SEOUL NATIONAL UNIVERSITY
SCHOOL OF MECHANICAL AND AEROSPACE ENGINEERING

SYSTEM CONTROL **Fall 2014**

Final Exam **Date: December 11, 2014 (Th)**
Closed book, closed note **11:00-12:30 am**

Student ID: ____________________ Name:__________________

Problem	Points
1(20)	
2(15)	
3(15)	
4(20)	
5(15)	
6(10)	
7(20)	
Total (115)	

[1] (20 points) Describe followings:

Solution

(1) Linear Control Systems

Systems: combination of components working together

Control : Applying inputs to the system to correct or limit deviation of the output values from desired values

Linear superposition

(+1)Open-loop/closed- loop control systems
(+1)feedback control system Block diagram

선형성/superposition 설명 +2
Control 설명 +1
System 설명 +2

(2) Stability

Definition: equilibrium, linear asymptotic stability
Linear system stability: Re(Pole)<0

Nyquist stability criterion (+1)

설명 틀리면 0 점
Equilibrium point 로 수렴 빠지면 -1
선형 시스템 안정성 설명 빠지면 -3

(3) controllability and observability

Definition : state transfer
A system is said to be controllable at time t0 if it is possible by means of an unconstrained control vector to transfer the system from any initial state x(t0) to any other state in a finite interval of time.
State equation $\dot{x} = Ax + Bu$
State feedback, controllability condition, controllability matrix $\underline{C} = [B \quad AB \quad A^2B \quad \cdots \quad A^{n-1}B]$
State-feedback, regulator pole (A-BK)
(+1) separation property

Definition: observability
A system is said to be observable at time t0 if, with the system in state x(t0), it is possible to determine this state from the observation of the output over a finite time interval.

Observer equation $\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}$

Observability matrix $\underline{O} = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$

Observer pole (A-LC)

각 cont/obs matrix 설명 없으면 -2
(Cont/obs matrix 틀리면 -1)
각 기본 개념 설명 없으면 -3
추가 설명 있으면 +1(closed loop pole 등)

(4) State, state equation, state space

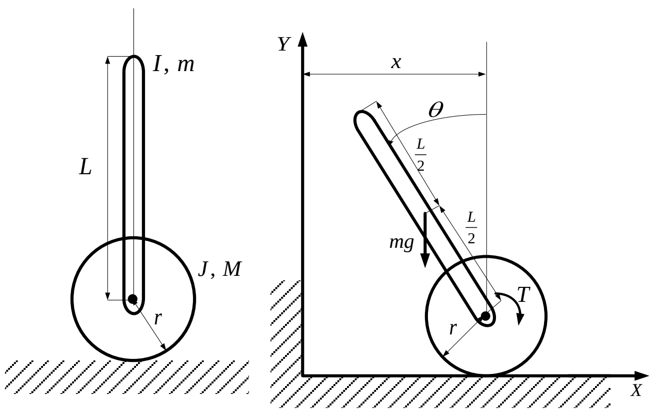
State x:
The smallest set of variables such that knowledge of these variables at t=t0 , together with the knowledge of the input for t>t0 , completely determines the behavior of the system at any time t>t0.

State equation
$\dot{x} = Ax + Bu$
$y = Cx$

State space:
The n-dimensional space, whose coordinate axes consist of the x_1 axis, x_2 axis, …, x_n axis is called a state space. Any state can be represented by a point in the state space

State 설명 +2
State equation +2
state space 설명 +1

[2] An inverted bar mounted on a motor-driven cart is shown in Figure below. The objective of the control problem is to transfer the position of the cart to any desired place and to keep the bar in a vertical position. The inverted bar is unstable in that it may fall over any time in any direction unless a suitable control force is applied. Here we consider only a two-dimensional problem in which the pendulum moves only in the X-Y plane. The control force F is applied at the cart. Assume that the center of gravity of the bar is at its geometric center.



$$I\ddot{\theta} = m\ddot{x}\frac{L}{2}\cos\theta - m\left(\frac{L}{2}\right)^2\ddot{\theta} + mg\frac{L}{2}\sin\theta$$

The velocity of the cart can be controlled by a motor and the motor angular speed can be represented as

$$\omega(s) = \frac{k}{1+\tau s}u(s)$$

Where

$$\omega(t) = \frac{d\theta_M(t)}{dt},$$

$x(t) = r \cdot \theta_M(t)$, and $u(t)$ is the control input.

