Final Exam Solutions, Fall '06

1.

a)

$$\left\{ \begin{array}{c} \frac{\partial u}{\partial x} = 2x + ay \\ \frac{\partial v}{\partial y} = dx + 2y \end{array} \right\} \Rightarrow \left\{ \begin{array}{c} a = 2 \\ d = 2 \end{array} \right\}$$

$$\left\{ \begin{array}{c} -\frac{\partial u}{\partial y} = -(ax + 2by) \\ \frac{\partial v}{\partial x} = 2cx + dy \end{array} \right\} \Rightarrow \left\{ \begin{array}{c} b = -\frac{d}{2} = -1 \\ c = -\frac{a}{2} = -1 \end{array} \right\}$$

b)

$$\nabla u(x,y) = [\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}]$$

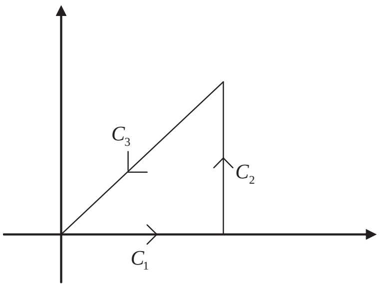
$$\nabla v(x,y) = [\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}] = [-\frac{\partial u}{\partial y}, \frac{\partial u}{\partial x}]$$

$$\nabla u \cdot \nabla v = -(\frac{\partial u}{\partial x})(\frac{\partial u}{\partial y}) + (\frac{\partial u}{\partial y})(\frac{\partial u}{\partial x}) = 0$$

(trivial when derivatives are zero)

2.

a)



$$\int_C z^2 dz = 0, \int_C \frac{1}{z-2} = 0, \int_{C1} x dz = \frac{1}{2}$$

where $z = x$

$$\int_{C2} x dz = \int_0^1 1 i dy = i$$

where $z = 1 + yi$

$$\int_{C3} x dz = \int_1^0 x[1+i] dx = \frac{-1}{2}(1+i)$$

where $z = x + x_1$

$$ans = \int_{C1} + \int_{C2} + \int_{C3} = \frac{i}{2}$$

b)

$$f(z) = Ln\frac{1}{1-z} = -Ln(1-z)$$

analytic for $Re(1-z) > 0$

$$f(0) = 0$$

$$f'(0) = \frac{1}{1-z}|_{z=0} = 1$$

$$f''(0) = (\frac{1}{1-z})^2|_{z=0} = 1$$

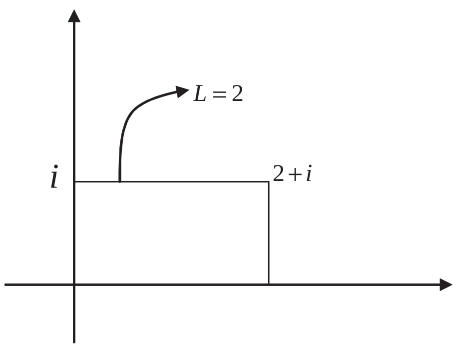
$$\vdots$$

$$f^{(n)}(0) = (n-1)!$$

$$\therefore \sum_{n=1}^{\infty} \frac{f^{(n)}(0)}{n!} z^n = \sum_{n=1}^{\infty} \frac{z^n}{n}$$

$$|\frac{A_{n+1}}{A_n}| = |\frac{\frac{1}{n+1} z^{n+1}}{\frac{1}{n} z^n}| \to |z| < 1 \therefore R = 1$$

c)



$$\int Ln(1+z)dz$$

$$Ln(1+z) = \ln|z+1| + i\arg(1+z)$$

$$|\ln|1+z|| \leq \ln\sqrt{10} \triangleq M_1$$

$$|\arg(1+z)| \leq \frac{\pi}{4} \triangleq M_2$$

$$ans \leq M_1 L + M_2 L = \ln 10 + \frac{\pi}{2}$$

3.

a)

