Electromagnetics

Mideterm 2 Solution

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Midterm 2 | Question 1(a): Parallel-plate waveguide

Q1. A parallel-plate waveguide consists of two perfectly conducting plates that are parallel to each other. The conductors are departed by a dielectric medium characterized by (μ, ε) with thickness b. For simplicity, assume that the plates are infinite in extent in x-direction such that fields do not vary in that direction and the effect by fringing fields are neglected.

(a) Derive the electric and magnetic fields of the allowed modes for TE ***(25 points)***, TM ***(20 points)***, and TEM ***(15 points)*** waves that propagate in the parallel-plate waveguide. Find the cut-off frequency for each case.

For TM mode *(20 points)*

- Longitudinal fields (E_z and H_z)

$$\begin{cases} H_z = 0 \\ \nabla^2 E_z + k^2 E_z = 0, \quad \text{where } E_z(y,z) = E_z^0(y) e^{-\gamma z} \end{cases}$$

$$\rightarrow \frac{d^2 E_z^0}{dy^2} + h^2 E_z^0 = 0 \quad (\because h^2 = k^2 + \gamma^2)$$

- Bound condition (E_t = 0 at conducting interface)

$$\boxed{E_z^0(y) = 0, \quad \text{where } y = 0 \text{ and } y = d}$$

- Solution *(Deduce -5 points if B.C. not used)*

$$\boxed{E_z^0(y) = A_n \sin(hy) = A_n \sin\left(\frac{n\pi}{d} y\right), \quad (n = 1, 2, \ldots)}$$

- Transverse fields (E_x, E_y, H_x, H_y)

$$\begin{cases} E_x^0 = -\dfrac{1}{h^2}\left(\gamma \cancel{\dfrac{\partial E_z^0}{\partial x}} + j\omega\mu \cancel{\dfrac{\partial H_z^0}{\partial y}}\right) \\ E_y^0 = -\dfrac{1}{h^2}\left(\gamma \dfrac{\partial E_z^0}{\partial y} - j\omega\mu \cancel{\dfrac{\partial H_z^0}{\partial x}}\right) \\ H_x^0 = -\dfrac{1}{h^2}\left(\gamma \cancel{\dfrac{\partial H_z^0}{\partial x}} - j\omega\varepsilon \dfrac{\partial E_z^0}{\partial y}\right) \\ H_y^0 = -\dfrac{1}{h^2}\left(\gamma \cancel{\dfrac{\partial H_z^0}{\partial y}} + j\omega\varepsilon \cancel{\dfrac{\partial E_z^0}{\partial x}}\right) \end{cases} \rightarrow$$

(15 points)

$$\boxed{\begin{cases} E_x^0(y) = 0 \\ E_y^0(y) = -\dfrac{\gamma}{h^2} A_n \cos\left(\dfrac{n\pi y}{d}\right) \\ H_x^0(y) = \dfrac{j\omega\varepsilon}{h} A_n \cos\left(\dfrac{n\pi y}{d}\right) \\ H_y^0(y) = 0 \end{cases}}$$

- Cutoff frequency (γ = 0)

$$\gamma = \sqrt{h^2 - k^2} = \sqrt{\left(\frac{n\pi}{d}\right)^2 - \omega^2\mu\varepsilon} \rightarrow$$

(5 points)

$$\boxed{f_c = \frac{h}{2\pi\sqrt{\mu\varepsilon}} = \frac{n}{2d\sqrt{\mu\varepsilon}} \text{ (Hz)}}$$

Midterm 2 | Question 1(a): Parallel-plate waveguide

For TE mode *(25 points)*

- Longitudinal fields (E_z and H_z)

$$\begin{cases} E_z = 0 \\ \nabla^2 H_z + k^2 H_z = 0, \quad \text{where } H_z(y,z) = H_z^0(y)e^{-\gamma z} \end{cases}$$

$$\rightarrow \frac{d^2 H_z^0}{dy^2} + h^2 H_z^0 = 0 \quad (\because h^2 = k^2 + \gamma^2)$$

By using B.C. on the right, we get

$$\boxed{\therefore H_z^0(y) = B_n \cos(hy) = B_n \cos\left(\frac{n\pi y}{b}\right), \quad n = 1,2\cdots}$$

(n cannot be 0!)

- B.C. provided by transverse fields (E_t = E_x = 0) at conducting interface

$$\begin{cases} E_x^0 = -\frac{1}{h^2}\left(\gamma \cancel{\frac{\partial E_z^0}{\partial x}} + j\omega\mu \frac{\partial H_z^0}{\partial y}\right) \\ E_y^0 = -\frac{1}{h^2}\left(\gamma \cancel{\frac{\partial E_z^0}{\partial y}} - j\omega\mu \cancel{\frac{\partial H_z^0}{\partial x}}\right) \\ H_x^0 = -\frac{1}{h^2}\left(\gamma \cancel{\frac{\partial H_z^0}{\partial x}} - j\omega\varepsilon \cancel{\frac{\partial E_z^0}{\partial y}}\right) \\ H_y^0 = -\frac{1}{h^2}\left(\gamma \frac{\partial H_z^0}{\partial y} + j\omega\varepsilon \cancel{\frac{\partial E_z^0}{\partial x}}\right) \end{cases}$$

$$\rightarrow E_x^0(y) = -\frac{j\omega\mu}{h^2}\frac{dH_z^0(y)}{dy} = 0 \Big|_{y=0 \text{ and } y=b}$$

$$\downarrow$$

(at the surface of conducting plates)

$$\boxed{\therefore \frac{dH_z^0(y)}{dy} = 0 \quad \text{at } y = 0 \text{ and } y = b}$$

(Deduce -10 points if B.C. not used)

- Transverse fields

$$\boxed{\begin{cases} E_x^0(y) = -\frac{j\omega\mu}{h^2}\frac{dH_z^0(y)}{dy} = \frac{j\omega\mu}{h}B_n \sin\left(\frac{n\pi y}{b}\right) \\ E_y^0(y) = 0 \\ H_x^0(y) = 0 \\ H_y^0(y) = -\frac{\gamma}{h^2}\frac{dH_z^0(y)}{dy} = \frac{\gamma}{h}B_n \sin\left(\frac{n\pi y}{b}\right) \end{cases}}$$

(20 points)

- Cutoff frequency (γ = 0)

$$\gamma = \sqrt{h^2 - k^2} = \sqrt{\left(\frac{n\pi}{d}\right)^2 - \omega^2\mu\varepsilon} \rightarrow \boxed{f_c = \frac{h}{2\pi\sqrt{\mu\varepsilon}} = \frac{n}{2d\sqrt{\mu\varepsilon}} \text{ (Hz)}}$$

(5 points)

Midterm 2 | Question 1(a): Parallel-plate waveguide

For TEM mode *(15 points)*

- Method (1): From TM results,

$$\begin{cases} E_z^0(y) = A_n \sin\left(\dfrac{n\pi}{d} y\right) \\ H_z^0(y) = 0 \end{cases}$$

$$\begin{cases} E_x^0(y) = 0 \\ E_y^0(y) = -\dfrac{\gamma}{h^2} A_n \cos\left(\dfrac{n\pi y}{d}\right) \\ H_x^0(y) = \dfrac{j\omega\varepsilon}{h} A_n \cos\left(\dfrac{n\pi y}{d}\right) \\ H_y^0(y) = 0 \end{cases}$$

$$f_c = \frac{h}{2\pi\sqrt{\mu\varepsilon}} = \frac{n}{2d\sqrt{\mu\varepsilon}} \text{ (Hz)}$$

$\xrightarrow{n = 0}$

$$\begin{cases} E_z^0(y) = 0 \\ H_z^0(y) = 0 \end{cases}$$

$$\begin{cases} E_x^0(y) = 0 \\ E_y^0(y) = -\lim\limits_{n\to 0} \dfrac{\gamma}{h^2} A_0 = E_0 \\ H_x^0(y) = \lim\limits_{n\to 0} \dfrac{j\omega\varepsilon}{h} A_0 = \dfrac{E_0}{\eta} \\ H_y^0(y) = 0 \end{cases}$$

(10 points)

You have to justify that as n = 0, E_y^0 and H_x^0 converges to constant values. Otherwise, -5 points deduced!

$$f_c = \frac{n}{2d\sqrt{\mu\varepsilon}} = 0 \text{ (Hz)}$$

(5 points)

- Method (2): Analytically solve it starting from Maxwell's equations (Refer to lecture note 12-1)

Midterm 2 | Question 1(b): Parallel-plate waveguide

(b) Discuss the relationship of wave impedance vs. frequency for the allowed modes of the parallel-plate waveguides. **(40 points)**

For TEM mode

- From TM mode equations,

$$Z_{TEM} = \frac{E_x^0}{H_y^0} = \frac{j\omega\mu}{\gamma_{TEM}} \quad (\Omega) \quad \cdots(1)$$

Propagation contant vs. Frequency

- For TEM modes,

$$\gamma = \sqrt{-k^2} = jk = j\omega\sqrt{\mu\varepsilon} \quad \cdots(2)$$

- For TE & TM modes,

$$\gamma = \sqrt{h^2 - k^2} = \sqrt{h^2 - \omega^2\mu\varepsilon} \quad \text{and} \quad f_c = \frac{h}{2\pi\sqrt{\mu\varepsilon}}$$

$$\gamma_{TE} \text{ or } \gamma_{TM} = h\sqrt{1 - \left(\frac{f}{f_c}\right)^2} \quad \cdots(3)$$

