School of Mech & Aero Eng
Seoul National University

Eng Probability
April 17, 2008

MIDTERM

- Do not open exam until told to do so.
- 'How you arrived at your answer' is much more important than the answer itself. Read the following problems carefully, and make sure you show your work *step by step*.
- You can attach extra pages if necessary. Please use a separate sheet for each problem.
- Ask questions if you don't understand what you are being asked, and **GOOD LUCK !**

Student ID: ________________

Name: ________________

1	/ 10
2	/ 15
3	/ 10
4	/ 15
5	/ 10
6	/ 20
7	/ 15
8	/ 10
Total	/ 100

1. [`5+5=10 pts`]

(a) You want to find someone with the same birthday as yours (out of 365 days per year). What is the least number of people you need to ask to have a 50 % chance of finding at least one matches?

Sol. $P = 1 - (364/365)^n \geq 1/2$
$\therefore n \geq \frac{\log 2}{\log(365/364)}$

(b) A girl and her friend are supposed to meet between 1 and 2 PM. Each comes at a random moment between 1 and 2 PM and waits for exactly 10 minutes. The meeting is successful only when the other person arrives within the 10-minute interval. What is the probability that the two people successfully meet?

Sol.

$$P(|X - Y| \leq 1/6) = 1 - (5/6)^2 = 11/36.$$

2. [15 pts] A biased coin is tossed repeatedly. Each toss is independent with a probability p of a head. Show that the probability that there is a run of r heads in a row before there is a run of s tails is

$$\frac{p^{r-1}(1-q^s)}{p^{r-1}+q^{s-1}-p^{r-1}q^{s-1}} \ ,$$

where r and s are positive integers.

Sol.

A: r heads before s tails
B: first toss is a head
C: first s tosses are tails
Then

$$\begin{aligned} P(A|B^c) &= P(A|B^c \cap C)P(C|B^c) + P(A|B^c \cap C^c)P(C^c|B^c) \\ &= 0 + P(A|B)(1-q^{s-1}) \end{aligned}$$

where $q = 1 - p$. Similarly,

$$P(A|B) = p^{r-1} + P(A|B^c)(1-p^{r-1}) \ .$$

Solve for $P(A|B)$ and $P(A|B^c)$, and use

$$\begin{aligned} P(A) &= P(A|B)p + P(A|B^c)q \\ &= Ans. \end{aligned}$$

3. [`5+5=10 pts`] Let X and Y be independent random variables with common distribution function F and density function f.

(a) Compute the distribution function and density function of $V = \max(X, Y)$.

(b) Compute the distribution function and density function of $U = \min(X, Y)$.

Sol.

$$P(V \leq v) = P(X \leq v, Y \leq v) = P(X \leq v)P(Y \leq v) = F(v)^2$$

$$f_V(v) = 2F(v)f(v)$$

$$P(U \leq u) = 1 - P(U > u) = 1 - P(X \leq u)P(Y \leq u) = 1 - [1 - F(u)]^2$$

$$f_U(u) = 2f(u)(1 - F(u))$$

4. [`7+8=15 pts`]Let X have the normal distribution $N(0,1)$.

(a) Compute the density function of $Y = e^X$.

(b) Let $Z = \mu + \sigma X$. Show that $E[(Z-\mu)g(Z)] = \sigma^2 E[g'(Z)]$ for some function g.

