Eng Math. Final Term (12/12/2007)

(Closed book and note: 120 min.)

[1] prove that $|\sin z|^2 = \sin^2 x + \sinh^2 y$, $z = x + yi$

[20 points]

(solution)

$\sin z = \sin x \cosh y + i \cos x \sinh y$ 이고

$\cosh^2 y = 1 + \sinh^2 y$ 이므로

$$|\sin z|^2 = \left(\sqrt{(\sin x \cosh y)^2 + (\cos x \sinh y)^2}\right)^2$$

$$= \sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y$$

$$= \sin^2 x (1 + \sinh^2 y) + \cos^2 x \sinh^2 y$$

$$= \sin^2 x + \sin^2 x \sinh^2 y + \cos^2 x \sinh^2 y$$

$$= \sin^2 x + (\sin^2 x + \cos^2 x) \sinh^2 y$$

$$= \sin^2 x + \sinh^2 y$$

] Evaluate [40 points]
20 points×2

$$\int_0^{2\pi} e^{\cos\theta}\cos(\sin\theta)\,d\theta \quad , \quad \int_0^{2\pi} e^{\cos\theta}\sin(\sin\theta)\,d\theta$$

< Solution >

$$\int_c \frac{e^z}{z}\,dz = 2\pi i\left[e^z\right]_{z=0} = 2\pi i$$

by Cauchy's Integral formula

$$\int_c \frac{e^z}{z}\,dz = \int_0^{2\pi} \frac{e^{\cos\theta + i\sin\theta}}{\cancel{e^{i\theta}}}\, i\cancel{e^{i\theta}}\,d\theta$$

put $z = e^{i\theta}$

$0 \le \theta \le 2\pi$

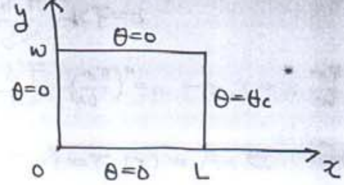
$$e^{\cos\theta + i\sin\theta} = e^{\cos\theta} e^{i\sin\theta} = e^{\cos\theta}\left[\cos(\sin\theta) + i\sin(\sin\theta)\right]$$

$$= \underbrace{i\int_0^{2\pi} e^{\cos\theta}\cos(\sin\theta)\,d\theta}_{\text{허수부}} - \underbrace{\int_0^{2\pi} e^{\cos\theta}\sin(\sin\theta)\,d\theta}_{\text{실수부} = 0} = 2\pi i$$

$$\therefore \int_0^{2\pi} e^{\cos\theta}\cos(\sin(\theta))\,d\theta = 2\pi$$

$$\int_0^{2\pi} e^{\cos\theta}\sin(\sin\theta)\,d\theta = 0$$

[40points]



$$\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}=0$$

Find $\theta(x,y)=?$

BC's
$\theta(0,y)=0 \quad (0<y<w)$
$\theta(0,L)=\theta_c \quad (0<y<w)$
$\theta(x,0)=0 \quad (0<x<L)$
$\theta(x,w)=0 \quad (0<x<L)$

(solution)

$$\theta(x,y)=F(x)G(y) \rightarrow \frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}=0 \rightarrow \theta_{xx}=-\theta_{yy}$$

$$G\frac{d^2F}{dx^2}+F\frac{d^2G}{dy^2}=0$$

$$G\frac{d^2F}{dx^2}=-F\frac{d^2G}{dy^2}$$

양변을 FG로 나누고 정리하면

$$\frac{1}{G}\frac{d^2G}{dy^2}=-\frac{1}{F}\frac{d^2F}{dx^2}=-K \text{ (constant)}$$

$$\frac{1}{G}\frac{d^2G}{dy^2}=-K$$

$$\frac{d^2G}{dy^2}+KG=0 \rightarrow \lambda=\pm\sqrt{K}$$

$$G(y)=A\cos\sqrt{K}y+B\sin\sqrt{K}y$$

$$G(0)=A=0 \rightarrow A=0$$

$$G(w)=B\sin\sqrt{K}w=0$$

$$\sqrt{K}w=n\pi \rightarrow \sqrt{K}=\frac{n\pi}{w} \rightarrow K=\left(\frac{n\pi}{w}\right)^2$$

