Midterm Exam Solutions, Fall '06

1.

a) B, B, A
b) $I - vv^T$ is symmetric. $\therefore$ diagonalizable
c) Let $Ax = \lambda x$. $x \neq 0$ by Definition.

$$x^T x = x^T A^T A x = (\lambda x)^T A x = \lambda^2 x^T x.$$
$$\therefore \lambda = \pm 1$$

d) $AC^T = (detA)I = 0$ if A is singular.

2.

In a quadratic form, $q(x) = x^T A x$

$$\text{where} A = \begin{bmatrix} 19 & -14 & 0 \\ -14 & 19 & 0 \\ 0 & 0 & 17 \end{bmatrix}.$$
$$det(A - \lambda I) = (17 - \lambda)[(19 - \lambda)^2 - 14^2] = 0$$
$$\Longrightarrow \lambda = 17, 5, 33$$
$$\therefore A \text{ is positive definite.}$$

Thus we can transform to a principal axis form using the normalized eigenvector matrix, i.e., $x = Xy$, $X^T X = I$, and then

$$q(x) = y^T \Lambda y = \sum_{i=1}^{3} \lambda_i y_i^2, \qquad (*)$$
$$\text{where } \Lambda = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}.$$

Since $\sum_{i=1}^{3} y_i^2 \sum_{i=1}^{3} x_i^2 = 1$, the maximum value of (*) is the maximum eigenvalue of A, which is 33.

3.

Elementary Row operation

$$\left[\begin{array}{ccc|c} 1 & 2 & -2 & b_1 \\ 2 & 5 & -4 & b_2 \\ 4 & 9 & -8 & b_3 \end{array}\right] \Longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & -2 & b_1 \\ 0 & 1 & 0 & b_2 - 2b_1 \\ 0 & 1 & 0 & b_3 - 4b_1 \end{array}\right] \Longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & -2 & b_1 \\ 0 & 1 & 0 & b_2 - 2b_1 \\ 0 & 0 & 0 & b_3 - b_2 - 4b_1 \end{array}\right].$$

Then x_3: arbitrary, $x_2 = b_2 - 2b_1$, $x_1 = b_1 - 2x_2 + 2x_3 = 5b_1 - 2b_2 + 2x_3$.

$$\therefore x = \begin{bmatrix} 5b_1 - 2b_2 \\ b_2 - 2b_1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}.$$

4.

This is in the form of Fourier Cosine Integral

$$g(x) = \int_0^\infty A(w) \cos wx \; dw \text{ (where } g(x) \text{ is conti.)}$$
$$A(w) = \frac{2}{\pi} \int_0^\infty g(v) \cos wv \; dv = \frac{\sin w}{w}. \quad (1)$$

Since $\frac{2}{\pi} \int_0^1 \frac{\pi}{2} \cos wv \; dv = \frac{\sin w}{w}$, the equation in the problem is satisfied by $g(x) = \begin{cases} \frac{\pi}{2} & 0 \leq x < 1 \\ 0 & x > 1 \end{cases}$
And at $x = 1$, where $g(x)$ is discontinuous, the value of the integral converges to

$$\frac{1}{2}(g(1^{+1}) + g(1^{-1})) = \frac{\pi}{4}.$$

5.

Let

$$\hat{u}(w,t) := \mathbb{F}\{u(x,t)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty u(x,t) e^{-iwx} dx.$$

Taking $\mathbb{F}$ of the first equation in the problem,

$$\hat{u}_{++}(w,t) = -c^2 w^2 \hat{u}(w,t)$$
$$\therefore \hat{u}(w,t) = A \sin cwt + B \cos cwt.$$

Taking $\mathbb{F}$ of I.C.s,

$$\hat{u}(w,0) = \mathbb{F}\{f(x)\} = B$$
$$\hat{u}_t(w,0) = \mathbb{F}\{g(x)\} = cwA$$
$$\therefore \hat{u}(w,t) = \mathrm{F}(w) \cos cwt + \frac{\mathrm{G}(w)}{cw} \sin cwt$$
$$\text{where } \mathrm{F}(w) = \mathbb{F}\{f(x)\}, \; \mathrm{G}(w) = \mathbb{F}\{g(x)\}$$

Taking $\mathbb{F}^{-1}$

$$u(x,t) = \mathbb{F}^{-1}\{\hat{u}(w,t)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty (\mathrm{F}(w) \cos cw + \frac{\mathrm{G}(w)}{cw} \sin cwt) e^{iwx} \; dw.$$
$$\therefore u_{mn}(x,y,z) = A_{mn} \sinh(\sqrt{(\tfrac{m\pi}{a})^2 + (\tfrac{n\pi}{b})^2} z) \sin(\tfrac{m\pi x}{a}) \sin(\tfrac{n\pi y}{b}).$$

The remaining b, c

Thus A_{mn} are the double Fourier Series coefficients.

$$\therefore u(x,y,z) = \sum_{m=1}^\infty \sum_{n=1}^\infty A_{mn} \sinh(w_{mn} z) \sin(\tfrac{m\pi x}{a}) \sin(\tfrac{n\pi y}{b})$$
$$\text{where } w_{mn} = \sqrt{(\tfrac{m\pi}{a})^2 + (\tfrac{n\pi}{b})^2}$$
$$A_{mn} = \frac{4}{ab \sinh(cw_{mn})} \int_0^b \int_0^a f(x,y) \sin(\tfrac{m\pi x}{a}) \sin(\tfrac{n\pi y}{b}) \; dx \; dy.$$

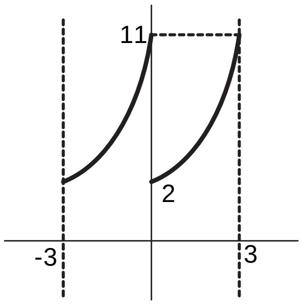
7.

a) True, because

$$\int_{-\pi}^{\pi} f(x) g(x) \; dx = 0$$

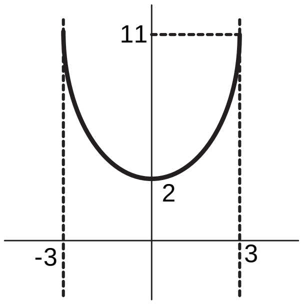
b)

Fourier series



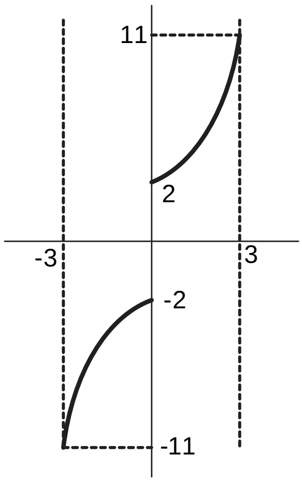
$\Longrightarrow$ converges to $\frac{2+11}{2} = 6.5$

Cosine Series



$\Longrightarrow$ converges to 2.

Sine series



$\Longrightarrow$ converges to 0.