrt{\frac{3}{10}} \le x \le \sqrt{\frac{3}{10}} \\ 0, & otherwise \end{cases}.$

3) Gaussian wave $u(x,0) = e^{-300x^2}.$

Boundary conditions: Periodic condition (i.e., $u(-1,t) = u(1,t)$).

Compute the numerical solution up to *t=4.0* with Δx*=0.05, 0.025, 0.0125* for above initial conditions, and discuss the results.
(Use CFL condition to determine the time-step size.)

2. Consider the 1-D Burger's equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0, \quad -1 \le x \le 1.$$

Initial conditions:

1) shock wave $u(x,0) = \begin{cases} 1.2, & x \le -0.3 \\ 0.4, & x > -0.3 \end{cases}.$

2) expansion shock $u(x,0) = \begin{cases} 0, & x \le -0.5 \\ 1, & x > -0.5 \end{cases}.$

3) sine wave $u(x,0) = 1 + 0.5\sin(\pi x).$

Boundary conditions:
1) Transmissive condition (i.e., $u(-1,t) = 1.2$ and $u(1,t) = 0.4$).
2) Transmissive condition (i.e., $u(-1,t) = 0$ and $u(1,t) = 1$).
3) Periodic condition (i.e., $u(-1,t) = u(1,t)$).

Compute the numerical solution up to *t=1.0* with Δx*=0.05, 0.025, 0.0125* for above initial conditions, and discuss the results.
(Use CFL condition to determine the time-step size.)

3. For the following 2-D linear advection equation,

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} = 0,$$

consider a solid body rotation of a cut-off cylinder (see Fig.1).

Computational domain is given by $0 \le x \le 100,\ 0 \le y \le 100$. The domain contains a cut-off cylinder centered at (*50,75*). The value of *u* inside the cut-off cylinder is *1.0*, while outside the cylinder *u = 0*. The solid-body rotation is defined with wave speed component:

$$a = \frac{\pi}{314}(50 - y),\ b = \frac{\pi}{314}(x - 50).$$

For the given initial condition, compute the solution up to *t=628.0* (one full revolution) on 100 by 100 mesh points.

1) Discuss the result of each method in Class (I) and Class (II). Compare the result with the analytical solution.
2) Conduct grid refinement study with WENO and MLP3 (or MLP5) on four (or more) different grids by changing the number of mesh points. Compare the order of accuracy of two methods based on L_1 errors.

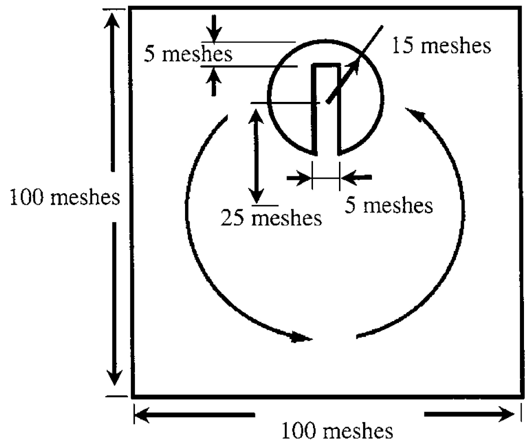

Fig.1 Initial profile of 2-D cut-off cylinder rotation

Optional) Consider the sinusoidal initial condition for the same 2-D linear advection equation:

$$u(x, y, 0) = \frac{1}{2}\sin\left(\pi(x + y)/50\right).$$

The wave speeds are *a = b = 20*. With the same computational domain, periodic boundary condition is imposed in both directions. Repeat 2) grid refinement study with WENO and MLP3 (or MLP5) at *t=5.0*. Compare the order of accuracy between two initial conditions (cut-off cylinder rotation and sine wave translation).

As relevant references to carry out this project, the following papers are recommended. If you have enquiries or need help with your term project, please send an email to TA.

MUSCL Van Leer, B. (1979) "Towards the Ultimate Conservative Difference Scheme. V. A Second-Order Sequel to Godunov's Method" Journal of Computational Physics 32: 101-136.

TVD Harten, A. (1983) "High Resolution Schemes for Hyperbolic Conservation Laws," Journal of Computational Physics 49(3): 357-393.

TVD Flux limiter Sweby, P. K. (1984) "High Resolution Schemes Using Flux Limiters for Hyperbolic Conservation Laws," SIAM Journal on Numerical Analysis 21(5): 995-1011.

WENO Liu, X.-D. *et al*. (1994) "Weighted Essentially Non-oscillatory Schemes," Journal of Computational Physics 115: 200-212.

WENO Jiang G.-S. and Shu, C.-W. (1996) "Efficient Implementation of Weighted ENO Schemes," Journal of Computational Physics 126: 202-228.

MLP Kim, K. H. and Kim, C. (2005) "Accurate, Efficient and Monotone Numerical Methods for Multi-dimensional Compressible Flows Part II: Multi-dimensional Limiting Process," Journal of Computational Physics 208: 570-615.

MLP Yoon, S.-H., Kim, C. and Kim, K.-H. (2008) "Multi-dimensional Limiting Process for Three-dimensional Flow Physics Analyses," Journal of Computational Physics 227:6001-6043.