(1) Define the state as follows:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \theta \\ \dot{\theta} \\ x \\ \dot{x} \end{bmatrix}$$

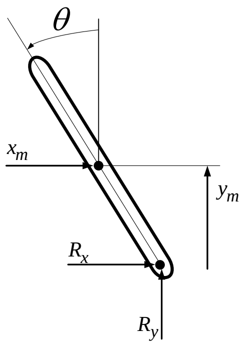
If we consider θ and x as the outputs of the system, then

$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \theta \\ x \end{bmatrix} = \begin{bmatrix} x_1 \\ x_3 \end{bmatrix}$$

Obtain a state-space model of the system. The model is of nonlinear dynamics. And then obtain a linear state-space model assuming that the θ is small.

Solution)

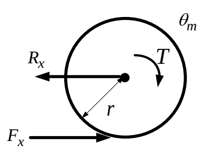
Bar of mass, m, and, moment of inertia, I



Equation of motion of the bar:

$$\sum F_x = R_x = m\ddot{x}_m$$
$$\sum F_y = R_y - mg = m\ddot{y}_m$$
$$\sum M = R_x \frac{L}{2}\cos\theta + R_y \frac{L}{2}\sin\theta = I\ddot{\theta}$$

Equation of motion of the motor; no slip condition



$$J\frac{d^2\theta_m}{dt^2} = T - F_x \cdot r$$
$$M\frac{d^2x}{dt^2} = F_x - R_x \qquad (*)$$

No slip condition: $x = r\theta_m \;\Rightarrow\; \ddot{x} = r\ddot{\theta}_m$

Combining the equation (*) with no slip condition

$$J\ddot{\theta}_m = T - r\left(M\frac{d^2x}{dt^2} + R_x\right)$$
$$(J + r^2 M)\ddot{\theta}_m = T - r \cdot R_x$$
$$\left(\frac{J}{r^2} + M\right)\ddot{x} = \frac{T}{r} - R_x$$

Combining the equation of motion of bar and motor

$$I\ddot{\theta} = m\ddot{x}\frac{L}{2}\cos\theta - m\left(\frac{L}{2}\right)^2\ddot{\theta} + mg\frac{L}{2}\sin\theta$$
$$\left(\frac{J}{r^2} + M + m\right)\ddot{x} + m\frac{L}{2}(\dot{\theta}^2\sin\theta - \ddot{\theta}\cos\theta) = F = \frac{T}{r}$$

Motor speed control can be represented as

$$\omega(s) = \frac{k}{1+\tau s}u(s)$$
$$\tau\ddot{\theta}_m + \dot{\theta}_m = k \cdot u$$
$$\tau\ddot{x} + \dot{x} = r \cdot k \cdot u$$

Then

$$I\ddot{\theta} = m\frac{1}{\tau}(-\dot{x} + r \cdot k \cdot u)\frac{L}{2}\cos\theta - m\left(\frac{L}{2}\right)^2\ddot{\theta} + mg\frac{L}{2}\sin\theta$$
$$\tau\ddot{x} + \dot{x} = r \cdot k \cdot u$$

Set state as

$$x = \begin{bmatrix}\theta & \dot{\theta} & x & \dot{x}\end{bmatrix}^T$$
$$\dot{x}_1 = \dot{\theta} = x_2$$
$$\dot{x}_2 = \frac{1}{\left[I + m\left(\frac{L}{2}\right)^2\right]}\left[-\frac{m}{\tau}x_4 \cdot \frac{L}{2}\cos x_1 + mg\frac{L}{2}\sin x_1 + \frac{m}{\tau}r \cdot k \cdot \frac{L}{2}\cos x_1 \cdot u\right]$$
$$x_3 = \dot{x} = x_4$$
$$\dot{x}_4 = -\frac{1}{\tau}x_4 + \frac{1}{\tau}r \cdot k \cdot u$$