- By plugging in (2) into (1),

$$\boxed{Z_{TEM} = \frac{j\omega\mu}{\gamma_{TEM}} == \sqrt{\frac{\mu}{\varepsilon}} = \eta}$$

(10 points)

For TM mode

- From given equations,

$$\begin{cases} E_x^0 = -\dfrac{1}{h^2}\left(\gamma \cancel{\dfrac{\partial E_z^0}{\partial x}} + j\omega\mu \cancel{\dfrac{\partial H_z^0}{\partial y}}\right) \\ E_y^0 = -\dfrac{1}{h^2}\left(\gamma \dfrac{\partial E_z^0}{\partial y} - j\omega\mu \cancel{\dfrac{\partial H_z^0}{\partial x}}\right) \\ H_x^0 = -\dfrac{1}{h^2}\left(\gamma \cancel{\dfrac{\partial H_z^0}{\partial x}} - j\omega\varepsilon \dfrac{\partial E_z^0}{\partial y}\right) \\ H_y^0 = -\dfrac{1}{h^2}\left(\gamma \cancel{\dfrac{\partial H_z^0}{\partial y}} + j\omega\varepsilon \cancel{\dfrac{\partial E_z^0}{\partial x}}\right) \end{cases}$$

$$Z_{TM} = \frac{E_x^0}{H_y^0} = -\frac{E_y^0}{H_x^0} = \frac{\gamma_{TM}}{j\omega\varepsilon} \quad (\Omega) \quad \cdots(4)$$

- By plugging (4) into (3),

$$\boxed{Z_{TM} = \frac{\gamma_{TM}}{j\omega\varepsilon} = \eta\sqrt{1 - \left(\frac{f_c}{f}\right)^2}}$$

(10 points)

For TE mode

- From given equations,

$$\begin{cases} E_x^0 = -\dfrac{1}{h^2}\left(\gamma \cancel{\dfrac{\partial E_z^0}{\partial x}} + j\omega\mu \dfrac{\partial H_z^0}{\partial y}\right) \\ E_y^0 = -\dfrac{1}{h^2}\left(\gamma \cancel{\dfrac{\partial E_z^0}{\partial y}} - j\omega\mu \cancel{\dfrac{\partial H_z^0}{\partial x}}\right) \\ H_x^0 = -\dfrac{1}{h^2}\left(\gamma \cancel{\dfrac{\partial H_z^0}{\partial x}} - j\omega\varepsilon \cancel{\dfrac{\partial E_z^0}{\partial y}}\right) \\ H_y^0 = -\dfrac{1}{h^2}\left(\gamma \dfrac{\partial H_z^0}{\partial y} + j\omega\varepsilon \cancel{\dfrac{\partial E_z^0}{\partial x}}\right) \end{cases}$$

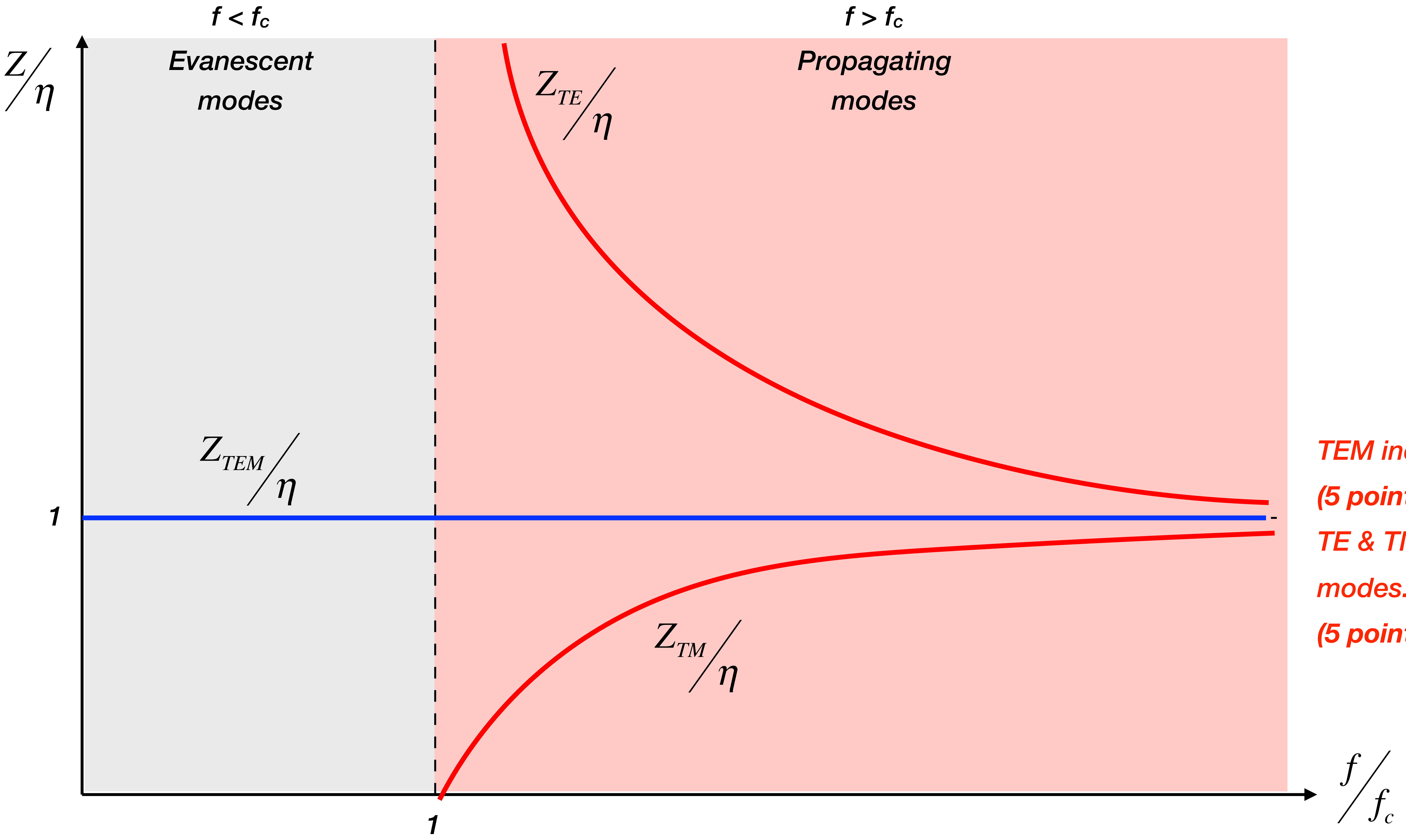
$$Z_{TE} = \frac{E_x^0}{H_y^0} = -\frac{E_y^0}{H_x^0} = \frac{j\omega\mu}{\gamma_{TE}} \quad (\Omega) \quad \cdots(5)$$

- By plugging (5) into (3),

$$\boxed{Z_{TE} = \frac{j\omega\mu}{\gamma_{TE}} = \frac{\eta}{\sqrt{1 - \left(\frac{f_c}{f}\right)^2}}}$$

(10 points)

Midterm 2 | Question 1(b): Parallel-plate waveguide



TEM independent of frequency.

(5 points)

TE & TM below f_c are evanescent modes.

(5 points)

Midterm 2 | Question 2(a): Triangular waveguide

Q2. A waveguide is constructed so that the cross section of the guide forms a triangle with sides of length a, a and sqrt(2a) as shown below. The walls are perfect conductors and inside is air with (μ_0, ε_0).

(a) Determine electric and magnetic fields of the allowed modes for TE, TM and TEM waves propagating in the guide ***(120 points)***

1. Start with wave equation for E_z

$$\left(\nabla_{xy}^2+\nabla_z^2\right)E_z+k^2E_z=0,\quad \text{where}\ E_z(x,y,z)=E_z^0(x,y)e^{-\gamma z}$$

$$\rightarrow$$

$$\nabla_{xy}^2E_z^0+h^2E_z^0=0,\quad \text{where}\ h^2=k^2+\gamma^2$$

2. Separation of variables

$$E_z^0(x,y)=X(x)Y(y)\quad\rightarrow\quad\begin{cases}\dfrac{d^2X(x)}{dx^2}+k_x^2X(x)=0\\[2mm]\dfrac{d^2Y(y)}{dy^2}+k_y^2Y(y)=0\end{cases}$$

where $h^2=k_x^2+k_y^2$

3. Solution form

$$X(x)=A\sin(k_xx)+B\cos(k_xx)$$

$$Y(y)=C\sin(k_yy)+D\cos(k_yy)$$

4. Three boundary conditions

- At y = 0 and a, E_z^0 = 0

$$\begin{cases}Y(0)=D=0\\Y(a)=C\sin(k_ya)=0\end{cases}$$

$$\rightarrow Y(y)=C\sin\left(\frac{m\pi}{a}x\right),\ \mathrm{m}=1,2,3,\cdots$$

- At x = 0 and a, E_z^0 = 0 (similarly as above)