Sol.

$$f_X(x) = \frac{1}{\sqrt{2\pi}} \exp^{-x^2/2}$$

(a)

$$P(Y \le y) = P(X \le \log y)$$

$$f_Y(y) = \frac{1}{y} f_X(\log y) = \frac{1}{y\sqrt{2\pi}} e^{-\frac{1}{2}(\log y)^2}$$

(b)

$$\begin{aligned}
P(Z \le z) &= P(X \le \frac{z-\mu}{\sigma}) \\
\therefore\ f_Z(z) &= \frac{1}{\sigma} f_X(\frac{z-\mu}{\sigma}) \\
E[(Z-\mu)g(Z)] &= \int_{-\infty}^{\infty} (z-\mu) g(z) f_Z(z)\, dz \\
&= \int_{-\infty}^{\infty} g(z) \left\{ \frac{z-\mu}{\sigma} f_X(\frac{z-\mu}{\sigma}) \right\} dz \\
&= -\left[g(z)\sigma f_X(\frac{z-\mu}{\sigma}) \right]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} g'(z)\sigma f_X(\frac{z-\mu}{\sigma})\, dz \\
&= \sigma^2 \int_{-\infty}^{\infty} g'(z) f_Z(z)\, dz \\
&= RHS.
\end{aligned}$$

5. [`10 pts`] A point (X, Y) is chosen uniformly at random in the unit circle. find the joint density function of $R^2 = X^2 + Y^2$ and X.

Sol.

$$F(r, x) = P(R \le r, X \le x) = \frac{2}{\pi} \int_{-r}^{x} \sqrt{r^2 - u^2} du$$

$$f(r, x) = \frac{2r}{\pi \sqrt{r^2 - u^2}} \quad |x| < r < 1$$

6. [`15 pts`] A random number N of dice is thrown. Let $P(N = i) = 2^{-i},\ i \geq 1$, and S denote the sum of the scores.find the probability that

(a) S=4 given N=even.

(b) the largest number shown by any die is less than or equal to m, where S is unknown.

(c) the largest number shown by any die is equal to m, where S is unknown.

Sol.

(a)

$$\begin{aligned} P(S=4|Neven) &= \frac{P(S=4|N=2)\frac{1}{4} + P(S=4|N=4)\frac{1}{16}}{P(N\ even)} \\ &= \frac{\frac{1}{12}\frac{1}{4} + \frac{1}{6^4}\frac{1}{16}}{\frac{1}{4} + \frac{1}{16} + \cdots} = \frac{4^2 3^3 + 1}{4^4 3^3} \end{aligned}$$

(b) M: the largest number shown by any die.

$$P(M \leq m) = \sum_{j=1}^{\infty} P(M \leq m|N=j)2^{-j} = \frac{r}{12}(1 - \frac{r}{12})^{-1} = \frac{r}{r-12}$$

(c)

$$P(M = m) = P(M \leq m) - P(M \leq m-1)$$

7. [`15 pts`] Let X and Y have joint density function

$$f(x,y) = \frac{1}{x}, \quad 0 \le y \le x \le 1.$$

(a) compute the density functions of X.

(b) compute the density function of $X+Y$.

Sol.

(a) $f_X(x) = \int_0^x \frac{1}{x} dy = 1.$

(b) $f_{X+Y}(z) = \int_A \frac{1}{x} dx$, where $A = \{x : 0 \le z - x \le x \le 1\} = [z/2,\ \min(z,1)]$.

$$f_{X+Y}(z) = \begin{cases} \log 2 & 0 \le z \le 1 \\ \log(2/z) & 1 \le z \le 2 \\ 0 & else \end{cases}$$

8. [`10 pts`] If the density of X is given by

$$f(x) = \begin{cases} ax + bx^2 & 1 > x > 0 \\ 0 & else \end{cases},$$

and $E[X] = 0.7$, compute $var(X)$ and $P(X > 0.9)$.

Sol.

$$1 = \int_{-\infty}^{\infty} f(x)dx = a/2 + b/3$$
$$.7 = \int_{-\infty}^{\infty} xf(x)dx = a/3 + b/4$$
$$\therefore a = b = 1.2$$

and

$$var(X) = \int_{-\infty}^{\infty} x^2 f(x)dx - (E[X])^2 = 0.05$$
$$P(X > 0.9) = \int_{.9}^{1} ax + bx^2 \; dx = .6(1 - .9^2) + .4(1 - .9^3)$$