Linearized model, $\theta = x_1 \approx 0 \Rightarrow \sin x_1 \approx x_1,\ \cos x_1 \approx x_1$

$$\dot{x}_1 = \dot{\theta} = x_2$$

$$\dot{x}_2 = \frac{1}{\left[I + m\left(\frac{L}{2}\right)^2\right]}\left[mg\frac{L}{2}x_1 - \frac{m}{\tau}\cdot\frac{L}{2}x_4 + \frac{m}{\tau}r\cdot k\frac{L}{2}u\right]$$

$$x_3 = \dot{x} = x_4$$

$$\dot{x}_4 = -\frac{1}{\tau}x_4 + \frac{1}{\tau}r\cdot k\cdot u$$

State equation form trial +1
Nonlinear State equation +1
Linearization +3

(2) Assume that the measurements of $x_1(t), x_2(t),$ and $x_4(t)$ are available. Therefore, you can use a control law of following form:

$u(t) = -k_1\cdot x_1(t) - k_2\cdot x_2(t) - k_4\cdot x_4(t)$
Using the linearized model obtained in problem (2), determine the gains, $k_1(t), k_2(t),$ and $k_4(t)$ so that $x_1(t)$ tends to zero as t increases, i.e.,
$\lim_{t\to\infty} x_1(t) = 0$.

Solution)

$$\dot{x}_2 = Ax_1 - Bx_4 + C\cdot u$$
$$= -2\zeta\omega_n x_2 - \omega_n{}^2 x_1$$
$$where,\ A = \frac{mg\frac{L}{2}}{\left[I + m\left(\frac{L}{2}\right)^2\right]}\ B = \frac{\frac{m}{\tau}\frac{L}{2}}{\left[I + m\left(\frac{L}{2}\right)^2\right]}\ C = \frac{\frac{m}{\tau}r\cdot k\frac{L}{2}}{\left[I + m\left(\frac{L}{2}\right)^2\right]}$$

$$Since\ \dot{x}_2 = \ddot{x}_1,\ x_2 = \dot{x}_1$$
$$\ddot{x}_1 + 2\zeta\omega_n\dot{x}_1 + \omega_n{}^2 x_1 = 0$$

Therefore,

$$u(t) = \frac{1}{C}(-2\zeta\omega_n x_2 - \omega_n{}^2 x_1 - Ax_1 + Bx_4)$$
$$= -\frac{A + \omega_n{}^2}{C}x_1 - \frac{2\zeta\omega_n}{C}x_2 + \frac{B}{C}x_4$$
$$\therefore k_1 = \frac{A + \omega_n{}^2}{C}\ k_2 = \frac{2\zeta\omega_n}{C}\ k_4 = -\frac{B}{C},\ B > 0\ \ C > 0$$

Concept suggestion (u = -kx) +1
Explanation +2
Input Calculation +2

(3) It is possible to have both $x_1(t)$ and $x_2(t)$ tends to zero by the control law of the form in the problem (2). What happens, in this case, to the position of the system? Analyze the behavior of the system.

Solution)

$$\dot{x}_4 = -\frac{1}{\tau}x_4 + \frac{1}{\tau}r\cdot k(-k_1x_1 - k_2x_2 - k_4x_4)$$
$$= -\left(\frac{1}{\tau} + \frac{1}{\tau}r\cdot k\cdot k_4\right)x_4 + \frac{1}{\tau}r\cdot k(-k_1x_1 - k_2x_2)$$
$$\dot{x}_3 = x_4$$

Since $x_1(t) \to 0$ *and* $x_2(t) \to 0$ *as* $t \to \infty$

$$\dot{x}_4 = -\frac{1}{\tau}(1 + r\cdot k\cdot k_4)x_4 = 0 \quad (\because k_4 = -\frac{1}{r\cdot k})$$

