Eng Math. Final Term (12/12/2007)

(Closed book and note: 120 min.)

1. Prove the following equation [20 points].

$|\sin z|^2 = \sin^2 x + \sinh^2 y$, where $z = x + iy$

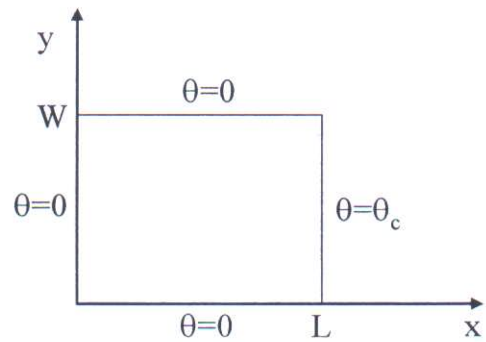
2. Evaluate the following integrals [40 points].

(1) $\int_0^{2\pi} e^{\cos\theta}\cos(\sin\theta)d\theta$

(2) $\int_0^{2\pi} e^{\cos\theta}\sin(\sin\theta)d\theta$

3. Find a temperature profile [$\theta = \theta(x, y)$] for the following two-dimensional heat conduction problem. Boundary conditions are given below [40 points].

$$\nabla^2\theta = 0$$



4. (25 pts) Give proper answers to each question:

(a) (20 pts) Solve $\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2}$ with BC and IC, u(0,t)=10, u(3,t)=40 and u(x,0)=25, respectively. (0<x<3, t>0) (Hint: use $u(x,t) = v(x,t) + \varphi(x)$)

(b) (5 pts) Find the temperature at steady state

5. (20 pts) With a positive integer n and $w = e^{2\pi i/n}$, evaluate $\sum_{j=0}^{n-1} (-1)^j w^j$.

6. (30 pts) Evaluate each integral.

(a) (10 pts) $\int_0^\infty e^{-s^2} \cos 2bs\,ds$. Show the detail.

(b) (20 pts) $\int_0^{2\pi} \frac{d\theta}{5+3\sin\theta} = ?$

7. (25 pts) Electrostatic potential of a sphere would reduce to Poisson equation $\nabla^2 u = -\frac{\rho}{\varepsilon_0}$. Given the Laplacian in spherical coordinate as

$$\nabla^2 u = \left[\left(r^2 u_r\right)_r + \frac{1}{\sin\phi}\left(\sin\phi\, u_\phi\right)_\phi + \frac{1}{\sin^2\phi} u_{\theta\theta} \right]$$

and with the conditions "$u(R,\phi) = \cos\phi$" and "$\lim_{r\to\infty} u(r,\phi) = 0$", find the potential of the sphere in the interior of the sphere. (Here the radius of the sphere is 1, and the charge density vanishes.) Show the detail.

You may use some of the followings:

Hint 1. $\sum_{m=1}^{\infty} A_m J_0(\frac{\alpha_m}{R} r) = f(r),\ A_m = \frac{2}{R^2 J_1^2(\alpha_m)} \int_0^R r f(r) J_0(\frac{\alpha_m}{R} r)dr$

Hint 2. $\sum_{m=1}^{\infty} A_n P_n(\cos\alpha) = f(\alpha),\ A_n = \frac{2n+1}{2} \int_0^\pi f(\phi) P_n(\cos\phi)\sin\phi\, d\phi$

Hint 3. $J_\alpha(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m+\alpha+1)} \left(\frac{x}{2}\right)^{2m+\alpha}$

Hint 4. Legendre's polynomial

$$P_0(x) = 1,$$
$$P_1(x) = x,$$
$$P_2(x) = \left(3x^2 - 1\right)/2$$

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