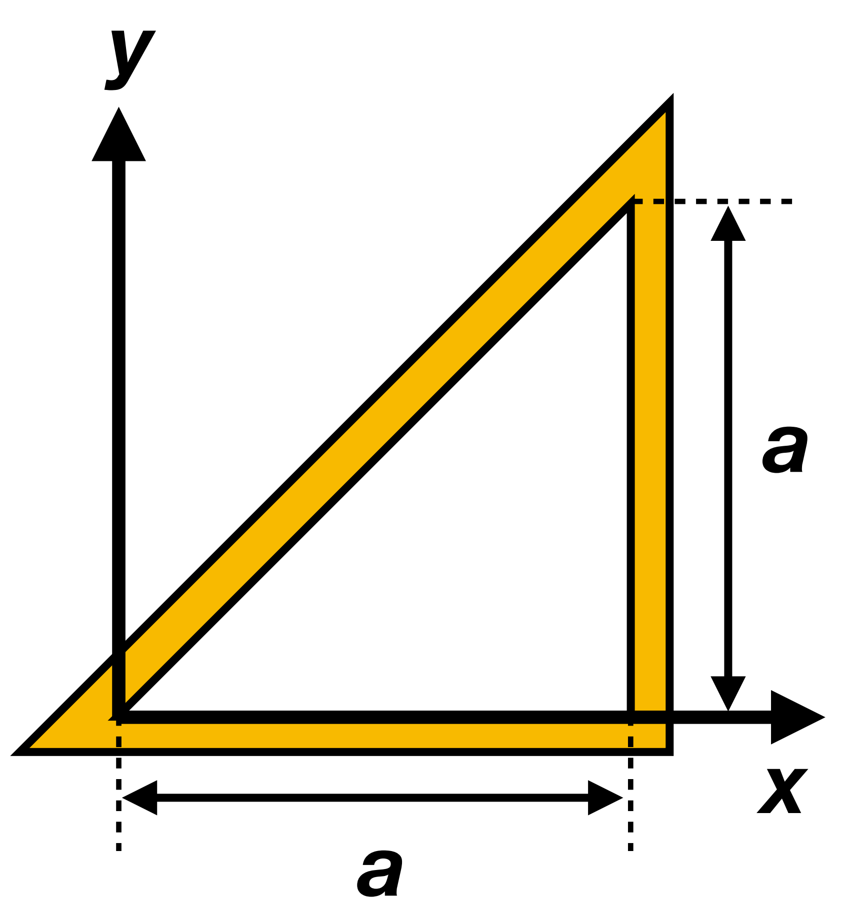
$$\rightarrow X(x)=A\sin\left(\frac{n\pi}{a}x\right),\ \mathrm{n}=1,2,3,\cdots$$

- At $y = x$, E_z^0 = 0

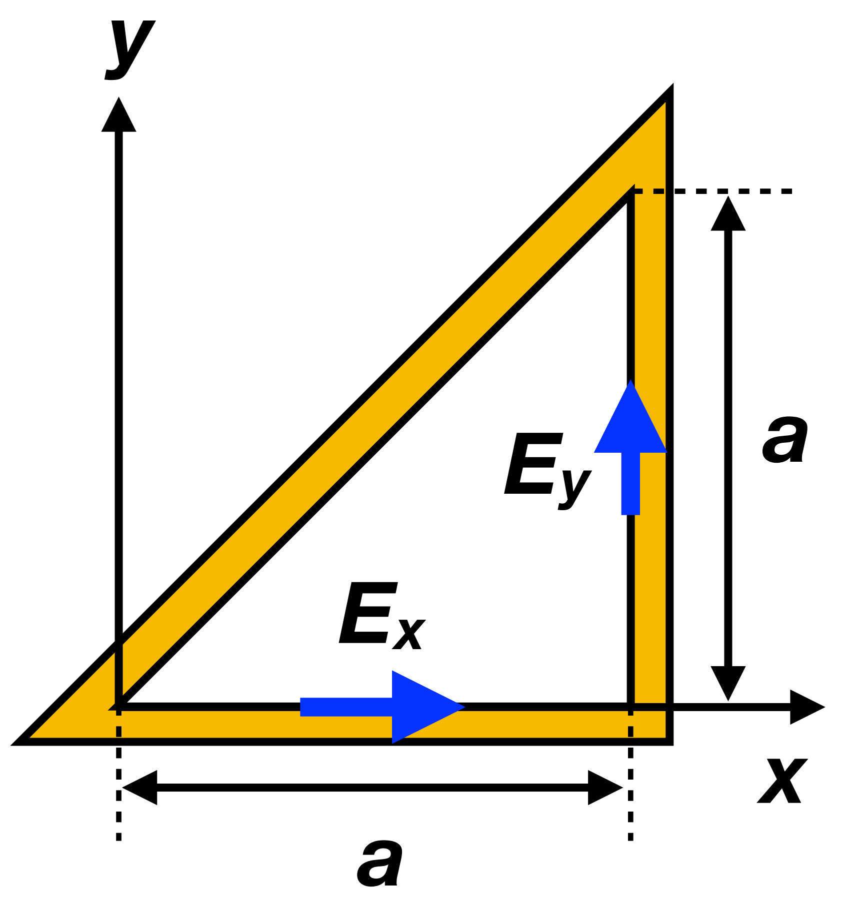
$$E_z^0(x,x)=E_{z0}\sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{m\pi}{a}x\right)=0\quad\text{(For all } 0\le x\le a\text{)}$$

$$\boxed{\therefore E_{z0}=0\quad\rightarrow\quad E_z^0(x,y)=E_{z0}\sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{m\pi}{a}x\right)=0}$$

TM cannot exist!
Only TE modes!
(40 points)



Midterm 2 | Question 2(a): Triangular waveguide



5. Derivation of TE mode

- Start with wave equations with H_z *(as previously)*

$$H_z^0(x,y) = X(x)Y(y) \quad \rightarrow \quad \begin{cases} X(x) = A\sin(k_x x) + B\cos(k_x x) \\ Y(y) = C\sin(k_y y) + D\cos(k_y y) \end{cases}$$

6. Three B.C. provided by transverse fields

$$\begin{cases} E_x^0 = -\frac{1}{h^2}\left(\gamma \cancel{\frac{\partial E_z^0}{\partial x}} + j\omega\mu \frac{\partial H_z^0}{\partial y}\right) \\ E_y^0 = -\frac{1}{h^2}\left(\gamma \cancel{\frac{\partial E_z^0}{\partial y}} - j\omega\mu \frac{\partial H_z^0}{\partial x}\right) \\ H_x^0 = -\frac{1}{h^2}\left(\gamma \frac{\partial H_z^0}{\partial x} - j\omega\varepsilon \cancel{\frac{\partial E_z^0}{\partial y}}\right) \\ H_y^0 = -\frac{1}{h^2}\left(\gamma \frac{\partial H_z^0}{\partial y} + j\omega\varepsilon \cancel{\frac{\partial E_z^0}{\partial x}}\right) \end{cases}$$

- *At x = 0 and a, $E_y^0 = 0$*

$$\left.\frac{\partial H_z^0}{\partial x}\right|_{x=0 \text{ and } a} = \left.\frac{\partial X(x)}{\partial x} Y(y)\right|_{x=0 \text{ and } a} = 0$$

$$\frac{\partial X(x)}{\partial x} = Ak_x \cos(k_x x) - Bk_x \sin(k_x x)$$

$$\begin{cases} \frac{\partial X(0)}{\partial x} = Ak_x = 0 \\ \frac{\partial X(a)}{\partial x} = Bk_x \sin(k_x a) = 0 \end{cases} \rightarrow \quad X(x) = B\cos\left(\frac{n\pi}{a}x\right)$$

- *At y = 0 and a, $E_x^0 = 0$* (Similarly)

$$\rightarrow Y(y) = D\cos\left(\frac{m\pi}{a}x\right)$$

$$\therefore H_z^0(x,y) = X(x)Y(y) = H_{0z}\cos\left(\frac{n\pi}{a}x\right)\cos\left(\frac{m\pi}{a}y\right)$$

$$\begin{cases} E_x^0(x,y) = E_{0x}\cos\left(\frac{n\pi}{a}x\right)\sin\left(\frac{m\pi}{a}y\right) \\ E_y^0(x,y) = E_{0y}\sin\left(\frac{n\pi}{a}x\right)\cos\left(\frac{m\pi}{a}y\right) \end{cases}$$