Therefore, $x_4(t) = const.$
i.e. $\dot{x}_3(t) = x_4(t) = const.$

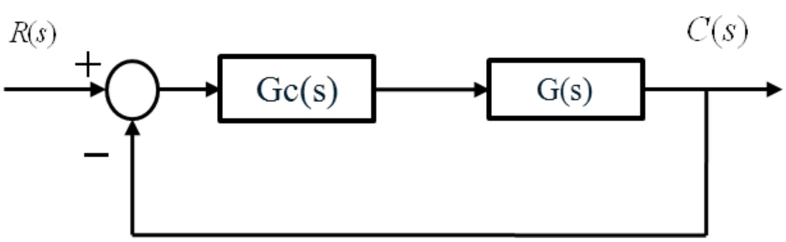
$\therefore$ $x_3(t)$ and $x_4(t)$ do not converge to zero.

Concept Explanation +2

$x_3(t)$ and $x_4(t)$ do not converge to zero. +1

Whole process +2

[3] (15 points) Consider a unity feedback closed-loop control system shown below



When the controller is proportional control, i.e.,
$G_c(s) = K = 10$

The Bode diagram of the system is shown in Figure 5.

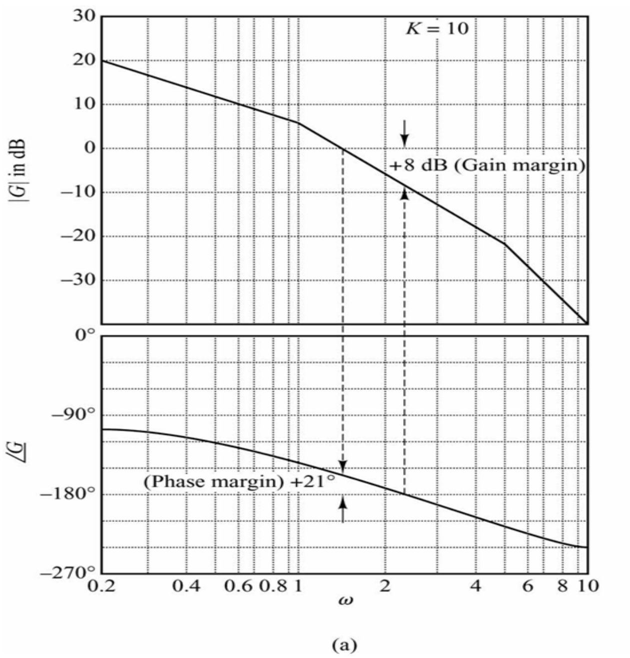
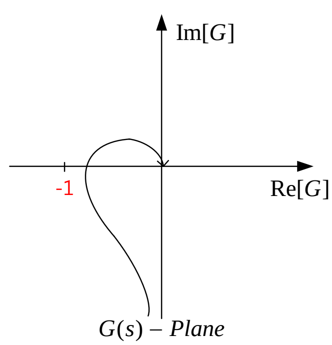


Fig. 5 Bode diagram of the system

(1) Sketch the Nyquist plot of the system.

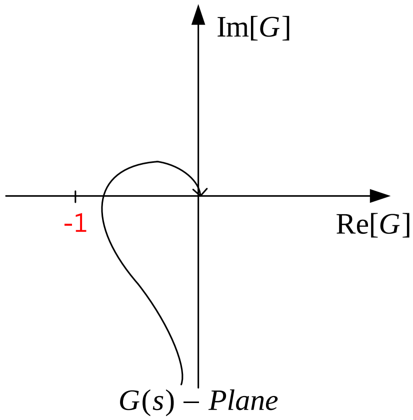
Solution:



(2) Determine the stability of the system using Nyquist Stability Criterion.

Solution: stable ;

Since the Nyquist plot of G(s) makes no enclosure of the -1+ j0 point, the system is stable.



(3) When the control gain is increased from 10 to 100, obtain the phase and gain margins of the system.