$$\therefore G_n=G_n(y)=\sin\frac{n\pi}{w}y \quad -①$$

$$\frac{d^2F}{dx^2} - kF = 0$$

$$\frac{d^2F}{dx^2} - \left(\frac{n\pi}{w}\right)^2 = 0 \rightarrow \lambda = \pm \frac{n\pi}{w}$$

$$F(x) = F_n(x) = A_n e^{n\pi x/w} + B_n e^{-n\pi x/w}$$

$$F(0) = 0 \rightarrow F(0) = A_n + B_n = 0 \rightarrow B_n = -A_n$$

$$F_n(x) = A_n\left(e^{n\pi x/w} - e^{-n\pi x/w}\right)$$

$$= 2A_n \sinh\frac{n\pi}{w}x \qquad \because \sinh x = \frac{1}{2}(e^{x} - e^{-x})$$

$$= A_n^{*} \sinh\frac{n\pi}{w}x \quad \text{—②}$$

by ① and ②

$$\therefore \theta(x,y) = \sum_{n=1}^{\infty} A_n^{*} \sin\frac{n\pi y}{w} \sinh\frac{n\pi}{w}x \quad \text{—③}$$

$\theta(L,y) = \theta_c$ 이므로

$$\theta_c = \sum_{n=1}^{\infty} A_n^{*} \sin\frac{n\pi y}{w} \sinh\frac{n\pi L}{w} = \sum_{n=1}^{\infty} \underbrace{A_n^{*} \sinh\frac{n\pi L}{w}}_{\downarrow} \sin\frac{n\pi}{w}y$$

θ_c의 Fourier sine series

put $b_n = A_n^{*} \sinh\frac{n\pi L}{w} = \frac{2}{w}\int_0^w \theta_c \sin\frac{n\pi y}{w}\,dy$

$$A_n^{*} = \frac{2}{w}\,\frac{\theta_c}{\sinh\frac{n\pi L}{w}}\int_0^w \sin\frac{n\pi}{w}y\,dy$$

이를 식 ③에 대입하면

$$\theta(x,y) = \theta_c \frac{2}{\cancel{w}}\,\frac{\sin\frac{n\pi y}{w}\sinh\frac{n\pi x}{w}}{\sinh\frac{n\pi L}{w}}\underbrace{\int_0^w \sin\frac{n\pi y}{w}\,dy}_{\downarrow}$$

$$= \left(\frac{-w}{n\pi}\right)\left[\cos\frac{n\pi y}{w}\right]_0^1$$

$$= \left(\frac{-w}{n\pi}\right)(\cos n\pi - \cos 0)$$

$$= \left(\frac{-\cancel{w}}{n\pi}\right)\left[\cos n\pi - 1\right]$$

↳ $n = 0, 2, 4$ 일때 → 1

전체 → 0

$$\theta(x,y) = \theta_c \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}+1}{n}\,\frac{\sinh\frac{n\pi x}{w}}{\sinh\frac{n\pi L}{w}}\sin\frac{n\pi y}{w}$$

4. (25 pts) (a) (20 pts)   (b) (5 pts)

(a) $u_t = 2 \cdot u_{xx}$ $u(0,t) = 10$

$u(3,t) = 40$

$u(x,0) = 25$

$u(x,t) = v(x,t) + \phi(x)$

$u(0,t) = 10 = v(0,t) + \phi(0) = 10$

$u(3,t) = 40 = v(3,t) + \phi(3) = 40$

$u_t = v_t = 2u_{xx} = 2[u_{xx} + \phi_{xx}]$

let $\phi(0) = 10$, $\phi(3) = 40$, $\phi''(x) = 0$, then $\phi = 10x + 10$

$v(0,t) = 0$, $v(3,t) = 0$

& $v_t = 2v_{xx}$

Since $L = 3$, $v(x,t) = \sum_1^{\infty} v_n = \sum_1^{\infty} \left[\frac{2}{3}\int_0^3 f(x) \cdot \sin\frac{n\pi x}{3}\,dx\right] \cdot \sin\frac{n\pi x}{3} \cdot e^{-\frac{2n^2\pi^2}{9}t}$

$\to v(x,0) = 15 - 10x$

$u(x,0) = v(x,0) + \phi(x) = 25$

$v(x,0) = 25 - \phi = 15 - 10x$

$\frac{2}{3}\int_0^3 (15 - 10x)\sin\frac{n\pi x}{3}\,dx = \frac{30}{n\pi}(\cos n\pi - 1)$

$\therefore u(x,t) = v + \phi = 10x + 10 + \sum_{n=1}^{\infty}\left[\frac{2}{3}\int_0^3 (15 - 10x) \cdot \sin\frac{n\pi x}{3}\,dx\right]\sin\frac{n\pi x}{3} \cdot e^{-\frac{2n^2\pi^2 t}{9}}$

or

$= 10x + 10 + \sum_{n=1}^{\infty}\frac{30}{n\pi}(\cos n\pi - 1)\sin\frac{n\pi x}{3} \cdot e^{-\frac{2n^2\pi^2}{9}t}$

(b) at steady state $10x + 10$.