Midterm 2 | Question 2(a): Triangular waveguide

6. Three B.C. provided by transverse fields

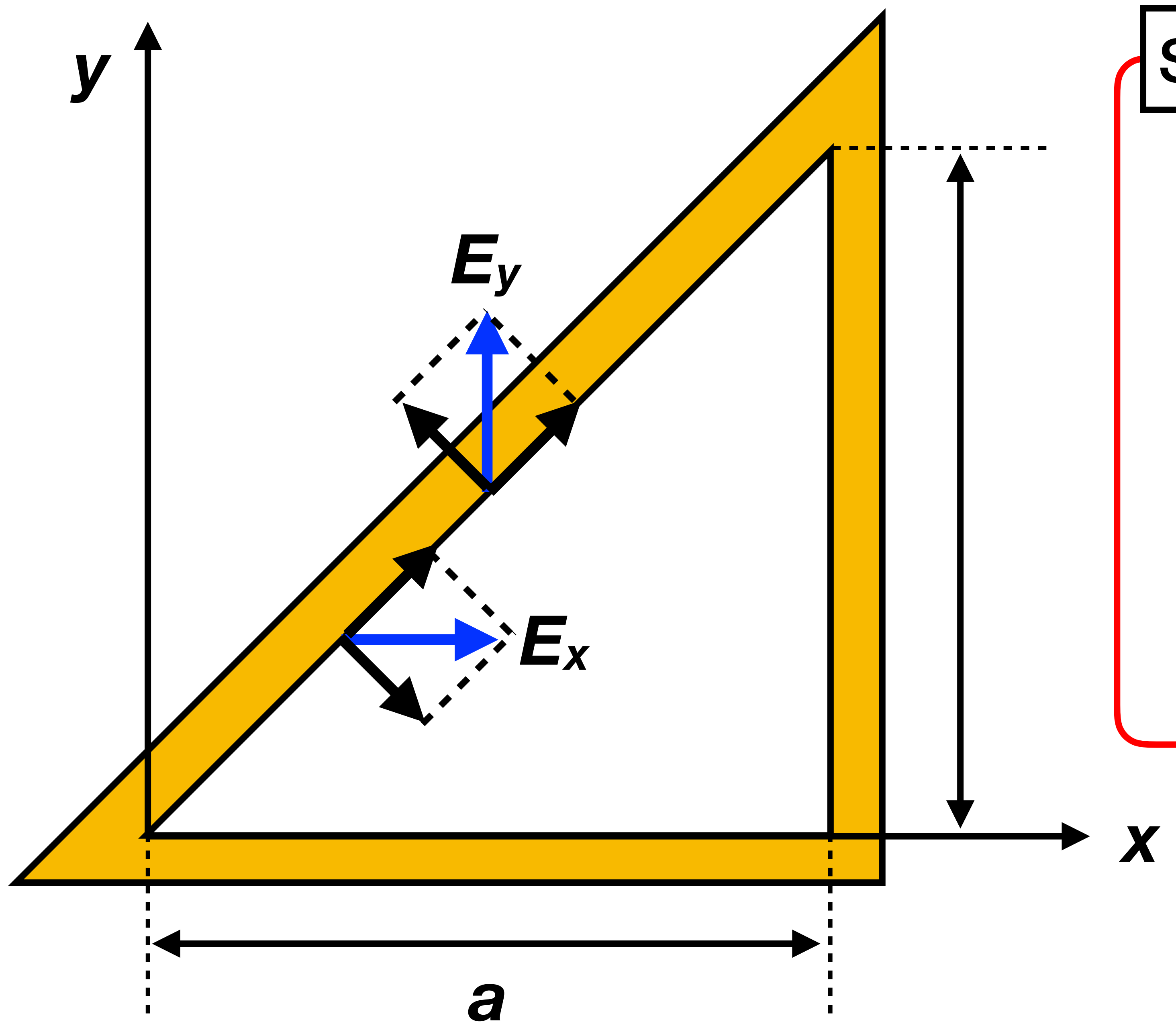
- At $x = y$

E-field parallel to the diagonal side = 0 → $E_x^0(x,x)\cos\frac{\pi}{4} + E_y^0(x,x)\cos\frac{\pi}{4} = 0$

$$\left(\because \begin{cases} E_x^0(x,y) = E_{0x}\cos\left(\frac{n\pi}{a}x\right)\sin\left(\frac{m\pi}{a}y\right) \\ E_y^0(x,y) = E_{0y}\sin\left(\frac{n\pi}{a}x\right)\cos\left(\frac{m\pi}{a}y\right) \end{cases}\right)$$

$$\rightarrow \quad E_{0x}\cos\left(\frac{n\pi}{a}x\right)\sin\left(\frac{m\pi}{a}x\right) + E_{0y}\sin\left(\frac{n\pi}{a}x\right)\cos\left(\frac{m\pi}{a}x\right) = 0$$

$$\rightarrow \quad E_{0x}\tan\left(\frac{m\pi}{a}x\right) = -E_{0y}\tan\left(\frac{n\pi}{a}x\right)$$



Solution (1): $E_{0x} = E_{0y}$ and $m = -n$ ($TE_{n,-n}$ mode)

$$H_z^0(x,y) = H_{0z}\cos\left(\frac{n\pi}{a}x\right)\cos\left(\frac{n\pi}{a}y\right)$$

$$\begin{cases} E_x^0(x,y) = -E_{0x}\cos\left(\frac{n\pi}{a}x\right)\sin\left(\frac{n\pi}{a}y\right) \\ E_y^0(x,y) = E_{0x}\sin\left(\frac{n\pi}{a}x\right)\cos\left(\frac{n\pi}{a}y\right) \end{cases}$$

Solution (2): $E_{0x} = -E_{0y}$ and $m = n$ ($TE_{n,n}$ mode)

$$H_z^0(x,y) = H_{0z}\cos\left(\frac{n\pi}{a}x\right)\cos\left(\frac{n\pi}{a}y\right)$$

$$\begin{cases} E_x^0(x,y) = -E_{0x}\cos\left(\frac{n\pi}{a}x\right)\sin\left(\frac{n\pi}{a}y\right) \\ E_y^0(x,y) = E_{0x}\sin\left(\frac{n\pi}{a}x\right)\cos\left(\frac{n\pi}{a}y\right) \end{cases}$$

- Note that n SHOUD NOT be 0.
- The rest (H_x^0, H_y^0, and relationship between constants H_{z0} and E_{x0}) can be easily calculated by substituting above to transverse field equations.

Midterm 2 | Question 2(b): Triangular waveguide

(b) If some modes are not allowed, explain why not. ***(30 points)***

- *In (a), we showed that due to the B.C. at the diagonal side, TM cannot be supported. (10 points)*
- *A given waveguide is one type of single conductor and therefore, TEM cannot be supported. (20 points)*

▸ *Proof of 2nd statement*

- Suppose that TEM wave exists in such a guide
- Its ***B*** and ***H*** should form ***a closed loop in a transverse plane (xy)***

$\because \nabla \cdot \boldsymbol{B} = 0$ (*Magnetic flux lines close upon themselves*)

- According to ***Ampere's circuital law (see <Fig. 1> and <Fig. 2>),***

$$\oint_C \boldsymbol{H} \cdot d\boldsymbol{l} = \int_S \left(\boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t} \right) \cdot d\boldsymbol{s}$$

*Line integral of **H** around any closed loop C **in a transverse plane***

=

Longitudinal** conduction & displacement currents **through the loop C

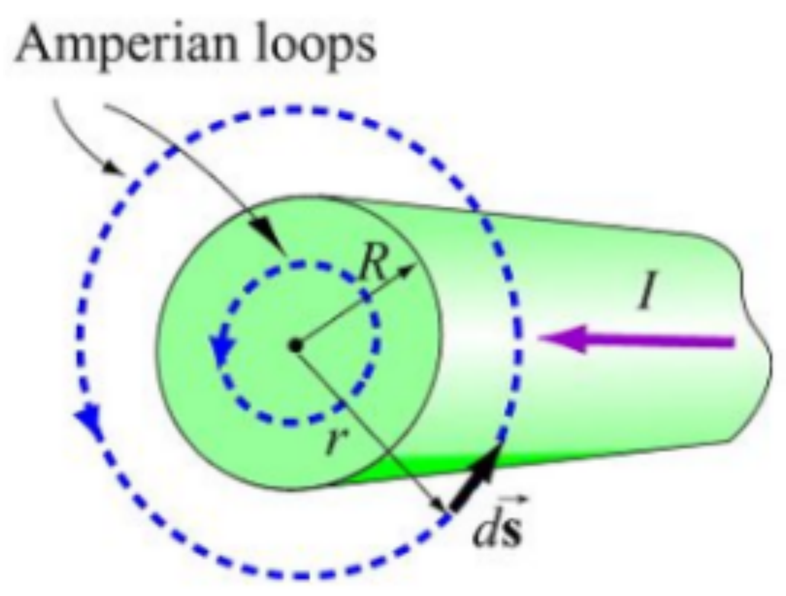
- By definition, TEM wave does not have longitudinal E_z

