Electromagnetics

Midterm 1 | Date: Apr. 9, 2019 | Duration: 75 minutes | Full marks: 500 points

*** Follow the notation used in the textbook** (ε: permittivity, μ: permeability, ρ: charge density, σ: conductivity, $\boldsymbol{E}$, $\boldsymbol{D}$, $\boldsymbol{B}$, $\boldsymbol{H}$**:** electric field, electric flux, magnetic flux, magnetic field intensity vectors, $\boldsymbol{J}$: Volume current density vector). Note that the **Bold** characters denote the vector quantities.

Q1 (*130 points*)

(a) Write down Maxwell's equations and the two constitutive equations. (***20 points***)

$$\text{Maxwell's Eqns: } \begin{cases} \nabla \times \boldsymbol{E} = -\dfrac{\partial \boldsymbol{B}}{\partial t} \\ \nabla \times \boldsymbol{H} = \boldsymbol{J} + \dfrac{\partial \boldsymbol{D}}{\partial t} \\ \nabla \cdot \boldsymbol{D} = \rho \\ \nabla \cdot \boldsymbol{B} = 0 \end{cases}, \quad \text{Constitutive relation: } \begin{cases} \boldsymbol{D} = \varepsilon \boldsymbol{E} = \varepsilon_0 \boldsymbol{E} + \boldsymbol{P} \\ \boldsymbol{B} = \mu \boldsymbol{H} = \mu_0 (\boldsymbol{H} + \boldsymbol{M}) \end{cases}$$

6 points are given per a correct equation. Give **final 2 points** if all the equations are correct.

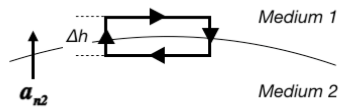
(b) Derive general boundary conditions for the field quantities from Maxwell's equations. (***40 points***)

B.C. for tangential components of ***E*** and ***H***

$$\oint_C \boldsymbol{E} \cdot d\boldsymbol{l} = -\int_S \cancel{\frac{\partial \boldsymbol{B}}{\partial t}} \cdot d\boldsymbol{s} \rightarrow \boxed{E_{1t} = E_{2t} \quad (\text{V/m})}$$

$$\oint_C \boldsymbol{H} \cdot d\boldsymbol{l} = \int_S \left(\boldsymbol{J} + \cancel{\frac{\partial \boldsymbol{D}}{\partial t}} \right) \cdot d\boldsymbol{s} \rightarrow \boxed{\boldsymbol{a}_{n2} \times (\boldsymbol{H}_1 - \boldsymbol{H}_2) = \boldsymbol{J}_S \quad (\text{A/m})}$$

Time-varying terms vanishes as S goes to 0!

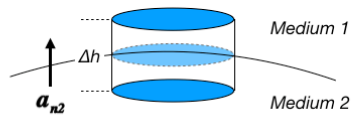


Scheme 1

B.C. for normal components of ***D*** and ***B***

$$\nabla \cdot \boldsymbol{D} = \rho \rightarrow \boxed{\boldsymbol{a}_{n2} \cdot (\boldsymbol{D}_1 - \boldsymbol{D}_2) = \rho_S \quad (\text{C/m}^2)}$$

$$\nabla \cdot \boldsymbol{B} = 0 \rightarrow \boxed{B_{1n} = B_{2n} \quad (\text{T})}$$



Scheme 2

10 points are given per a correct equation. deduct **3 points** for each if integration schemes are not stated.

(c) Derive non-homogeneous wave equations for vector magnetic and scalar electric potentials under Lorentz condition. Assume that the wave exists in the simple medium (linear, isotropic, and homogeneous). (***30 points***)

$$\boxed{\begin{array}{l} \boldsymbol{H} = \dfrac{1}{\mu}\boldsymbol{B} = \dfrac{1}{\mu}\nabla \times \boldsymbol{A} \\ \boldsymbol{D} = \varepsilon \boldsymbol{E} = \varepsilon\left(-\nabla V - \dfrac{\partial \boldsymbol{A}}{\partial t}\right) \end{array}} \rightarrow \nabla \times \boldsymbol{H} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t}$$