5. (20 pts)

With a positive integer n and $w = e^{2\pi i/n}$, evaluate $\sum_{j=0}^{n-1} (-1)^j w^j$.

Sol)

Let w ($\neq 1$) be any nth root of unity. Then, $\sum_{j=0}^{n-1} (-1)^j w^j = \sum_{j=0}^{n-1} (-w)^j$.

Considering three cases:

For odd n,

$$\sum_{j=0}^{n-1} (-w)^j = \frac{1-(-w)^n}{1-(-w)} = \frac{1+w^n}{1+w} = \frac{2}{1+w}$$

(since $w^n = 1$)

For even n with w=−1,

$$\sum_{j=0}^{n-1} (-w)^j = \sum_{j=0}^{n-1} 1 = n$$

For even n with w≠−1,

$$\sum_{j=0}^{n-1} (-w)^j = \frac{1-(-w)^n}{1-(-w)} = \frac{1-w^n}{1+w} = 0$$

(since $w^n = 1$)

6. (30 pts) Evaluate the integral.

(a) (10 pts) $\int_0^\infty e^{-s^2} \cos 2bs ds$. Show the detail.

Sol)

$$\left[\int_0^\infty e^{-x^2} dx\right]^2 = \frac{1}{4}\int_{-\infty}^\infty e^{-x^2} dx \int_{-\infty}^\infty e^{-y^2} dy$$
$$= \frac{1}{4}\int_{-\infty}^\infty \int_{-\infty}^\infty e^{-(x^2+y^2)} dxdy = \frac{1}{4}\int_0^{2\pi}\int_0^\infty e^{-r^2} rdrd\theta$$
$$= \frac{\pi}{2}\int_0^\infty e^{-r^2} rdr = \frac{\pi}{4}$$
$$Therefore,\ \int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$$
$$\int_0^\infty e^{-s^2} \cos 2bsds = \int_0^\infty e^{-s^2} e^{2ibs} ds$$
$$= e^{(ib)^2}\int_0^\infty e^{-s^2+2ibs-(ib)^2} ds$$
$$= e^{(ib)^2}\int_0^\infty e^{-(s-ib)^2} ds, \text{let } p = s - ib$$
$$= \frac{e^{(ib)^2}}{2}\int_{-\infty}^\infty e^{-p^2} dp = \frac{e^{-b^2}\sqrt{\pi}}{2}$$

(b) (20 pts) $\int_0^{2\pi} \frac{d\theta}{5+3\sin\theta} = ?$

Sol) Let $z = e^{i\theta}$, then $\sin\theta = \frac{z-1/z}{2i}$ and $d\theta = \frac{dz}{iz}$

$$\int_0^{2\pi} \frac{d\theta}{5+3\sin\theta} = \oint_C \frac{dz/iz}{5+3\frac{z-z^{-1}}{2i}}, \text{ where C is unit circle.}$$

$$Then, \oint_C \frac{dz/iz}{5+3\frac{z-z^{-1}}{2i}} = \oint_C \frac{2dz}{3z^2+10iz-3} = \oint_C \frac{2dz}{3(z+3i)(z+i/3)}$$

$$\text{let } f(z) = \frac{1}{3(z+3i)}$$

$$= \oint_C \frac{2f(z)dz}{(z+i/3)} = 2\times 2\pi i f(-i/3), f(-i/3) = \frac{1}{8i}$$

$$= \pi/2$$

7. (25 pts)

Sol)

The PDE after separating variables becomes Legendre ODE.

Therefore,

$$\tilde{f} = w, A_n = \frac{2n+1}{2}\int_{-1}^{1} wP_n(w)dw$$

Due to the orthogonal $P_1(w)(=w)$ on the interval from −1 to 1, we can obtain $A_1=1$ and $A_n=0$ (n=2,3,4⋯). So, the answer after integration is $u = r\cos\phi$.