→ No longitudinal current (***J*** and *δ**D**/δt*) can flow

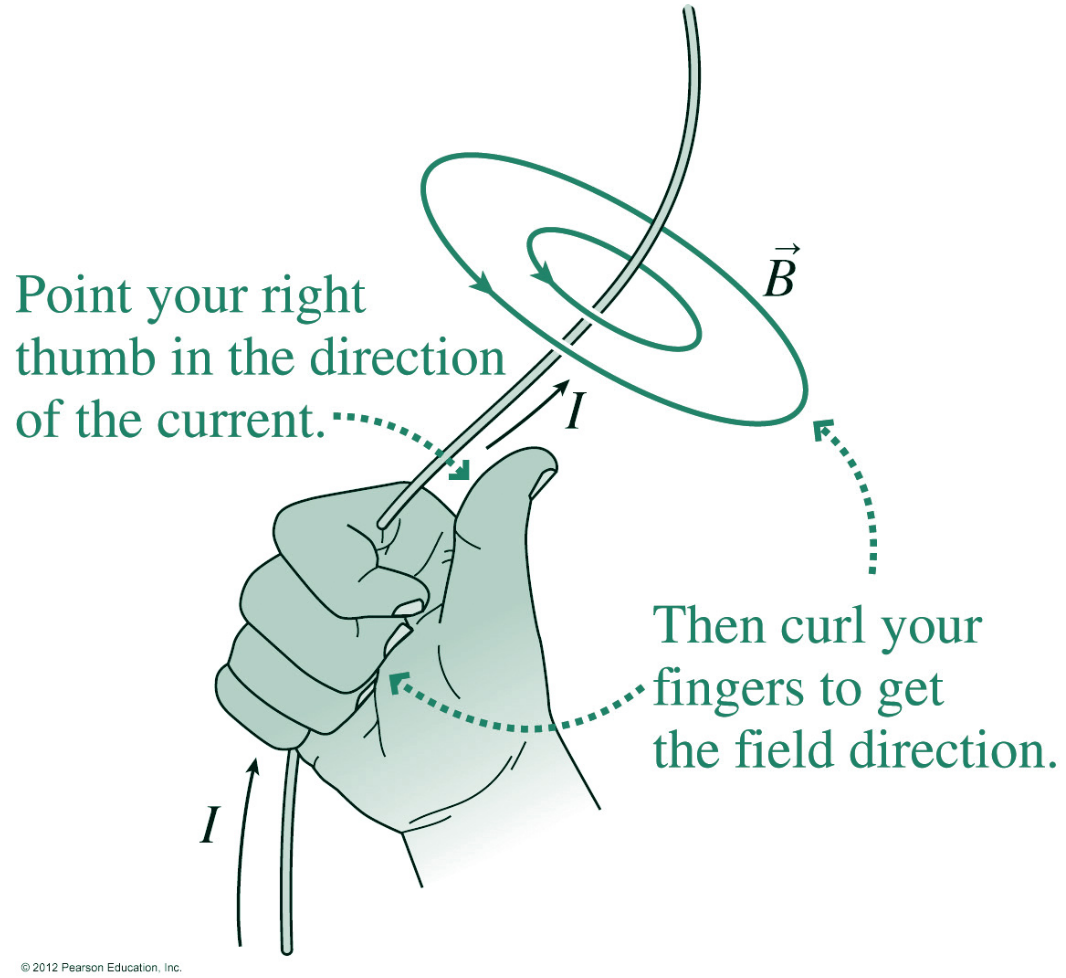
→ Thus, ***B*** and ***H*** *do not exist* in a transverse plane

→ Thus, ***E*** and ***D*** also *do not exist* in a transverse plane

∴ TEM cannot exist in a single conductor waveguide!



Ampere's circuital law
<Fig. 1>
(Img src: Toppr)



Ampere's circuital law
<Fig. 2>
(img src: Pearson Education)

Midterm 2 | Question 3(a): Circular waveguide

Q3. A circular waveguide is a circular metal pipe having a uniform cross section of radius *a*. Inside is filled with a dielectric medium characterized by (μ, ε). Assume the metal pipe is a perfect conductor for simplicity.

(a) Derive the general expressions for electric and magnetic fields for TE and TM modes. ***(140 points)***

Derivation of TM mode

- Start with wave equations for longitudinal fields

$$\begin{cases} H_z = 0 \quad \textit{(By definition)} \\ \left(\nabla_{r\phi}^2 + \nabla_z^2\right)E_z + k^2 E_z = 0 \quad \text{where } E_z(r,\phi,z) = E_z^0(r,\phi)e^{-\gamma z} \end{cases}$$

$$\rightarrow \nabla_{r\phi}^2 E_z^0 + h^2 E_z^0 = 0 \quad \text{where } h^2 = k^2 + \gamma^2$$

- Separation of variables

$$E_z^0(r,\phi) = R(r)\Phi(\phi)$$

- Two ODEs obtained

$$\begin{cases} \dfrac{d^2\Phi(\phi)}{d^2\phi} + n^2\Phi(\phi) = 0 \\ \dfrac{r}{R(r)}\dfrac{d}{dr}\left(r\dfrac{dR(r)}{dr}\right) + h^2r^2 = n^2 \end{cases}$$

- Solutions to Bessel's differential equation

$$R(r) = C_n J_n(hr) + D_n Y_n(hr)$$

where J_n and Y_n are the first and second kind of Bessel's functions. The solution should include the axis region where $r = 0$ and $Y_n(0)$ diverges to infinity. Thus, $D_n = 0$. *(Deduce -15 points if not mentioned)*

$$\therefore R(r) = C_n J_n(hr)$$

- Solution to 2nd-order ODE for ϕ

$$\therefore \Phi(\phi) = \cos n\phi$$

All the field components for a circular waveguide are periodic with respect to ϕ. Thus, Φ should be in a sinusoidal form and because of periodicity, n should be an integer value. *(Deduce -15 points if not mentioned about n)*

Midterm 2 | Question 3(a): Circular waveguide

Derivation of transverse fields

- Transverse *E*-fields can be expressed in terms of longitudinal *E*-field as below

$$\left(\boldsymbol{E}_T^0\right)_{TM} = \boldsymbol{a}_r E_r^0 + \boldsymbol{a}_\phi E_\phi^0 = -\frac{\gamma}{h^2}\nabla_T E_z^0 \quad \text{where} \quad \nabla_T = \boldsymbol{a}_r \frac{\partial}{\partial r} + \boldsymbol{a}_\phi \frac{\partial}{r\,\partial\phi}$$

$$\rightarrow \left(\boldsymbol{E}_T^0\right)_{TM} = \boxed{\boldsymbol{a}_r E_r^0 + \boldsymbol{a}_\phi E_\phi^0 = \boldsymbol{a}_r\left(-\frac{\gamma}{h^2}\frac{\partial E_z^0}{\partial r}\right) + \boldsymbol{a}_\phi\left(-\frac{\gamma}{h^2 r}\frac{\partial E_z^0}{\partial \phi}\right)}$$

- Transverse *H*-fields can be expressed in terms of transverse *E*-field as below

$$\left(\boldsymbol{H}_T\right)_{TM} = \frac{1}{Z_{TM}}\left[\boldsymbol{a}_z \times \left(\boldsymbol{E}_T\right)_{TM}\right] \quad \text{where} \quad Z_{TM} = \frac{\gamma}{j\omega\varepsilon} \quad (\Omega)$$

$$\left[\left(\boldsymbol{H}_T\right)_{TM} = \boldsymbol{a}_r H_r^0 + \boldsymbol{a}_\phi H_\phi^0\right] = \frac{j\omega\varepsilon}{\gamma}\boldsymbol{a}_z \times \left(\boldsymbol{a}_r E_r^0 + \boldsymbol{a}_\phi E_\phi^0\right)$$

$$\boxed{\boldsymbol{a}_r H_r^0 + \boldsymbol{a}_\phi H_\phi^0 = \boldsymbol{a}_r\left(-\frac{j\omega\varepsilon}{\gamma}E_\phi^0\right) + \boldsymbol{a}_\phi\left(\frac{j\omega\varepsilon}{\gamma}E_r^0\right)}$$

Final form

- By using two relations as given left,

$$\boxed{\begin{aligned}
&\left\{\begin{aligned}
E_r^0 &= -\frac{j\beta}{h^2}\frac{\partial E_z^0}{\partial r} = -\frac{j\beta}{h}C_n J_n'(hr)\cos n\phi \\
E_\phi^0 &= -\frac{j\beta}{h^2 r}\frac{\partial E_z^0}{\partial \phi} = \frac{j\beta n}{h^2 r}C_n J_n(hr)\sin n\phi \\
H_r^0 &= -\frac{\omega\varepsilon}{\beta}E_\phi^0 = -\frac{j\omega\varepsilon n}{h^2 r}C_n J_n(hr)\sin n\phi \\
H_\phi^0 &= \frac{\omega\varepsilon}{\beta}E_r^0 = -\frac{j\omega\varepsilon}{h}C_n J_n'(hr)\cos n\phi
\end{aligned}\right. \\
&\text{where } E_z^0(r,\phi) = C_n J_n(hr)\cos n\phi
\end{aligned}}$$

If all procedures correct, give 70 points

Midterm 2 | Question 3(a): Circular waveguide

Derivation of TE mode

- Longitudinal fields

$$\begin{cases} E_z = 0 \\ H_z(r,\phi,z) = H_z^0(r,\phi)e^{-\gamma z} \quad \text{where } \nabla_{r\phi}^2 H_z^0 + h^2 H_z^0 = 0 \text{ and } H_z^0(r,\phi) = R(r)\Phi(\phi) \end{cases}$$

- Similarly to the TM case,

$$\boxed{\therefore H_z^0(r,\phi) = D_n J_n(hr)\cos n\phi \quad \text{(TE modes)}}$$

- Transverse magnetic fields:

$$\left[\left(\boldsymbol{H}_T^0\right)_{TE} = \boldsymbol{a}_r H_r^0 + \boldsymbol{a}_\phi H_\phi^0\right] = \left[-\frac{\gamma}{h^2}\nabla_T H_z^0 = -\frac{\gamma}{h^2}\left(\boldsymbol{a}_r \frac{\partial}{\partial r} + \boldsymbol{a}_\phi \frac{\partial}{r\,\partial\phi}\right)H_z^0\right]$$