Maxwell's Equations

$$\nabla \cdot \boldsymbol{D} = \rho \leftarrow \boxed{\boldsymbol{D} = \varepsilon \boldsymbol{E} = \varepsilon\left(-\nabla V - \frac{\partial \boldsymbol{A}}{\partial t}\right)}$$

$$\nabla^2 \boldsymbol{A} - \mu\varepsilon \frac{\partial^2 \boldsymbol{A}}{\partial t^2} = -\mu \boldsymbol{J} + \nabla\left(\nabla \cdot \boldsymbol{A} + \mu\varepsilon \frac{\partial V}{\partial t}\right)$$

$$\nabla^2 V + \frac{\partial}{\partial t}(\nabla \cdot \boldsymbol{A}) = -\frac{\rho}{\varepsilon}$$

$$\nabla \cdot \boldsymbol{A} + \mu\varepsilon \frac{\partial V}{\partial t} = 0$$

Lorenz condition

$$\boxed{\nabla^2 \boldsymbol{A} - \mu\varepsilon \frac{\partial^2 \boldsymbol{A}}{\partial t^2} = -\mu \boldsymbol{J}}$$

Non-homogeneous equation for A

$$\boxed{\nabla^2 V - \mu\varepsilon \frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\varepsilon}}$$

Non-homogeneous equation for V

15 points are given per correct derivation of each equation. Deduct **3 points** if constitutive relation is not properly used. Deduct **3 points** if Lorentz condition is not used. Deduct additional **2 points** if no derivation at all stated and only final equations are written.

(d) Suppose that there is time-varying charge density at the origin, which is the only source of the electromagnetic wave and is given by

$$\rho(\boldsymbol{R},t)=\rho_0\delta(\boldsymbol{R})\cos\omega t$$

Here, $\delta(\boldsymbol{R})$ denotes a Dirac-delta function specified in three-dimensional space.

Q: Determine the scalar electric potential V at a distance of $|\boldsymbol{R}|$ from the origin caused by $\rho(\boldsymbol{R},t)$ under Lorentz condition. (***20 points***)

As a first step, we need to derive the electric potential V from the equation obtained in (c). Following the derivation step we went through in the class, electric potential V is given by

$$V(R,t)=\frac{1}{4\pi\varepsilon}\int\frac{\rho(R,t')}{|\boldsymbol{R}-\boldsymbol{R}'|}d^3R'=\frac{1}{4\pi\varepsilon}\int\frac{\rho(R,t-|\boldsymbol{R}-\boldsymbol{R}'|/u)}{|\boldsymbol{R}-\boldsymbol{R}'|}d^3R'\quad\cdots(1)$$ (**10 points** if correct).

Now by plugging $\rho(\boldsymbol{R},t)=\rho_0\delta(\boldsymbol{R})\cos\omega t\quad\cdots(2)$ into the derived electric potential as in (1), we get

$$V(R,t)=\frac{1}{4\pi\varepsilon}\int\frac{\rho_0\delta(\boldsymbol{R}')\cos\omega\left(t-|\boldsymbol{R}-\boldsymbol{R}'|/\sqrt{\mu\varepsilon}\right)}{|\boldsymbol{R}-\boldsymbol{R}'|}d^3R'=\frac{\rho_0}{4\pi\varepsilon}\frac{\cos\left(t-R/\sqrt{\mu\varepsilon}\right)}{R}$$. **10 points** are given if the final form is correct.

Q: Describe the process how you will obtain the electric and magnetic fields by using the given electric potential [No derivation required!]. (***20 points***)

Electric and magnetic fields can be obtained using their potentials as given by

$\boldsymbol{B}=\nabla\times\boldsymbol{A},\ \boldsymbol{E}=-\nabla V-\frac{\partial\boldsymbol{A}}{\partial t}$ (***10 points***). Since we already got electric potential V as above, we only need to obtain the magnetic potential $\boldsymbol{A}$. According to **Helmholtz theorem**, the curl and divergence of $\boldsymbol{A}$ should be given to determine it everywhere. Its divergence can be provided by **Lorentz condition** as $\nabla\cdot\boldsymbol{A}=-\mu\varepsilon\frac{\partial V}{\partial t}$. Hence, we additionally need to verify $\nabla\times\boldsymbol{A}$ to completely determine the electric and magnetic fields. (***5 points** given if Lorentz condition stated and additional **5 points** if Helmholtz theorem stated*)

Electromagnetics

Midterm 1 | Date: Apr. 9, 2019 | Duration: 75 minutes | Full marks: 500 points

Q2 (*70 points*)

Suppose that the medium is source-free, simple (linear & isotropic) and lossless.

