Plasma Electrodynamics 2 **Selected Solutions** Oct. 20, 2009

1. a) 3 b) 4 c) 1 d) 2 e) 4 f) 1, 3 g) 3 h) 4 i) 3 j) 2

2. a) Maxwell's eqns. with Ohm's law :

$$\nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t} \quad \cdots\cdots \; ①$$

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} = \sigma \vec{E} + \epsilon \frac{\partial \vec{E}}{\partial t} \quad \cdots\cdots \; ②$$

$$\nabla \cdot \vec{E} = 0 \quad \cdots\cdots \; ③ \qquad \nabla \cdot \vec{B} = 0 \quad \cdots ④$$

① in $\nabla \times$ ② using $\nabla \times \nabla \times \vec{H} = \nabla(\underbrace{\nabla \cdot \vec{H}}_{0 \text{ by } ④}) - \nabla^2 \vec{H}$:

$$\nabla^2 \vec{H} - \mu\sigma \frac{\partial \vec{H}}{\partial t} - \mu\epsilon \frac{\partial^2 \vec{H}}{\partial t^2} = \vec{0} \quad \cdots \; ⑤$$

: magnetic wave equation

b) For time-harmonic fields $\vec{H}(\vec{R},t) = Re[\vec{H}_s(\vec{R})\, e^{j\omega t}]$,

$\frac{\partial}{\partial t} \to j\omega$, $\frac{\partial^2}{\partial t^2} \to -\omega^2$ in ⑤

$$\nabla^2 \vec{H}_s - j\omega\mu\sigma \vec{H} + \omega^2 \mu\epsilon \vec{H} = \vec{0}$$

or $\nabla^2 \vec{H}_s + k_c^2 \vec{H}_s = \vec{0}$: Helmholtz's eqn. ⑥

or $\nabla^2 \vec{H}_s - \gamma^2 \vec{H}_s = \vec{0}$ ⑦

where

$$\gamma = jk_c = j\omega\sqrt{\mu\epsilon_c} = j\omega\sqrt{\mu\epsilon}\left(1 - j\frac{\sigma}{\omega\epsilon}\right)^{1/2} \equiv \alpha + j\beta \quad ⑧$$

c) Solution of ⑥ : $\vec{H}_s(\vec{R}) = \vec{H}_0\, e^{-j\vec{k}_c \cdot \vec{R}}$ ⑨

From ②, $\nabla \times \vec{H}_s = (\sigma + j\omega\epsilon)\vec{E}_s = j\omega\epsilon\left(1 - j\frac{\sigma}{\omega\epsilon}\right)\vec{E}_s = j\omega\epsilon_c \vec{E}_s$

$$\underset{\nabla \to -j\vec{k}_c}{\Longrightarrow} \vec{E}_s(\vec{R}) = \frac{-j\vec{k}_c \times \vec{H}_s}{j\omega\epsilon_c} = -\frac{\vec{k}_c \times \vec{H}_s}{\omega\epsilon_c} \underset{⑧}{=} -\sqrt{\frac{\mu}{\epsilon_c}}\, \hat{k}_c \times \vec{H}_s(\vec{R})$$

$$\Rightarrow \vec{E}_s = -\eta_c\, \hat{k}_c \times \vec{H}_s \quad ⑩$$

where $\eta_c = \sqrt{\frac{\mu}{\epsilon_c}} = \sqrt{\frac{\mu}{\epsilon}}\left(1 - j\frac{\sigma}{\omega\epsilon}\right)^{-1/2} = \eta\left(1 - j\frac{\sigma}{\omega\epsilon}\right)^{-1/2}$ ⑪

3. Given $\vec{E}(z,t) = \hat{y} E_0 \cos(\omega t - kz + \psi)$ in $\epsilon = 9\epsilon_0$, $\mu = \mu_0$, $\sigma = 0$

a) $\omega = 2\pi \times 10^9$

$$k = \omega\sqrt{\mu\epsilon} = (2\pi\times10^9)(3\sqrt{\mu_0\epsilon_0}) \overset{c=1/\sqrt{\mu_0\epsilon_0}}{=} \frac{2\pi\times10^9\times3}{3\times10^8} = 20\pi$$

$$\vec{E}(z,t) = \hat{y} E_0 \cos(2\pi\times10^9 t - 20\pi z + \psi) \quad ①$$

$E_{max}(1,0) = 5 \overset{①}{\Rightarrow} E_0 = 5$

$-20\pi + \psi = 0 \Rightarrow \psi = 20\pi$

$$\therefore \vec{E}(z,t) = \hat{y}\, 5\cos[2\pi\times10^9 t - 20\pi(z-1)] \quad ②$$

(V/m)

b) From $\nabla\times\vec{E} = -\mu\frac{\partial\vec{H}}{\partial t}$,

$$-j\vec{k}\times\vec{E} = -j\omega\mu\vec{H} \Rightarrow \vec{H} \overset{k=\omega\sqrt{\mu\epsilon}}{=} \frac{\sqrt{\mu\epsilon}}{\mu}\hat{k}\times\vec{E} = \frac{\hat{k}\times\vec{E}}{\eta} \quad ③$$

where $\eta = \sqrt{\frac{\mu}{\epsilon}}$: intrinsic impedance ④

$$\therefore \vec{H}(z,t) \overset{②,④}{=} \frac{\hat{z}\times\hat{y}\, 5\cos[2\pi\times10^9 t - 20\pi(z-1)]}{\underbrace{\sqrt{\mu_0/9\epsilon_0}}_{=\eta_0/3 = 40\pi}}$$

$$= -\hat{x}\frac{1}{8\pi}\cos[2\pi\times10^9 t - 20\pi(z-1)] \quad \text{(A/m)} \quad ⑤$$