- Transverse electric fields:

$$\left[\left(\boldsymbol{E}_T^0\right)_{TE} = \boldsymbol{a}_r E_r^0 + \boldsymbol{a}_\phi E_\phi^0\right] = \left[-Z_{TE}\left(\boldsymbol{a}_z \times \left(\boldsymbol{H}_T^0\right)_{TE}\right) = -\frac{j\omega\mu}{\gamma}\left(\boldsymbol{a}_r H_r^0 + \boldsymbol{a}_\phi H_\phi^0\right)\right]$$

$$\boxed{\begin{cases} H_r^0 = -\dfrac{j\beta}{h^2}\dfrac{\partial H_z^0}{\partial r} = -\dfrac{j\beta}{h}D_n J_n'(hr)\cos n\phi \\ H_\phi^0 = -\dfrac{j\beta}{h^2 r}\dfrac{\partial E_z^0}{\partial\phi} = \dfrac{j\beta n}{h^2 r}D_n J_n(hr)\sin n\phi \\ E_r^0 = -\dfrac{\omega\varepsilon}{\beta}H_\phi^0 = -\dfrac{j\omega\varepsilon n}{h^2 r}D_n J_n(hr)\sin n\phi \\ E_\phi^0 = \dfrac{\omega\varepsilon}{\beta}H_r^0 = -\dfrac{j\omega\varepsilon}{h}D_n J_n'(hr)\cos n\phi \end{cases}}$$

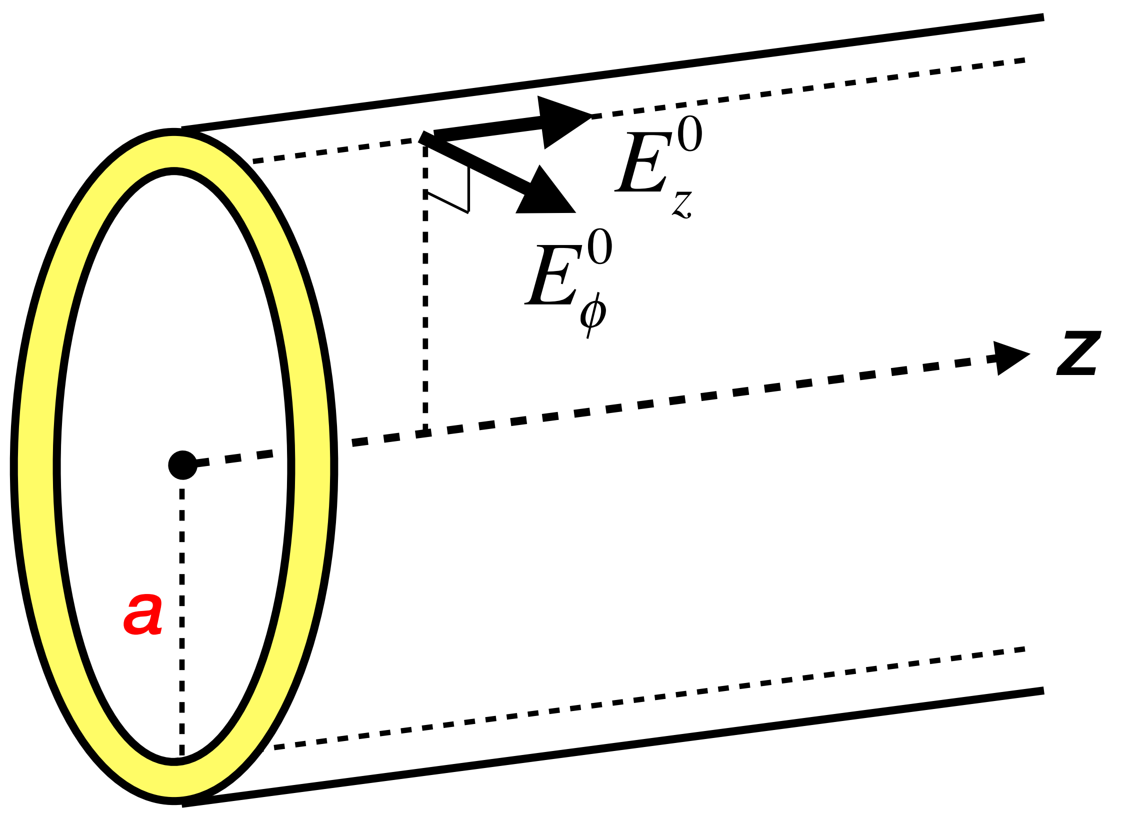
If all procedures correct, give 70 points

Midterm 2 | Question 3(b): Circular waveguide

(b) Identify the cut-off frequencies for propagating TE and TM modes. What is the dominant mode for the circular waveguide? ***(60 points)***

For TM mode

- From the boundary condition



$E_\phi^0 = E_z^0 = 0$ for all ϕ at $r = a$

$$E_z^0(a,\phi) = C_n J_n(ha)\cos n\phi = 0 \quad \rightarrow \quad J_n(ha) = 0$$

(15 points)

- Lowest ha satisfying $J_n(ha) = 0$:

$$ha = 2.405 \quad \rightarrow \quad h_{TM01} = \frac{2.405}{a}$$

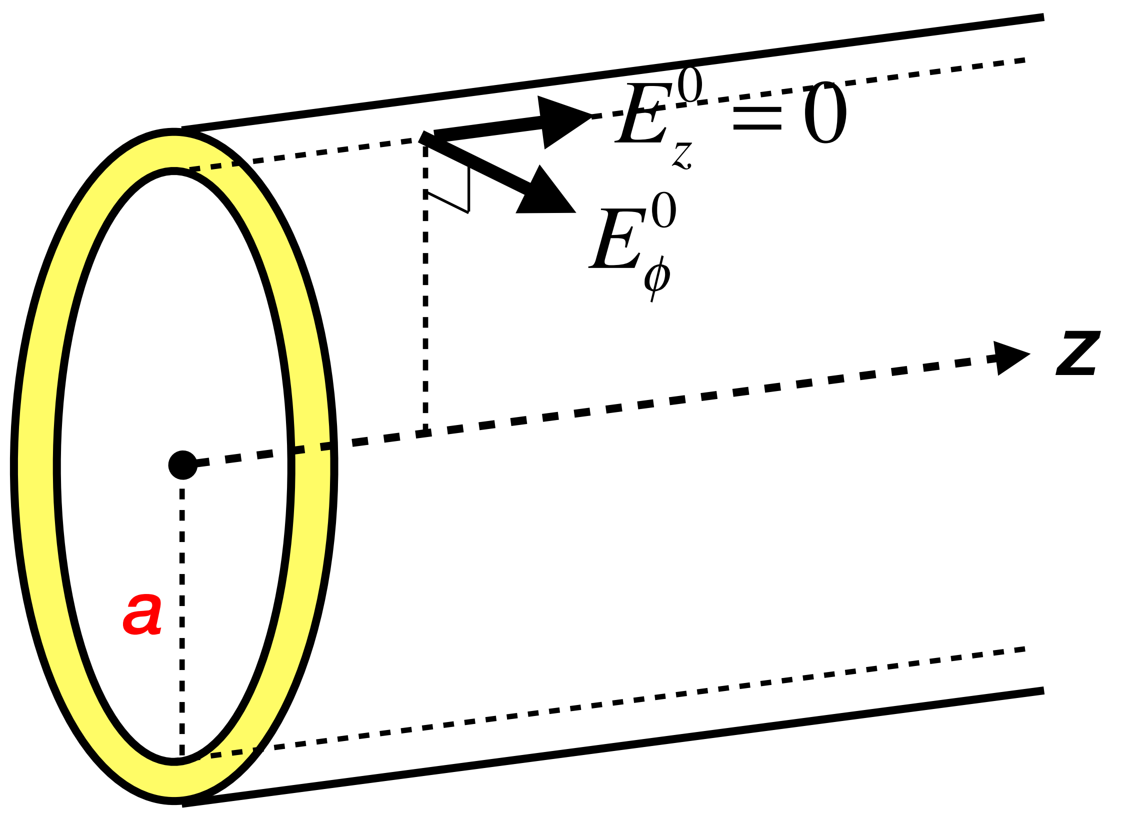
(10 points)

- Cutoff frequency

$$f_c = \frac{h}{2\pi\sqrt{\mu\varepsilon}} \quad \rightarrow \quad \therefore (f_c)_{TM01} = \frac{h_{TM01}}{2\pi\sqrt{\mu\varepsilon}} = \frac{0.383}{a\sqrt{\mu\varepsilon}}$$

For TE mode

- From the boundary condition



$E_\phi^0 = 0$ for all ϕ at $r = a$

$$E_\phi^0 = \frac{\omega\varepsilon}{\beta}H_r^0 = -\frac{j\omega\varepsilon}{h}D_n J_n'(hr)\cos n\phi = 0 \quad \rightarrow \quad J_n'(hr) = 0$$

(15 points)

- Lowest ha satisfying $J'_n(ha) = 0$:

$$ha = 1.841 \quad \rightarrow \quad h_{TE11} = \frac{1.841}{a}$$

(10 points)

- Cutoff frequency

Dominant mode! (10 points)

$$f_c = \frac{h}{2\pi\sqrt{\mu\varepsilon}} \quad \rightarrow \quad \boxed{\therefore (f_c)_{TE11} = \frac{h_{TE11}}{2\pi\sqrt{\mu\varepsilon}} = \frac{0.293}{a\sqrt{\mu\varepsilon}}}$$

Midterm 2 | Question 4(a): Dielectric waveguide

Q4.