(a) Derive the homogeneous wave equation for the electric field. (***20 points***)

• Let's take a curl to each side of $\nabla \times \boldsymbol{E} = -\mu \dfrac{\partial \boldsymbol{H}}{\partial t}$

$$\rightarrow \nabla \times \nabla \times \boldsymbol{E} = -\mu \frac{\partial}{\partial t}(\nabla \times \boldsymbol{H})$$

$$\text{(l.h.s)}\ \nabla \times \nabla \times \boldsymbol{E} = \nabla(\nabla \cdot \boldsymbol{E}) - \nabla^2 \boldsymbol{E} = -\nabla^2 \boldsymbol{E}$$

$$\text{(r.h.s)}\ -\mu \frac{\partial}{\partial t}(\nabla \times \boldsymbol{H}) = -\mu \frac{\partial}{\partial t}\left(\varepsilon \frac{\partial \boldsymbol{E}}{\partial t}\right) = -\mu\varepsilon \frac{\partial^2 \boldsymbol{E}}{\partial t^2}$$

$$\text{(l.h.s)} = \text{(r.h.s)} \rightarrow -\nabla^2 \boldsymbol{E} = -\mu\varepsilon \frac{\partial^2 \boldsymbol{E}}{\partial t^2}$$

$$\therefore \nabla^2 \boldsymbol{E} - \mu\varepsilon \frac{\partial^2 \boldsymbol{E}}{\partial t^2} = 0$$

15 points are given if the derivation makes sense (i.e. starting from Maxwell's equations and using the vector identity [curl of curl]). Additional **5 points** given if students used the fact that the divergence of *E* is zero (=source-free).

(b) Determine the solution to the equation obtained in (a) and briefly discuss its physical meaning. (***20 points***)

The solution to the above equation is given by $\boldsymbol{E}(\boldsymbol{R},t) = \boldsymbol{E}_0 e^{j(\omega t - \boldsymbol{k}\cdot\boldsymbol{R})}$ (**15 points**),

representing the plane wave propagating in an arbitrary direction. Here, the wave vector ***k*** represents the direction of propagation of the wave (**5 points**. Deduct this if the direction of the plane wave is not specified. *kz* is also ok as long its direction is mentioned).

(c) Describe the relationship between electric field, magnetic field, and propagation direction of the electromagnetic wave obtained in (b). [*No derivation required!*] (***15 points***)

The wave obtained in (b) is the TEM wave whose electric and magnetic fields are perpendicular to each other (**6 points**) and both are normal to the propagation direction (**6 points**). Additional **3 points** if both stated.

(d) For a simple medium whose constitutive parameters are not functions of time, one can derive the following rate equation from Maxwell's equations:

$$-\oint_S (\mathbf{E} \times \mathbf{H}) \cdot d\mathbf{s} = \frac{\partial}{\partial t}\int_V \left(\frac{1}{2}\varepsilon E^2 + \frac{1}{2}\mu H^2\right) dv + \int_V \sigma E^2\, dv.$$

Discuss the physical meaning of this rate equation. (***15 points***)

Above is the equation form of the Poynting's theorem. It states that power of the electromagnetic wave, represented by ***E*** and ***H***, flowing into a closed surface (*S*) enclosing a volume *V* at any instant (**5 points**) equals to the *time rate* of increase of stored electric and magnetic energies (**5 points**), and *Ohmic power* dissipated within a volume *V* (**5 points**). Deduct **2 points** if "time-rate" or *power* is not mentioned. Deduct **2 points** if the spatial info in integration is not specified (i.e. *S* and *V*)

Electromagnetics

Midterm 1 | Date: Apr. 9, 2019 | Duration: 75 minutes | Full marks: 500 points

Q3 (*60 points*)

Suppose that there is a plane interface (*xy plane*) at $z = 0$ between free space ($z < 0$) and a perfect conductor ($z > 0$). There exists *time-varying* current flowing along such interface and is given by $\boldsymbol{J_s} = \boldsymbol{a_x} J_{s0} \cos \omega t$. Under the given condition, determine the instantaneous electric and magnetic field expressions in free space (i.e. at $z < 0$).