4. Given $\epsilon_r = 36\pi$, $\mu_r = 1$, $\sigma = 20$ S/m

$\vec{H}(0,t) = \hat{y}\, 10\cos(10^8 t) \rightarrow \omega = 2\pi f = 10^8$

Loss tangent : $\tan\delta_c = \frac{\sigma}{\omega\epsilon} \underset{\epsilon=\epsilon_r\epsilon_0}{=} \frac{20}{(10^8)(36\pi)(10^{-9}/36\pi)} = 200$: Good conductor

a) Skin depth :

$$\delta = \frac{1}{\sqrt{\pi f\mu\sigma}} = \frac{1}{\sqrt{\pi(10^8/2\pi)(4\pi\times10^{-7})(20)}} = \frac{1}{20\sqrt{\pi}} = \frac{0.05}{\sqrt{\pi}} \text{ (m)} \quad ①$$

$$\Rightarrow \alpha = \beta = \frac{1}{\delta} = 20\sqrt{\pi} \quad ②$$

Intrinsic impedance :

$$\eta_c \approx \sqrt{\frac{\pi f\mu}{\sigma}}(1+j) = \frac{1}{\sigma\delta}(1+j) \overset{①}{=} \frac{20\sqrt{\pi}}{20}(1+j) = \sqrt{2\pi}\, e^{j\pi/4} \text{ (}\Omega\text{)} \quad ③$$

b) $\begin{cases} \vec{H}(z,t) = Re\left[\vec{H}(z)e^{j\omega t}\right] = \hat{y}\, H_0\, e^{-\alpha z}\cos(\omega t - \beta z) \\ \vec{H}(0,t) = \hat{y}\, 10\cos(10^8 t) \end{cases}$

$$\Rightarrow \quad H_0 = 10, \quad \omega = 10^8, \quad \alpha = \beta = \frac{1}{\delta} = 20\sqrt{\pi}$$

$$\therefore \quad \vec{H}(z,t) = \hat{y}\, 10\, e^{-20\sqrt{\pi}\, z}\cos(10^8 t - 20\sqrt{\pi}\, z) \quad (A/m) \quad ④$$

From Ampere's law (see 2.c) ⑩),

$$\vec{E}(z,t) = -\eta_c\, \hat{k}\times\vec{H}(z,t)$$

$$= -\eta_c\, \hat{z}\times\vec{H}(z,t)$$

$$\overset{③,④}{=} \hat{x}\, \sqrt{2\pi}\, 10\, e^{-20\sqrt{\pi}\, z}\cos\left(10^8 t - 20\sqrt{\pi}\, z + \frac{\pi}{4}\right)$$

$$= \hat{x}\, 10\sqrt{2\pi}\, e^{-20\sqrt{\pi}\, z}\cos\left(10^8 t - 20\sqrt{\pi}\, z + \frac{\pi}{4}\right) \quad (V/m) \quad ⑤$$

c) $\mathcal{P}_{av} = \frac{1}{2} Re(\vec{E}\times\vec{H}^*)$

$$\overset{⑤,④}{=} \frac{1}{2} Re\left[\hat{x}\, 10\sqrt{2\pi}\, e^{-20\sqrt{\pi}\, z}\, e^{j(10^8 t - 20\sqrt{\pi}\, z + \pi/4)} \times \hat{y}\, 10\, e^{-20\sqrt{\pi}\, z}\, e^{-j(10^8 t - 20\sqrt{\pi}\, z)}\right]$$

$$= \hat{z}\, Re\left[\frac{1}{2} 100\sqrt{2\pi}\, e^{-40\sqrt{\pi}\, z}\, e^{j\pi/4}\right]$$

$$= \hat{z}\, 50\sqrt{2\pi}\, e^{-40\sqrt{\pi}\, z}\cos\frac{\pi}{4} = \hat{z}\, 50\sqrt{\pi}\, e^{-40\sqrt{\pi}\, z} \quad (W/m^2)$$

5. Critical angle θ_c for total reflection on the fiber-cladding interface:

$$\theta_c = \sin^{-1}\left(\frac{\epsilon_2}{\epsilon_1}\right) = \sin^{-1}\left(\frac{n_2}{n_1}\right) \Rightarrow \sin\theta_c = \frac{n_2}{n_1} \quad ①$$

By Snell's law for refraction on the core's end face,

$$\frac{\sin\theta_a}{\sin(\frac{\pi}{2} - \theta_c)} = \frac{n_1}{n_0} \quad \Rightarrow \sin\theta_a = \frac{n_1}{n_0}\sin\left(\frac{\pi}{2} - \theta_c\right) = \frac{n_1}{n_2}\cos\theta_c$$

$$= \frac{n_1}{n_2}\sqrt{1 - \sin^2\theta_c} \quad ②$$

① in ②:

$$\sin\theta_a = \frac{n_1}{n_0}\sqrt{1 - \left(\frac{n_2}{n_1}\right)^2} = \frac{1}{n_0}\sqrt{n_1^2 - n_2^2}$$

$$\text{or} \quad \theta_a = \sin^{-1}\left(\frac{\sqrt{n_1^2 - n_2^2}}{n_0}\right)$$