(a) Derive the general expressions for electric and magnetic fields for the propagating TM modes in the dielectric waveguide ***(80 points)***. Explain under what condition and requirements the waves can propagate within the waveguide ***(40 points)***. (in the free space regions, derive the fields for the only one side, i.e. either above or below the guide.)

Longitudinal fields

$$\begin{cases} H_z = 0 \quad \textit{(By definition)} \\ \nabla^2 E_z + k^2 E_z = 0 \quad \text{where} \quad E_z(y,z) = E_z^0(y) e^{-\gamma z} \end{cases}$$

$$\rightarrow \quad \nabla_y^2 E_z^0 + h^2 E_z^0 = 0 \quad \text{where} \quad h^2 = \gamma^2 + k^2$$

Solution for dielectric region ($|y| \leq d/2$)

- Waves propagating in *z*-direction w/o attenuation
- Sinusoidal form → Bounded standing wave

$$\nabla_y^2 E_z^0 + h_d^2 E_z^0 = 0 \quad \text{where} \quad h_d^2 = \gamma^2 + k^2 = \omega^2 \mu_d \varepsilon_d - \beta^2 > 0$$

$$\boxed{E_z^0(y) = E_o \sin h_d y + E_e \cos h_d y}$$

(Deduce -5 points for each why the solution for dielectric / free-space should be in a sinusoidal / exponential form)

Solution for free-space region ($y \geq d/2$)

- Waves bounded only within dielectric by total internal reflection
- Waves not radiating away from dielectric (i.e. evanescent)

$$\nabla_y^2 E_z^0 + h_0^2 E_z^0 = 0 \quad \text{where} \quad h_0^2 = \gamma^2 + k^2 = \omega^2 \mu_0 \varepsilon_0 - \beta^2 < 0$$

Here, we denote $h_0^2 \triangleq -\alpha^2$

$$\boxed{E_z^0(y) = C e^{-\alpha\left(y - \frac{d}{2}\right)} + \cancel{D} e^{\alpha\left(y - \frac{d}{2}\right)} \quad \text{where } y \geq d/2}$$

($\because$Not to diverge to infinity)

Traverse field components

- There are two nonzero components, E_y^0 and H_x^0)

$$\begin{cases} E_x^0 = -\frac{1}{h^2}\left(\gamma \cancel{\frac{\partial E_z^0}{\partial x}} + j\omega\mu \cancel{\frac{\partial H_z^0}{\partial y}}\right) \\ E_y^0 = -\frac{1}{h^2}\left(\gamma \frac{\partial E_z^0}{\partial y} - j\omega\mu \cancel{\frac{\partial H_z^0}{\partial x}}\right) \end{cases} \qquad \begin{cases} H_x^0 = -\frac{1}{h^2}\left(\gamma \cancel{\frac{\partial H_z^0}{\partial x}} - j\omega\varepsilon \frac{\partial E_z^0}{\partial y}\right) \\ H_y^0 = -\frac{1}{h^2}\left(\gamma \cancel{\frac{\partial H_z^0}{\partial y}} + j\omega\varepsilon \cancel{\frac{\partial E_z^0}{\partial x}}\right) \end{cases}$$

Midterm 2 | Question 4(a): Dielectric waveguide

For odd TM modes

- In the dielectric region ($|y| \le d/2$)

$$E_z^0(y) = E_o \sin h_d y$$

$$E_y^0(y) = -\frac{\gamma}{h_d^2}\frac{\partial E_z^0}{\partial y} = -\frac{j\beta}{h_d} E_o \cos h_d y$$

$$H_x^0(y) = \frac{j\omega\varepsilon_d}{h_d^2}\frac{\partial E_z^0}{\partial y} = \frac{j\omega\varepsilon_d}{h_d} E_o \cos h_d y$$

- In the free-space region ($y \ge d/2$)
  - According to B.C. that E_z^0 should be continuous,

$$E_z^0(y) = Ce^{-\alpha\left(y-\frac{d}{2}\right)} \quad \rightarrow \quad E_z^0\left(\frac{d}{2}\right) = C = E_o \sin\frac{h_d d}{2}$$

$$E_y^0(y) = -\frac{\gamma}{h_0^2}\frac{\partial E_z^0}{\partial y} = -\frac{j\beta}{\alpha} Ce^{-\alpha\left(y-\frac{d}{2}\right)}$$

$$H_x^0(y) = \frac{j\omega\varepsilon_0}{h_0^2}\frac{\partial E_z^0}{\partial y} = \frac{j\omega\varepsilon_0}{\alpha} Ce^{-\alpha\left(y-\frac{d}{2}\right)}$$

Deduce -5 points if B.C. is not used

(If all procedures correct, Give 40 points)

For even TM modes

- In the dielectric region ($|y| \le d/2$)

$$E_z^0(y) = E_e \cos h_d y$$

$$E_y^0(y) = -\frac{\gamma}{h_d^2}\frac{\partial E_z^0}{\partial y} = \frac{j\beta}{h_d} E_e \sin h_d y$$

$$H_x^0(y) = \frac{j\omega\varepsilon_d}{h_d^2}\frac{\partial E_z^0}{\partial y} = -\frac{j\omega\varepsilon_d}{h_d} E_e \sin h_d y$$

- In the free-space region ($y \ge d/2$)
  - According to B.C. that E_z^0 should be continuous,

$$E_z^0(y) = Ce^{-\alpha\left(y-\frac{d}{2}\right)} \quad \rightarrow \quad E_z^0\left(\frac{d}{2}\right) = C = E_e \cos\frac{h_d d}{2}$$

$$E_y^0(y) = -\frac{\gamma}{h_0^2}\frac{\partial E_z^0}{\partial y} = -\frac{j\beta}{\alpha} Ce^{-\alpha\left(y-\frac{d}{2}\right)}$$

$$H_x^0(y) = \frac{j\omega\varepsilon_0}{h_0^2}\frac{\partial E_z^0}{\partial y} = \frac{j\omega\varepsilon_0}{\alpha} Ce^{-\alpha\left(y-\frac{d}{2}\right)}$$

Deduce -5 points if B.C. is not used

(If all procedures correct, Give 40 points)

Midterm 2 | Question 4(a): Dielectric waveguide

Condition and requirements for wave confinement (40 points)

- $\varepsilon_d > \varepsilon_0$ for having total internal reflection *(20 points)*
- $\omega^2 \mu_0 \varepsilon_0 - \beta^2 < 0$

  where $\beta^2 = k^2 - h_d^2 = \omega^2 \mu_d \varepsilon_d - h_d^2$

$$\rightarrow \quad \omega > \frac{h_d}{\sqrt{\mu_d \varepsilon_d - \mu_0 \varepsilon_0}}$$

Here, the smallest allowable h_d for n-th TE mode satisfies $\dfrac{h_d d}{2} = n\pi$

$$\therefore\ f > \frac{n}{d\sqrt{\mu_d \varepsilon_d - \mu_0 \varepsilon_0}}, \quad n = 0, 1, 2, \cdots$$ *(20 points)*

Midterm 2 | Question 4(a): Dielectric waveguide

(b)** Discuss the physical meaning of cutoff frequencies for the dielectric waveguide by comparing with those of other waveguide structures (parallel-plate and rectangle/circular waveguides). What factors do affect cutoff frequencies of these waveguides? **(40 points)

Commonality

- Waves of frequencies above f_c are confined within and propagate along the waveguide. *(5 points)*
- f_c is determined by propagating modes (TE, TM), dimension and materials of the waveguides (μ_d, ε_d). *(5 points)*

