Plasma Electrodynamics 2 **Solutions** Dec. 17, 2009

1.a) X (E-plane → H-plane or ϕ for $\theta=\pi/2$ → θ for a constant ϕ)

b) O c) O

d) X (a very short length compared → a length comparable)

e) X (Herzian dipole → linear dipole antenna

or depends → does not depend)

f) O

g) X (equal currents → unequal currents)

h) O i) O

j) X (proportional to the square → independent)

2. a) Spherical components of $\boldsymbol{A}$:

$$A_R = A_z\cos\theta = \frac{\mu_o I\,dl}{4\pi}\left(\frac{e^{-j\beta R}}{R}\right)\cos\theta, \quad A_\theta = -A_z\sin\theta = -\frac{\mu_o I\,dl}{4\pi}\left(\frac{e^{-j\beta R}}{R}\right)\sin\theta, \quad A_\phi = 0$$

Magnetic field from $\boldsymbol{B} = \nabla\times\boldsymbol{A}$ and $\boldsymbol{B} = \mu_o\boldsymbol{H}$:

$$\boldsymbol{H} = \hat{\phi}\frac{1}{\mu_o R}\left[\frac{\partial}{\partial R}(RA_\theta) - \frac{\partial A_R}{\partial\theta}\right] = -\hat{\phi}\frac{Idl}{4\pi}\beta^2\sin\theta\left[\frac{1}{j\beta R} + \frac{1}{(j\beta R)^2}\right]e^{-j\beta R}$$

Electric field from Faraday's law, $\boldsymbol{E} = \frac{1}{j\omega\epsilon_o}\nabla\times\boldsymbol{H}$:

$$E_R = -\frac{I\,d\ell}{4\pi}\eta_0\beta^2 2\cos\theta\left[\frac{1}{(j\beta R)^2} + \frac{1}{(j\beta R)^3}\right]e^{-j\beta R}$$

$$E_\theta = -\frac{I\,d\ell}{4\pi}\eta_0\beta^2\sin\theta\left[\frac{1}{j\beta R} + \frac{1}{(j\beta R)^2} + \frac{1}{(j\beta R)^3}\right]e^{-j\beta R}$$

$$E_\phi = 0$$

b) In the far-field zone ($R \gg \lambda/2\pi$, *i.e.*, $\beta R = 2\pi R/\lambda \gg 1$), neglecting $(\beta R)^{-2}$ and $(\beta R)^{-3}$ terms,

$$H_\phi = j\frac{Idl}{4\pi}\left(\frac{e^{-j\beta R}}{R}\right)\beta\sin\theta \qquad \text{(A/m)}$$

$$E_\theta = j\frac{Idl}{4\pi}\left(\frac{e^{-j\beta R}}{R}\right)\eta_o\beta\sin\theta = \eta_o H_\phi \qquad \text{(V/m)}$$

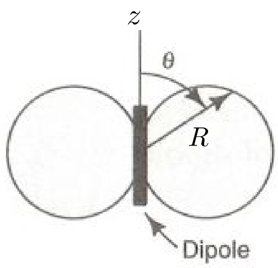
c) Pattern function:

$$E_\theta(\theta,\phi)_n = E_\theta(\theta,\phi)/E_\theta(\theta,\phi)_{\max}$$

E-plane pattern independent of ϕ at a given R:

$E_\theta(\theta,\phi)_n =$ Normalized $|E_\theta| = |\sin\theta|$

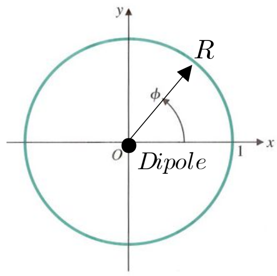
for $0 \le \theta \le \pi$, $0 \le \phi \le 2\pi$



d) H-plane pattern for for $\theta = \pi/2$ at a given R:

$$E_\theta(\theta,\phi)_n = |\sin\theta| = 1$$

for $\theta = \pi/2, \ 0 \le \phi \le 2\pi$



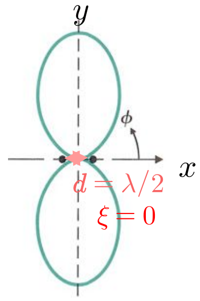
e) $|E| = \dfrac{2E_m}{R_o} |F(\theta,\phi)| \left|\cos\dfrac{\psi}{2}\right|$ where $\psi = \beta d \sin\theta \cos\phi + \xi$

Consider H-plane $(\theta = \pi/2)$ radiation patterns of two-element parallel dipole array directed in z and placed along the x-axis.

Broadside array factor for $d = \lambda/2 \ (\beta d = \pi), \ \xi = 0$ (in phase):

$$|A(\phi)|_n = \left|\cos\frac{\psi}{2}\right| = \left|\cos\frac{1}{2}(\beta d \cos\phi + \xi)\right|$$

$$= \left|\cos\left(\frac{\pi}{2}\cos\phi\right)\right|$$



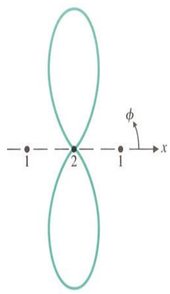
At $\phi = \pm\pi/2$, $\exists \ |E|_{\max}$.

At $\phi = 0, \ \pi$, $\exists \ |E|_{\min} = 0$.

