School of Mech & Aero Eng
Seoul National University

Eng Math 2
Fall '06

FINAL

- Do not open exam until told to do so.
- 'How you arrived at your answer' is much more important than the answer itself. Read carefully, and make sure you show your work *step by step*.
- Please use a separate sheet for each problem.
- Ask questions if you don't understand what the problem says.
- I thank you all, and wish you a wonderful winter break. But before that, **GOOD LUCK** tonight !

Student ID: ________________

Name: ________________

1	/ 10
2	/ 15
3	/ 15
4	/ 30
5	/ 20
6	/ 10
Total	/ 100

1. [10 pts]

(a) Find a, b, c, d such that $f(z) = x^2 + axy + by^2 + i(cx^2 + dxy + y^2)$ is analytic.

(b) Suppose that $f(z) = u(x, y) + iv(x, y)$ is analytic. Show that the family of curves $u(x, y) = c_1$ are orthogonal to $v(x, y) = c_2$.

2. [15 pts]

(a) Compute

$$\oint_C (z^2 + \frac{1}{z-2} + Re(z))\,dz\ , \quad C: \text{the triangle with vertices } 0, 1, 1+i, \text{ counterclockwise.}$$

(b) Represent $\mathrm{Ln}(\frac{1}{1-z})$ as a Maclaurin series and find the region of convergence of the series.

(c) C: straight line segment from i to $2+i$. Show that

$$\left| \int_C \mathrm{Ln}(z+1)dz \right| \leq \log_e 10 + \pi/2$$

3. [15 pts] For $f(z) = \log_e |z| + i\text{Arg } z = u(x, y) + iv(x, y)$.

(a) Check analyticity of $f(z)$.

(b) Are $u(x, y)$, $v(x, y)$ harmonic? Why?

4. [30 pts] Show or compute the following:

(a)

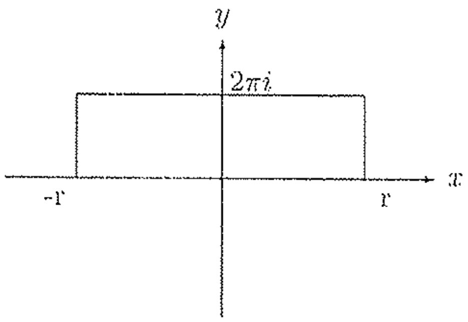
$$\int_0^{2\pi} \frac{\sin^2\theta}{a+b\cos\theta}\,d\theta = \frac{2\pi}{b^2}\left(a-\sqrt{a^2-b^2}\right) \qquad (a>b>0)$$

(b)

$$\text{p.v.} \int_{-\infty}^{\infty} \frac{\cos sx}{x^2(x^2+1)}\,dx = ?$$

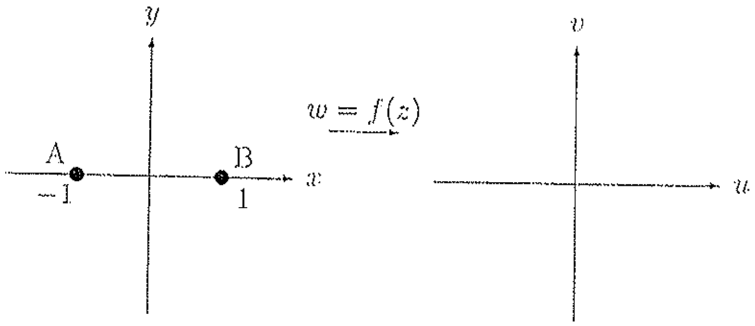
(c) Use the following contour to show that

$$\text{p.v.} \int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x}\,dx = \frac{\pi}{\sin a\pi} \qquad (0<a<1)$$



5. [20 pts]

(a) Describe why $f(z) = z + 1/z$ is conformal at all values of z except $z = 0, \pm 1$.



(b) Under the mapping $w = f(z)$,

(i) find the images of the points A and B, and the unit circle $|r| = 1$.

(ii) show that a circle of $r > 1$ is mapped to an ellipse.

(iii) show that a ray $\theta = const$ is mapped to a hyperbola.

(c) Are the hyperbolas and ellipses found in (b) are orthogonal? Why do you think so?

6. [10 pts]

(a) Show that $f(z) = \tan(1/z)$ has an infinite number of singularities. Are any of these isolated ?

(b) Does $f(z)$ have a Laurent series that converges in a region $0 < |z| < R$? Why?

$$\sin z = \sin x \cosh y + i \cos x \sinh y$$
$$\sin z = \frac{1}{2i}(e^{iz} - e^{-iz})$$
$$\cos z = \frac{1}{2}(e^{iz} + e^{-iz})$$
$$\cosh z = \frac{1}{2}(e^{z} + e^{-z})$$
$$\sinh z = \frac{1}{2}(e^{z} - e^{-z})$$