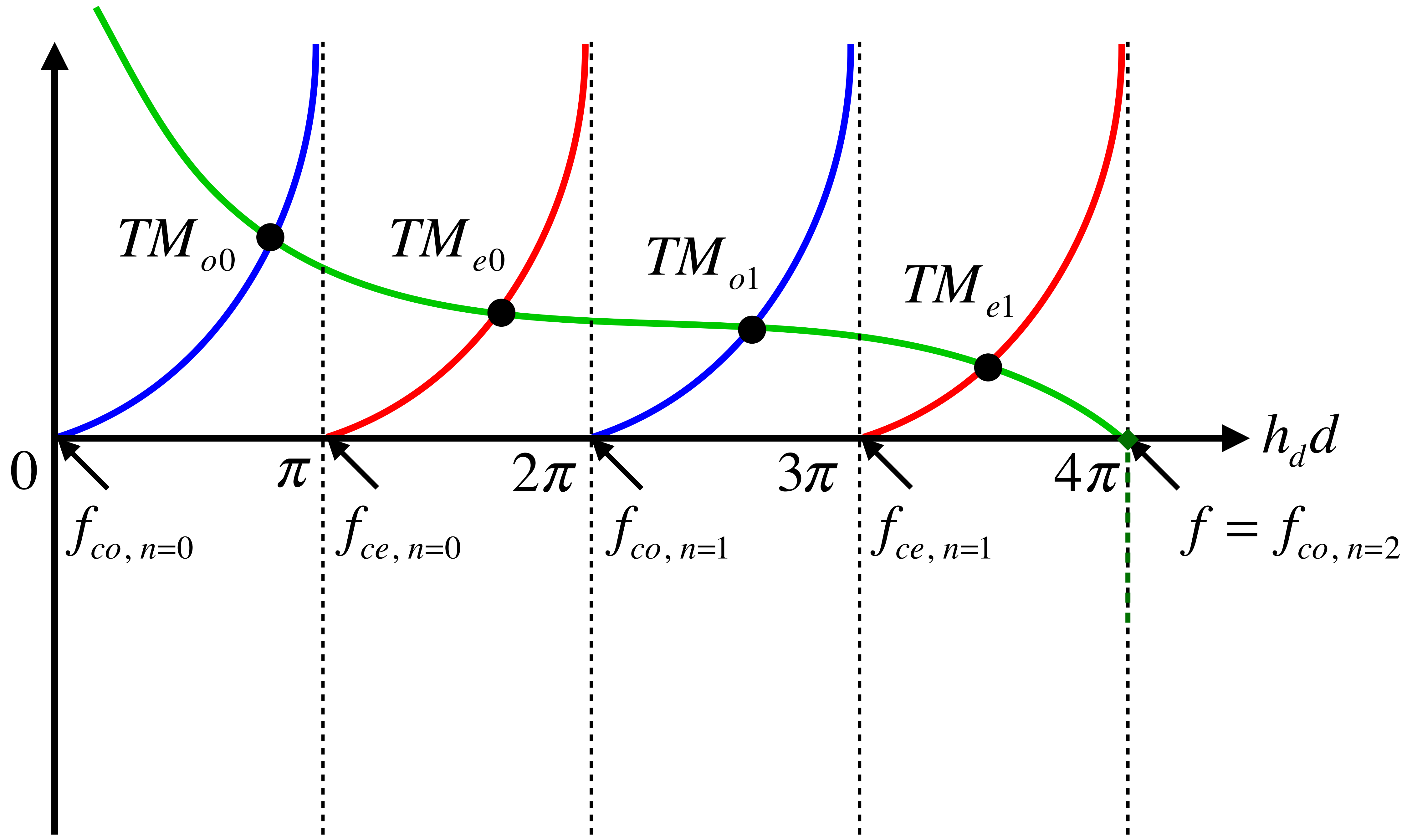
Difference

- For dielectric waveguide, waves of $f < f_c$ are no longer bounded to the dielectric and radiate away into free-space. *(5 points)*
  - Condition for f_c: $\alpha = 0$. *(5 points)*
  - Two cutoff frequency exists for even and odd modes $f_{co} = \dfrac{n}{d\sqrt{\mu_d\varepsilon_d - \mu_0\varepsilon_0}}$ and $f_{ce} = \dfrac{n+1/2}{d\sqrt{\mu_d\varepsilon_d - \mu_0\varepsilon_0}}$ *(5 points)*
- For rectangular/circular waveguide, waves of frequencies below f_c are attenuated. *(5 points)*
  - Condition for f_c: $\gamma = 0$. *(5 points)*
  - Common expression for cutoff frequency is given as $f_c = \dfrac{h}{2\pi\sqrt{\mu\varepsilon}}$ *(5 points)*

Midterm 2 | Question 4(a): Dielectric waveguide

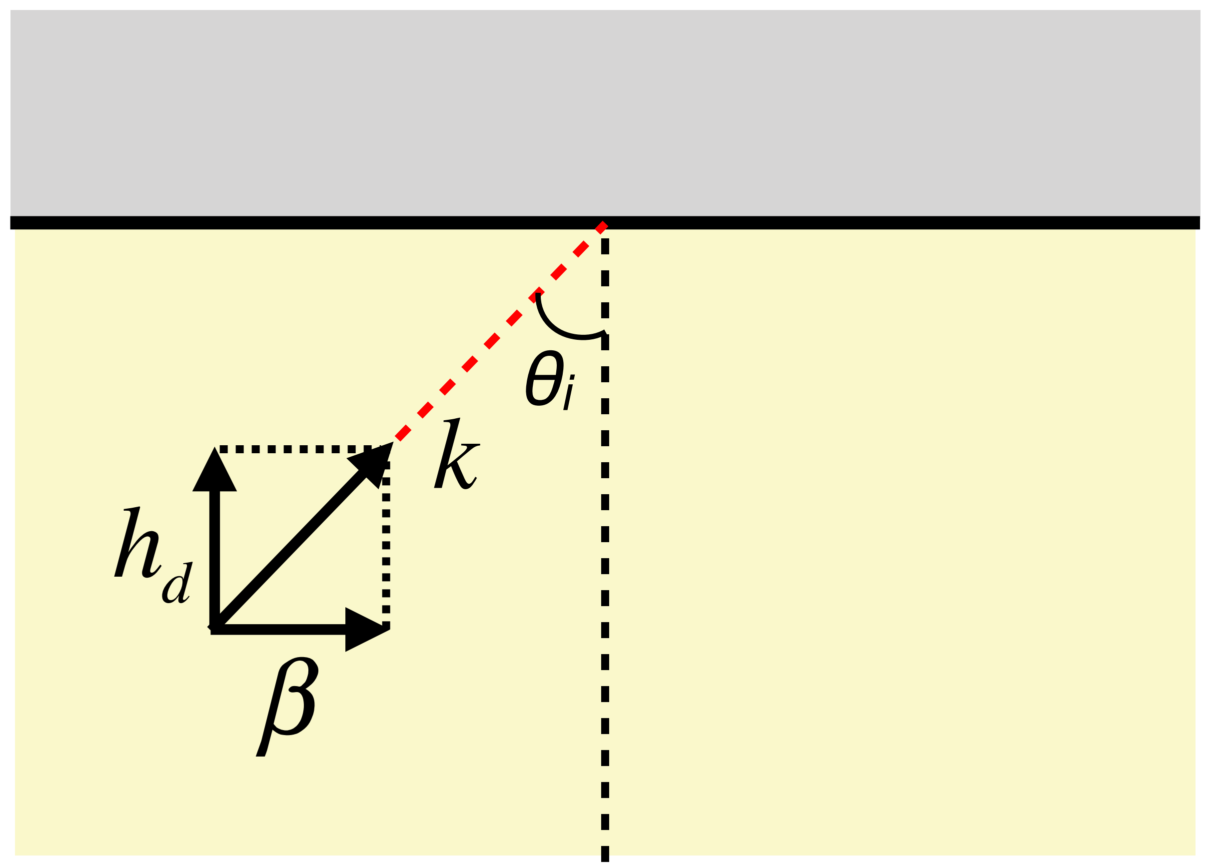
(c)** Transcendental equations for TM modes are given as below. If you want to send the TM-polarized waves with the frequency whose value coincides with **the cutoff frequency for the 2nd-order odd TM mode** along the dielectric waveguide, how many TM modes are allowed at that frequency and what is the difference among such modes? (Note that an order of the mode starts from n = 0) **(40 points)

$$\begin{cases} \dfrac{\alpha}{h_d} = \dfrac{\mu_0}{\mu_d}\tan\left(\dfrac{h_d d}{2}\right) & \cdots\text{for odd TM modes} \\ \dfrac{\alpha}{h_d} = -\dfrac{\mu_0}{\mu_d}\cot\left(\dfrac{h_d d}{2}\right) & \cdots\text{for even TM modes} \end{cases}$$



Answer

- There are total **4** allowed modes ***(20 points)*** *[$f_{co,n=2}$ should not be counted. Deduce -5 points if answer is 5 because of that. Otherwise, 0 points]*
- At a given frequency f, each mode has a distinct h_d value ***(5 points)*** and therefore, their incident angle at the dielectric/free-space interface is different as below. ***(15 points)***



$$\cos\theta_i = \frac{h_d}{k} = \frac{h_d}{2\pi f\sqrt{\mu_d\varepsilon_d}}$$

Midterm 2 | Question 5(a): Rectangular Cavity Resonator

(a) Discuss the physical meaning of the resonant frequency in a rectangular cavity resonator. (30 points)
(b) Is there power flow within the cavity? Why for your answer? (20 points)
(c) What is the dominant mode for a rectangular cavity resonator? (50 points)

(a) Resonant frequency

- Frequency at which resonance occurs (i.e., wave oscillates at f_r with greater amplitude than at other frequencies) ***(20 points)***
- It can have multiple values (m, n, p = 0, 1, 2, …) ***(5 points)***
- Determined by dimensions of the resonator (a, b, and d) ***(5 points)***

(b) No power flow ***(5 points)***

- All the E-fields are in time-phase, and E and H-fields are in time-quadrature ***(10 points)***
- Therefore, time-average Poynting vector becomes zero within the cavity ***(5 points)***

(c) Dominant mode depends on the dimension of the cavity ***(20 points)***

- If $a > b > d$, TM_{110}
- If $a > d > b$, TE_{101}
- If $a = d = b$, All three are degenerately dominant
- ***(Each example accounts for 10 points)***