Solution:

|G| : 10 times → 20log|G| : + 20dB

→ Gain margin : -8dB +20 dB → -12 dB (12dB above 0 dB)
→ Corresponding phase is -210 degrees
→ Phase margin : -30 degrees (30degrees below -180degrees)

[4] (20 points) The input u, and the output y, of the single-input, single-output system are related by

$$\frac{d^3}{dt^3}y(t)+6\frac{d^2}{dt^2}y(t)+2\frac{d}{dt}y(t)+3y(t)=2\frac{d^2}{dt^2}u(t)-5\frac{d}{dt}u(t)-5u(t)$$

(1) Find the transfer function from U to Y.

$$\frac{Y(s)}{U(s)}=\frac{2s^2-5s-5}{s^3+6s^2+2s+3}$$

(2) Is this system stable?

$$charac.eqn = s^3 + 6s^2 + 2s + 3 = 0$$

s^3	1	2
s^2	6	3
s^1	$\frac{3}{2} = \frac{12-3}{6}$	0
s^0	3	

$$\therefore \text{ Stable (No sign change)}$$

(3) If u(t)=2 for all $t \geq 0$, what is the steady state value of y(t)?

$$U(s) = \frac{2}{s}$$

$$Y(s) = \frac{Y(s)}{U(s)} \cdot U(s) = \frac{2s^2 - 5s - 5}{s^3 + 6s^2 + 2s + 3} \cdot \frac{2}{s}$$

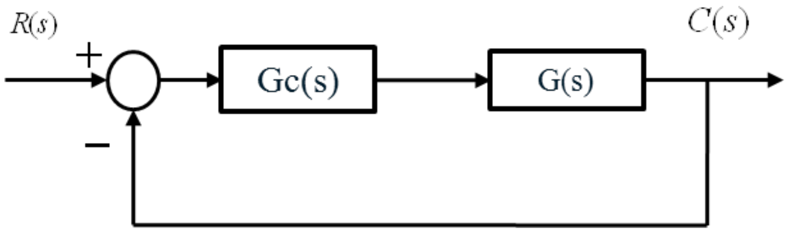
$$y_{ss}(t) = \lim_{s \to \infty} sY(s) = s \cdot \frac{2s^2 - 5s - 5}{s^3 + 6s^2 + 2s + 3} \cdot \frac{2}{s} = -\frac{10}{3}$$

(4) Suppose that the input is sinusoidal, $u(t) = \sin(10t)$. What is the steady state amplitude of the output y(t)?

$$u(t) = \sin(10t) = X \sin(10t) \therefore X = 1$$

$$|Y(jw)| = |G(jw)||U(jw)| = \left| \frac{-200 - 50j - 5}{-1000j - 600 + 20j + 3} \right| \cdot 1 = \left| \frac{-205 - 50j}{-980j - 507} \right| = \frac{\sqrt{205^2 + 50^2}}{\sqrt{980^2 + 597^2}}$$

[5] (15 points) Consider a system shown below.



$G_c(s) = K,$

$G(s) = \frac{1}{s}$

(1) Under what conditions is the system stable?

Solution)

$$\frac{C}{R} = \frac{K\frac{1}{s}}{1+K\frac{1}{s}} = \frac{K}{s+K}$$

The system is stable if $K > 0$
TF,답 맞으면 5 점
답만 맞으면 3 점

(2) If the system is stable, what is the time constant?

Time constant $T = \frac{1}{K}$

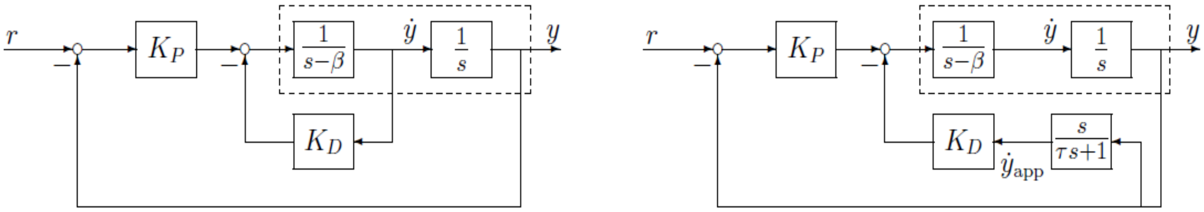
답 맞으면 5 점
과정 틀리면 2 점

(3) If the system is stable what is the steady-state gain from r to y ?