C-R eqns

$$\left\{ \begin{array}{c} U_r = \frac{1}{r} V_\theta \\ V_r = -\frac{1}{r} u_\theta \end{array} \right\}$$

$$\left\{ \begin{array}{c} U_r = \frac{1}{r} \\ V_\theta = 1 \end{array} \right\}$$

$$\left\{ \begin{array}{c} U_\theta = 0 \\ V_r = 0 \end{array} \right\}$$

b)

If $f(z)$ analytic, then u, v satisfy Laplace eqn, because $\left\{ \begin{array}{c} u_{xx} = v_{yx} \\ u_{yy} = -v_{xy} \end{array} \right\}, \left\{ \begin{array}{c} v_{xx} = -u_{yx} \\ v_{yy} = u_{xy} \end{array} \right\}$

4.

a)

$$z = e^{i\theta}$$

$$d\theta = \frac{dz}{iz}, \cos\theta = \frac{1}{2}(z + z^{-1}), \sin\theta = \frac{1}{2i}(z - z^{-1})$$

$$ans = \oint_C \frac{(z^2-1)^2}{z^2 + \frac{2a}{b}(z+z^{-1})} \frac{dz}{z^2}$$

double pole at $z_0 = 0$, simple pole at $z_1 = -\frac{a}{b} + \sqrt{(\frac{a}{b})^2 - 1}, z_2 = -\frac{a}{b} - \sqrt{(\frac{a}{b})^2 - 1}$. Let $f(z) = \frac{(z^2-1)^2}{z^2+\frac{2a}{b}(z+z^{-1})} \frac{1}{z^2}$

$$Res_0 f(z) = \lim_{z \to 0} \frac{d}{dz} z^2 f(z) = \lim_{z \to 0} \frac{d}{dz} \frac{(z^2-1)^2}{z^2 + \frac{2a}{b} z + 1} = -\frac{2a}{b}$$

$$Res_{z_1} f(z) = \lim_{z \to z_1} \frac{(z^2-1)^2}{(z-z_2)z^2} = \frac{(z_1^2-1)^2}{(z_1-z_2)z_1^2} = \frac{-4[\frac{a}{b}z_1+1]^2}{2\sqrt{(\frac{a}{b})^2-1}(\frac{a}{b}z_1+1)} = \frac{2}{b}\sqrt{a^2-b^2}$$

$$ans = \frac{2\pi i}{-2ib}[-\frac{2a}{b} + \frac{2}{b}\sqrt{a^2-b^2}] = \frac{\sqrt{2\pi}}{b^2}[a - \sqrt{a^2-b^2}]$$

$$(where \quad z_1^2 = -(\frac{2a}{b} z_1 + 1), z_1^2 - 1 = -2[\frac{a}{b} z_1 + 1])$$

b)

$$\int_{-\infty}^{\infty} \frac{\cos sx}{x^2(x^2+1)} dx$$

$$f(z) = \frac{\cos sz}{z^2(z^2+1)}$$

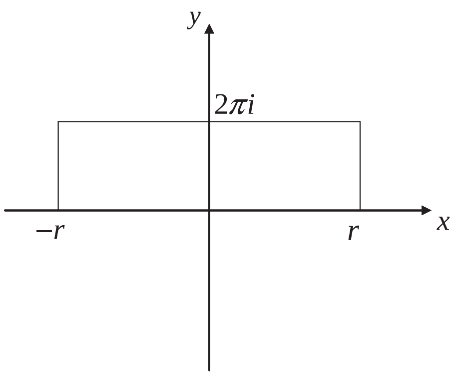
double pole at $z = 0$
simple poles at $\pm i$

$$Res_0 f(z) = \lim_{z\to 0} \frac{d}{dz}(\frac{\cos sz}{z^2+1}) = \lim_{z\to 0} \frac{-\sin sz \cdot (z^2+1) - \cos sz \cdot 2z}{(z^2+1)^2} = 0$$

$$Res_i f(z) = \lim_{z\to 0}(\frac{\cos sz}{z^2(z+i)}) = \frac{\cos is}{-1(2i)} = \frac{1}{2i}[\frac{e^{i(is)}e^{-i(is)}}{2}] = \frac{e^s + e^{-s}}{4i} = -\frac{1}{2i}\cosh s$$

$$ans = \pi i \cdot 0 + 2\pi i \cdot \frac{-1}{2i}\cosh h = -\pi \cosh s$$

c)



$$\int_{-r}^{r} \frac{e^{ax}}{1+e^x}dx - \int_{-r}^{r} \frac{e^{ax}e^{2\pi ai}}{1+e^x}dx$$