This is the situation that the electromagnetic wave, propagating in the z-direction, is reflected at a plane conducting boundary $z = 0$. That is, the incident wave in free space induces the surface current $\boldsymbol{J_s}$ at the surface of the perfect conductor, which in return, generates the electromagnetic wave propagating in the opposite direction (i.e., the reflected wave). Thus, there exist incident and reflected electromagnetic waves in free space, which are represented as

$$\text{Incident wave: } \begin{cases} \boldsymbol{E}_i(z) = \boldsymbol{a_x} E_{i0} e^{-j\beta_1 z} \\ \boldsymbol{H}_i(z) = \boldsymbol{a_y} \dfrac{E_{i0}}{\eta_1} e^{-j\beta_1 z} \end{cases}, \quad \text{Reflected wave: } \begin{cases} \boldsymbol{E}_r(z) = \boldsymbol{a_x} E_{r0} e^{+j\beta_1 z} \\ \boldsymbol{H}_r(z) = -\boldsymbol{a_y} \dfrac{E_{r0}}{\eta_1} e^{+j\beta_1 z} \end{cases}.$$

Then the total electric and magnetic fields in free space (denoted as medium 1) are given by

$$\begin{cases} \boldsymbol{E}_1(z) = -\boldsymbol{a_x} j2E_{i0} \sin(\beta_1 z) \\ \boldsymbol{H}_1(z) = \boldsymbol{a_y} 2\dfrac{E_{i0}}{\eta_1} \cos(\beta_1 z) \end{cases} \cdots(1).$$

Here, we used the fact that

$$\begin{cases} \boldsymbol{E}_1(z) = \boldsymbol{E}_2(z) = 0 \\ \boldsymbol{H}_1(z) = \boldsymbol{H}_2(z) = 0 \end{cases} \Bigg|_{z=0},$$

because the medium 2 is a perfect conductor, and hence both electric and magnetic fields in and at medium 2 vanish (**20 points** up to here).

On the other hand, the boudary condition yields $\boldsymbol{a_{n2}} \times \boldsymbol{H}_1 = \boldsymbol{J_S}$, where $\boldsymbol{a_{n2}} = -\boldsymbol{a_z}$ and $\boldsymbol{J_s} = \boldsymbol{a_x} J_{s0}$. Thus,

$$\left[\boldsymbol{H}_1(0) = \boldsymbol{a_y} 2\frac{E_{i0}}{\eta_1}\right] = \boldsymbol{a_y} J_{s0} \quad \rightarrow \quad 2\frac{E_{i0}}{\eta_1} = J_{s0} \quad \cdots(2) \quad \textbf{(25 points)}$$

Now, by plugging (2) into (1), we get

$$\begin{cases} \boldsymbol{E}_1(z) = -\boldsymbol{a_x} j\eta_1 J_{s0} \sin(\beta_1 z) \\ \boldsymbol{H}_1(z) = \boldsymbol{a_y} J_{s0} \cos(\beta_1 z) \end{cases} \cdots(3) \quad \textbf{(5 points)}$$

Finally, the instantaneous expressions for electric and magnetic fields are given by

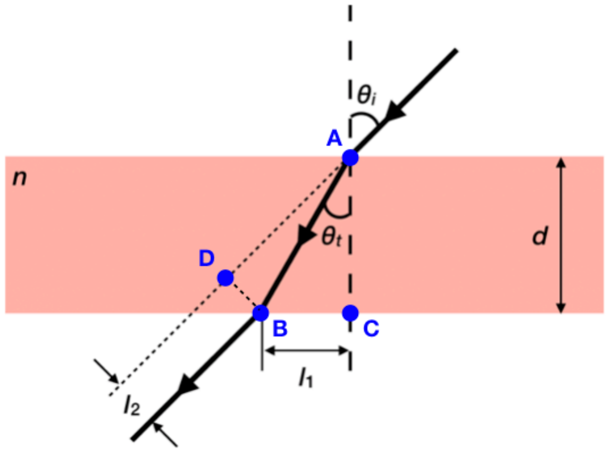
$$\begin{cases} \boldsymbol{E}_1(z,t) = \mathrm{Re}\left[\boldsymbol{E}_1(z) e^{j\omega t}\right] = \boldsymbol{a_x} \eta_1 J_{s0} \sin(\beta_1 z) \sin(\omega t) \\ \boldsymbol{H}_1(z,t) = \mathrm{Re}\left[\boldsymbol{H}_1(z) e^{j\omega t}\right] = \boldsymbol{a_y} J_{s0} \cos(\beta_1 z) \cos(\omega t) \end{cases}. \quad \textbf{(10 points)}$$