Steady state gain : 1

틀리면 0 점
과정 있으면 2 점
답 맞으면 5 점

[6] (10 points) Block diagram for two systems are shown below. The constant β is positive, $\beta > 0$. The system is stable if and only if $K_p > 0$ and $K_D > \beta$. What are the conditions on K_p, K_D and τ such that the system on the right is stable?



Solution)

Inner loop transfer function :

$$\frac{\dfrac{1}{s(s-\beta)}}{1+K_D\dfrac{s}{\tau s+1}\dfrac{1}{s(s-\beta)}}=\frac{\tau s+1}{s(s-\beta)(\tau s+1)+K_D s}$$

$$=\frac{\tau s+1}{s[(s-\beta)(\tau s+1)+K_D]}$$

Closed loop transfer function:

$$\frac{y}{r}=\frac{K_P(\tau s+1)}{s[(s-\beta)(\tau s+1)+K_D]+K_P(\tau s+1)}$$

$$=\frac{K_P(\tau s+1)}{s[\tau s^2+(1-\beta\tau)s-\beta+K_D]+K_P(\tau s+1)}$$

$$=\frac{K_P(\tau s+1)}{\tau s^3+(1-\beta\tau)s^2+(K_D+K_P\tau-\beta)s+K_p}$$

Routh's stability criterion

$$\begin{array}{lll} s^3 & \tau & K_D+K_P\tau-\beta \\ s^2 & 1-\beta\tau & K_P \\ s^1 & A & B \\ s^0 & C & \end{array}$$

$$A=\frac{(1-\beta\tau)(K_D+K_P\tau-\beta)-\tau\cdot K_P}{1-\beta\tau}$$

$$B=0$$

$$C=K_P$$

For stability

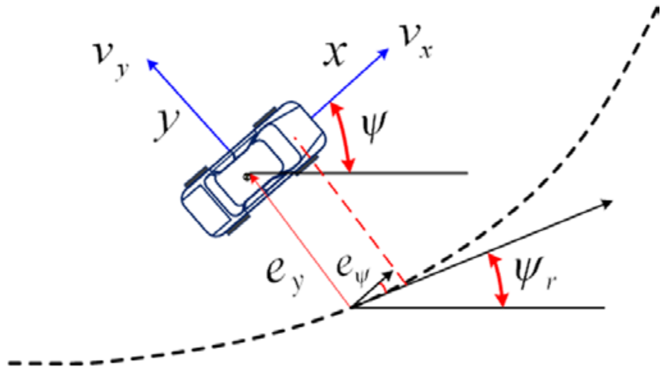
1) $1-\beta\tau>0,\ \tau<\dfrac{1}{\beta}$

2) $A>0$

3) $K_P>0$

전체 시스템의 Transfer function 을 구하면+ 5 점
Routh's Arrat 구하면 + 3 점
최종답 +2 점; 일부라도 쓰면 +1 점; $\tau=0$ 이라고 쓰면 0 점; 문제 조건 바로 인용하면 0 점.

[7] (20points) Consider an autonomous vehicle system shown in the Figure below:



The error dynamics are represented as follows:

$$\frac{de_y}{dt}=v_y+v_x\sin e_\psi-e_y\gamma\sin e_\psi$$
$$\frac{de_\psi}{dt}=\frac{d\psi}{dt}-\frac{d\psi_r}{dt}=\gamma-\frac{d\psi_r}{dt}=\gamma+w$$
$$e_\psi=\psi-\psi_r$$
$$w(t)=-\frac{d\psi_r}{dt}=-\frac{v_x}{R}$$

Where v_x is the vehicle speed and R is the radius of the path. The vehicle yaw rate, γ, is proportional to the steering input, δ, i.e.,

$$\frac{d\gamma(t)}{dt}=K_\gamma\delta(t)$$

(1) Assuming that the vehicle speed, v_x, is constant and the vehicle lateral velocity, v_y, is very small, obtain a linear state equation of the steering control system. Here, define the state and input as follows:

$x_1 = e_y$

$x_2 = e_\psi$

$y(t) = x_1(t)$

$u(t) = \delta(t)$

Solution)

Simplified model

$$\frac{de_y}{dt} = v_x \cdot e_\psi$$

$$\frac{de_\psi}{dt} = \gamma + w = K_r \cdot \delta(t) + w$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & v_x \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ K_r \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w \qquad u(t) = \delta(t)$$

$$y = x_1$$

Score: 4 점

simplified model state equation 두개 중 하나 틀릴 때 마다 -2 점

단순 계산 실수 -1 점

(2) Is the state equation model obtained in (1) controllable?

Solution)

Controllable $C = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 0 & v_x \\ 1 & 0 \end{bmatrix}$ $Rank(C) = 2$

Score: 3 점

Controllability matrix 정의 틀리면 -3 점

단순 계산 실수 -1 점

답만 쓰면 1 점

(3) Design a state feedback controller such that $\begin{matrix} x_1(t) \to 0 \\ x_2(t) \to 0 \end{matrix}$ as the time, t, increases when the disturbance, w, is zero.

Solution)

$$u = -k_1 x_1 - k_2 x_2$$

Score: 3 점

단순 실수 -1 점

(4) For the controller designed in (3), compute the steady state output error when the disturbance, w, is constant.

Solution)

$$\dot{x} = [A - BK]x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w$$
$$= \begin{bmatrix} 0 & v_x \\ -K_r \cdot k_1 & -K_r \cdot k_2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w$$

At steady state,

$$x(\infty) = -\begin{bmatrix} 0 & v_x \\ -K_r \cdot k_1 & -K_r \cdot k_2 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} w$$
$$= \frac{-1}{v_x K_r k_1} \begin{bmatrix} -K_r \cdot k_2 & -v_x \\ K_r \cdot k_1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} w$$
$$= \frac{-1}{v_x K_r k_1} \begin{bmatrix} -v_x \\ 0 \end{bmatrix} w$$
$$= \begin{bmatrix} \frac{1}{K_r k_1} w \\ 0 \end{bmatrix}$$

Score: 4 점
단순 계산 실수 -1 점

(5) Design a steering controller such that the steady state output error tends to zero when the disturbance, w, is constant.

Solution) zero with constant $w = -\frac{v_x}{R}$. i.e., R = constant, Vx = constant.

Let

$$\dot{\xi} = e_y = \dot{x}_3$$
$$\dot{x}_1 = v_x x_2$$
$$\dot{x}_2 = k_r u + w$$
$$\dot{x}_3 = e_y = x_1$$

$$\ddot{x}_1 = v_x \dot{x}_2$$
$$= v_x (k_r u + w)$$
$$u(t) = -k_1 x_1 - k_2 x_2 - k_3 x_3$$

$$\ddot{x}_1 = v_x k_r(-k_1 x_1 - k_2 x_2 - k_3 x_3) + v_x w$$
$$\dddot{x}_1 = -v_x k_r k_1 \dot{x}_1 - v_x k_r k_2 \dot{x}_2 - v_x k_r k_3 \dot{x}_3 + v_x \dot{w}$$

Since

$$\dot{x}_2 = \frac{\ddot{x}_1}{v_x}$$
$$\dot{x}_3 = x_1, \dot{w} = 0$$
$$\dddot{x}_1 = -v_x k_r k_1 \dot{x}_1 - v_x k_r k_2 \frac{\ddot{x}_1}{v_x} - v_x k_r k_3 x_1$$
$$\Rightarrow \quad \dddot{x}_1 + k_r k_2 \ddot{x}_1 + v_x k_r k_1 \dot{x}_1 + v_x k_r k_3 x_1 = 0$$

We can select k_1, k_2, k_3 that makes $x_1(t) \to 0 \quad as \quad t \to \infty$

Score: 6 점

I control term 을 만든 식을 써놓은 경우 부분점수 3 점
State x3 정의하지 않고 푸는 경우 -3 점
단순 계산 실수 -1 점