Main beams only, no side lobes

f) Broadside pattern by the principle of pattern multiplication using the above results:

$$|E| = \frac{4E_m}{R_o} \left|\cos\left(\frac{\pi}{2}\cos\phi\right)\right|^2$$



More directive than the two-element array

Main beams only, no sidelobes

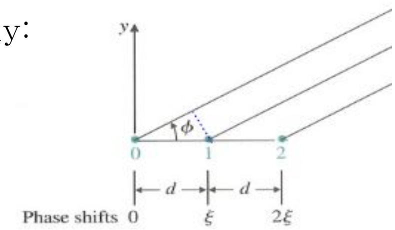
g) Array factor of an 3-element uniform linear array:

$$A(\psi) = 1 + e^{j\psi} + e^{j2\psi}$$

$$= \frac{1 - e^{j3\psi}}{1 - e^{j\psi}} = \frac{e^{j3\psi/2}}{e^{j\psi/2}}\left(\frac{e^{j3\psi/2} - e^{-j3\psi/2}}{e^{j\psi/2} - e^{-j\psi/2}}\right)$$

$$= e^{j\psi}\,\frac{\sin(3\psi/2)}{\sin(\psi/2)}$$

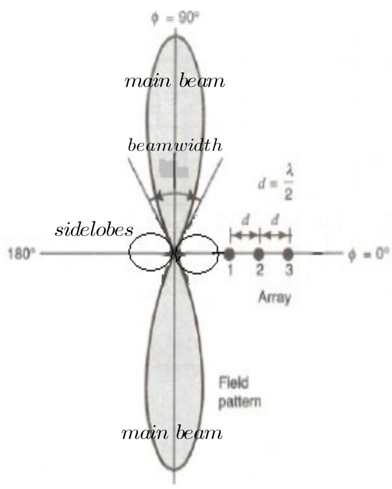
$$\Rightarrow \quad |A(\psi)| = \left|\frac{\sin(3\psi/2)}{\sin(\psi/2)}\right|$$



For $\psi = 0$, $|A(\psi)|_{\max} = 3$

Therefore, the normalized array factor becomes

$$|A(\psi)|_n \equiv \frac{|A(\psi)|}{|A(\psi)|_{\max}} = \frac{1}{3}\left|\frac{\sin(3\psi/2)}{\sin(\psi/2)}\right|$$



3. Given: *Two identical antennas* with $G_D = 1{,}000$, $r = 10\ km = 10^4\ m$,

$f = 300\ MHz = 3 \times 10^8\ Hz$, $P_t = 16\pi^2\ W$

a) $\lambda = c/f = 3\times10^8/3\times10^8 = 1\ (m)$

$$P_L = \left(\frac{G_D\lambda}{4\pi r}\right)^2 P_t = \left(\frac{10^3\times 1}{4\pi\times10^4}\right)^2(4\pi)^2 = 0.01\ (W) = 10\ (mW)$$

b) $$\mathscr{P}_{av} = \frac{P_t}{4\pi r^2}G_{D1} = \frac{E_i^2}{240\pi}$$

$$\Rightarrow\ E_i^2 = \frac{240\pi\ P_t}{4\pi r^2}G_D = \frac{240\pi\times(4\pi)^2}{4\pi\times(10^4)^2}\times 1000$$

$$\Rightarrow\ E_i = \frac{4\pi}{10^4}\sqrt{6\times10^4} = \frac{4\sqrt{6}\,\pi}{100}\ (V/m) \approx 0.308\ (V/m)$$

4. Given: a very *thin center-fed half-wave dipole* lying along the z-axis with $I(z) = I_o \cos 2\pi z$.

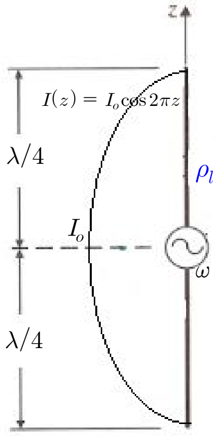
a) Current continuity equation (Charge conservation equation):

$$\frac{\partial\rho_v}{\partial t} + \nabla\cdot\boldsymbol{J} = 0$$

For time-harmonic fields ($e^{j\omega t}$) on a thin half-wave dipole,

$$j\omega\rho_l + \frac{dI(z)}{dz} = 0 \quad\Rightarrow\quad \text{Charge distribution: } \rho_l = \frac{j}{\omega}\frac{dI(z)}{dz}$$

$$\Rightarrow\ \rho_l = \frac{j}{\omega}\frac{d}{dz}(I_o\cos 2\pi z) = -\frac{j\beta}{\omega}I_o\sin 2\pi z = -j\frac{I_o}{c}\sin 2\pi z$$



b) $\beta = 2\pi/\lambda = 2\pi \quad\Rightarrow\quad \lambda = 1\ (m)$

5. *E-plane pattern* of a Hertzian dipole:

$E_\theta(\theta,\phi)_n = |\sin\theta|$ for $0 \le \theta \le \pi,\ 0 \le \phi \le 2\pi$

Maximum radiation field: $E_{\max} = E_\theta(\pi/2,\phi)_n = 1$

Half-power points: $$E_\theta(\theta_1,\phi)_n = \frac{E_{\max}}{\sqrt{2}} = \frac{1}{\sqrt{2}} = |\sin\theta_1|$$

$$\Rightarrow\ \theta_1 = \pi/4,\ 3\pi/4 \text{ or } (45^o,\ 135^o)$$

∴ Beamwidth: $\Delta\theta = 3\pi/4 - \pi/4 = \pi/2$ or 90^o