Electromagnetics

Midterm 1 | Date: Apr. 9, 2019 | Duration: 75 minutes | Full marks: 500 points

Q4 (***60 points***)

A light ray is incident from air obliquely on a transparent sheet of thickness d with an index of refraction n, as shown below. The angle of incidence is θ_i. Find θ_t, the distance l_1 at the point of exit, and the amount of the lateral displacement l_2 of the emerging ray.



According to Snell's law, $\dfrac{\sin\theta_t}{\sin\theta_i} = \dfrac{1}{n}$ and $\theta_t = \sin^{-1}\left(\dfrac{1}{n}\sin\theta_i\right)$. **(15 points)**

$$l_1 = \overline{BC} = \overline{AC}\tan\theta_t = d\frac{\sin\theta_t}{\cos\theta_t} = d\frac{{}^1\!/_n \sin\theta_i}{\sqrt{1 - {}^1\!/_{n^2}\sin^2\theta_i}} = d\frac{\sin\theta_i}{\sqrt{n^2 - \sin^2\theta_i}}$$ **(20 points)**

$$l_2 = \overline{BD} = \overline{AB}\sin(\theta_i - \theta_t) = \frac{d}{\cos\theta_t}(\sin\theta_i\cos\theta_t - \cos\theta_i\sin\theta_t) = d\sin\theta_i\left(1 - \frac{\cos\theta_i}{\sqrt{n^2 - \sin^2\theta_i}}\right)$$ **(25 points)**

Electromagnetics

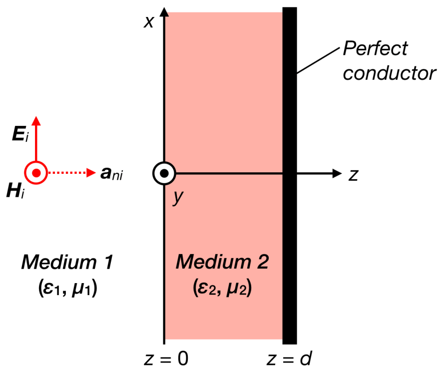
Midterm 1 | Date: Apr. 9, 2019 | Duration: 75 minutes | Full marks: 500 points

Q5 (***100 points***)

A uniform plane wave with

$$\boldsymbol{E_i}(x,z) = \boldsymbol{a_x} E_{i0} \cos\omega\left(t - \frac{z}{u_p}\right)$$

in medium 1 (ε_1, μ_1) is incident normally onto a lossless dielectric slab (ε_2, μ_2) of a thickness d backed by a perfectly conducting plane as shown below. Determine the thickness d that makes $\boldsymbol{E_1}(x,z)$ the same as if the dielectric slab were absent.



Total electric and magnetic fields in medium 1 and medium 2 are given by

$$\text{Medium 1: } \begin{cases} \boldsymbol{E_1} = \boldsymbol{a_x}\left(E_{i0}e^{-j\beta_1 z} + E_{r0}e^{j\beta_1 z}\right) \\ \boldsymbol{H_1} = \dfrac{\boldsymbol{a_y}}{\eta_1}\left(E_{i0}e^{-j\beta_1 z} - E_{r0}e^{j\beta_1 z}\right) \end{cases}, \quad \text{Medium 2: } \begin{cases} \boldsymbol{E_2} = \boldsymbol{a_x}\left(E_2^+e^{-j\beta_2 z} + E_2^-e^{j\beta_2 z}\right) \\ \boldsymbol{H_2} = \dfrac{\boldsymbol{a_y}}{\eta_2}\left(E_2^+e^{-j\beta_2 z} - E_2^-e^{j\beta_2 z}\right) \end{cases} \quad \cdots(1)$$