Let $z = x + 2\pi i$

$$f(z) = \frac{e^(az)}{1+e^z}$$

$$Res_{\pi i} f(z) = \lim_{z\to \pi i} \frac{e^{az}}{e^z} = \frac{e^{a\pi i}}{e^{\pi i}} = -e^{\pi ai}$$

$$(1 - e^{2\pi ai}) \int_{-r}^{r} \frac{2^{ax}}{1+e^x}dx + [\frac{e^{ar}}{1+e^r}i(2\pi i) - \frac{2^{-ar}}{1+e^{-r}}i(2\pi i)] = 2\pi i \cdot (-e^{\pi ai})$$

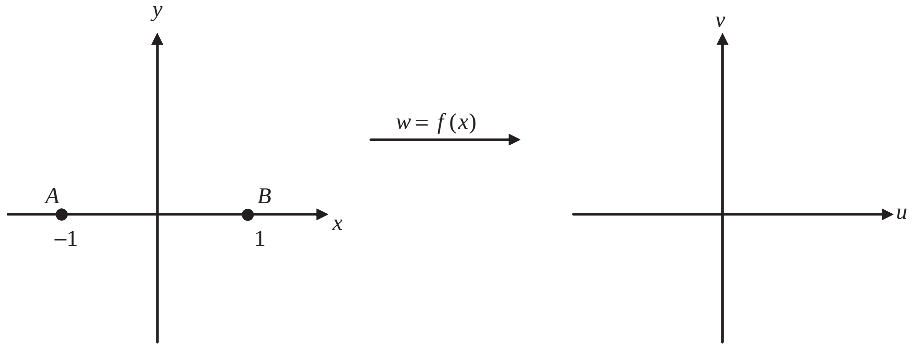
$$(where \lim_{r\to\infty}[\frac{e^{ar}}{1+e^r}i(2\pi i) - \frac{2^{-ar}}{1+e^{-r}}i(2\pi i)] = 0)$$

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x}dx = 2\pi i \frac{e^{\pi ai}}{1-e^{2\pi ai}} = \pi \frac{2i}{e^{-\pi ai} - e^{-\pi ai}} = \frac{\pi}{\sin a\pi}$$

5.

a)

$f(z)$ analytic except $z=0$, $f'(z) = 1 - \frac{1}{z^2} \neq 0$ except $z = \pm 1$.



b)

(i)

$$f(-1) = -2, f(1) = 2$$

$$f(r,\theta) = re^{i\theta} + \frac{1}{r}e^{-i\theta} = (r + \frac{1}{r})\cos\theta + i(r - \frac{1}{r})\sin\theta$$

Let $u = (r + \frac{1}{r})\cos\theta$, $v = i(r - \frac{1}{r})\sin\theta$.
If $r = 1$, then $v = 0, u = 2\cos\theta \Rightarrow$ line segment $[-2, 2]$.

(ii)

If $r \neq 1$, then

$$\frac{u^2}{(r+\frac{1}{r})^2} + \frac{v^2}{(r-\frac{1}{r})^2} = 1 \qquad \text{ellipse}$$

(iii)

If $z = te^{i\theta}, t \neq 1, \theta = const$ then,

$$\frac{u^2}{\cos^2\theta} - \frac{v^2}{\sin^2\theta} = (t + \frac{1}{t})^2 - (t - \frac{1}{t})^2 = 4 \qquad \text{hyperbola.}$$

c)

Circle and ray : orthogonal.
Their images under the conformal mapping $\rightarrow$ orthogonal

6.

a)

$\tan(\frac{1}{z}) = \frac{\pi}{2} \pm 2n\pi$: singularities, isolated, but $z = 0$ is nonisolated.

b)

NO!! Laurent series converges in an annular between singularities, but $z = 0$ is NOT isolated.