At $z = d$, $\boldsymbol{E_2}(d) = 0$ and thus, $E_2^- = -E_2^+e^{-j2\beta_2 d} \quad \cdots(2)$

By plugging (2) into (1), we get

$$\begin{cases} \boldsymbol{E_2} = \boldsymbol{a_x}E_2^+\left(e^{-j\beta_2 z} - e^{j\beta_2(z-2d)}\right) \\ \boldsymbol{H_2} = \boldsymbol{a_y}\dfrac{E_2^+}{\eta_2}\left(e^{-j\beta_2 z} + e^{j\beta_2(z-2d)}\right) \end{cases} \quad \cdots(3)$$

At z = 0,

$$\begin{cases} \boldsymbol{E_1} = \boldsymbol{E_2} \\ \boldsymbol{H_1} = \boldsymbol{H_2} \end{cases} \rightarrow \begin{cases} E_{i0} + E_{r0} = E_2^+\left(1 - e^{-j\beta_2 d}\right) \\ E_{i0} - E_{r0} = E_2^+\dfrac{\eta_1}{\eta_2}\left(1 + e^{-j\beta_2 d}\right) \end{cases} \quad \cdots(4)$$

Thus,

$$\begin{cases} \boldsymbol{E_1} = \boldsymbol{E_2} \\ \boldsymbol{H_1} = \boldsymbol{H_2} \end{cases} \rightarrow E_{r0} = -\left(\frac{\eta_1 - j\eta_2 \tan\beta_2 d}{\eta_1 + j\eta_2 \tan\beta_2 d}\right)E_{i0} \quad \cdots(5)$$

(total **60 points** up to here)

Electromagnetics

Midterm 1 | Date: Apr. 9, 2019 | Duration: 75 minutes | Full marks: 500 points

If total electric & magnetic fields are stated correctly as in (1), **30 points** are given. If the boundary conditions at $z = 0$ and $z = d$ are stated as in (2) and (4), **10 points** are given. If the final form of E_{r0} is correct, additional **20 points** is given.

Here $$E_{r0} = -\left(\frac{\eta_1 - j\eta_2 \tan\beta_2 d}{\eta_1 + j\eta_2 \tan\beta_2 d}\right)E_{i0} = -E_{i0} \quad \rightarrow \quad \tan\beta_2 d = 0 \quad \cdots(6),$$

$$\therefore d = \frac{n\pi}{\beta_2} = \frac{n\lambda_2}{2}, \quad n = 0, 1, 2, \cdots$$

The condition, $E_{r0} = -E_{i0}$, as given in (6) should be stated (**20 points**) and the final form of d should be correct (**20 points**).

Electromagnetics

Midterm 1 | Date: Apr. 9, 2019 | Duration: 75 minutes | Full marks: 500 points

Q6 (***50 points***)

Discuss the physical meaning of the critical angle (**15 points**) and Brewster's angle (**35 points**) for the TE and TM waves in non-magnetic media.

A critical angle is the angle of incidence at which all the incident wave, propagating in medium 1 and incident on medium 2, is reflected and there exists no refracted wave. (**4 points**) Above the critical angle, the refracted wave exists in the form of surface wave propagating along the surface whose intensity rapidly decays in the normal direction in medium 2 (i.e., an evanescent wave) (**5 points**). A critical angle occurs only if the wave in medium 1 is incident on a less dense medium 2 (**4 points**). A critical angle is not a function of polarization of the wave (**4 points**).

In non-magnetic media, only a TM wave has a Brewster's angle while a TE wave does not (**15 points**). If the TM wave is incident at its Brewster's angle, there exists no reflected wave (**10 points**). The Brewster's angle for the TM wave can occur for both cases of incidence from dense to less dense media and vice versa (**10 points**).

Q7 (***30 points***)

Describe two examples how polarization of light is exploited in real-world applications.

Linear polarizer in sunglasses, surface inspection via polarization detection at a Brewster's angle, 3D display glasses using two linear polarizers arranged perpendicularly to each other, two linear polarizers that arranged perpendicularly to each other in LCD, AR coating in OLED display comprised of linear polarizer and a phase retarder to prevent ambient illumination (e.g. sunlight) and so on. **15 points** are given for each example. Deduct **5 points** for each if description is missing how they are actually used.